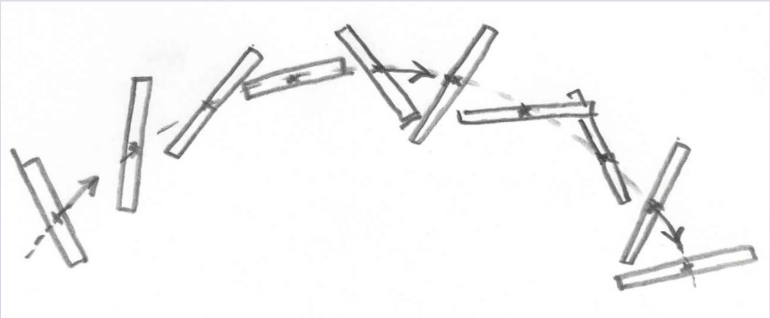
Translation and Rotation Kinematics

Overview: Rotation and Translation of Rigid Body



Thrown Rigid Rod

Translational Motion: the gravitational external force acts on center-of-mass

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt} = m^{\text{total}}\frac{d\vec{\mathbf{V}}_{\text{cm}}}{dt} = m^{\text{total}}\vec{\mathbf{A}}_{\text{cm}}$$

Rotational Motion: object rotates about center-of-mass. Note that the center-of-mass may be accelerating

Overview: Rotation about the Center-of-Mass of a Rigid Body

The total external torque produces an angular acceleration about the center-of-mass

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} = I_{\text{cm}}\vec{\alpha}_{\text{cm}} = \frac{d\vec{\mathbf{L}}_{\text{cm}}}{dt}$$

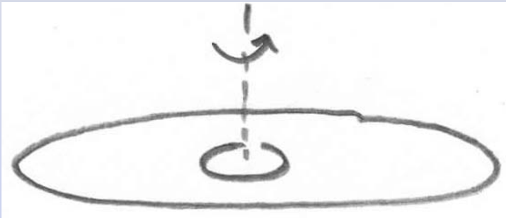
I_{cm} is the moment of inertial about the center-of-mass

α_{cm} is the angular acceleration about the center-of-mass

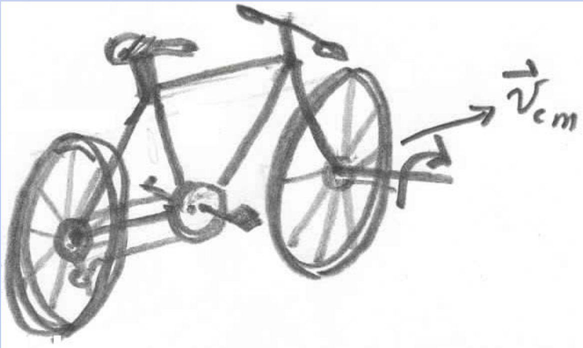
$\vec{\mathbf{L}}_{\text{cm}}$ is the angular momentum about the center-of-mass

Fixed Axis Rotation

- CD is rotating about axis passing through the center of the disc and is perpendicular to the plane of the disc.

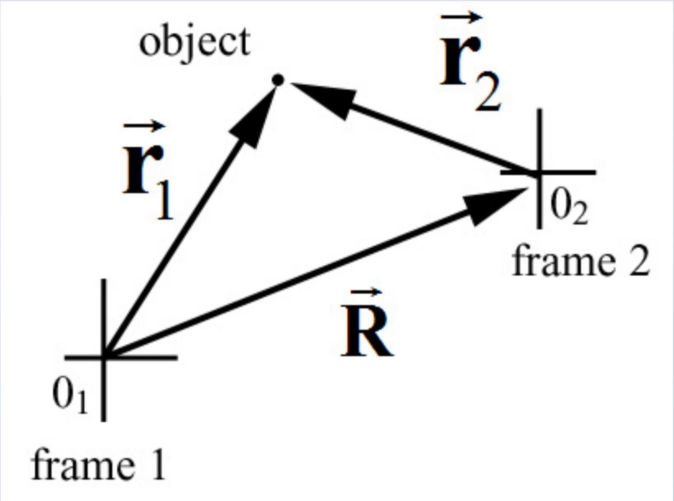


- For straight line motion, bicycle wheel rotates about fixed direction and center of mass is translating



Review: Relatively Inertial Reference Frames

Two reference frames.



Origins need not coincide.

One moving object has different position vectors in different frames

$$\vec{\mathbf{r}}_1 = \vec{\mathbf{R}} + \vec{\mathbf{r}}_2$$

Relative velocity between the two reference frames

$$\vec{\mathbf{V}} = d\vec{\mathbf{R}}\big/dt$$

is constant since the relative acceleration is zero

$$\vec{\mathbf{A}} = d\vec{\mathbf{V}}\big/dt = \vec{\mathbf{0}}$$

Review: Law of Addition of Velocities

Suppose the object is moving; then, observers in different reference frames will measure different velocities

Velocity of the object in Frame 1: $\vec{\mathbf{v}}_1 = d\vec{\mathbf{r}}_1 / dt$

Velocity of the object in Frame 2: $\vec{\mathbf{v}}_2 = d\vec{\mathbf{r}}_2 / dt$

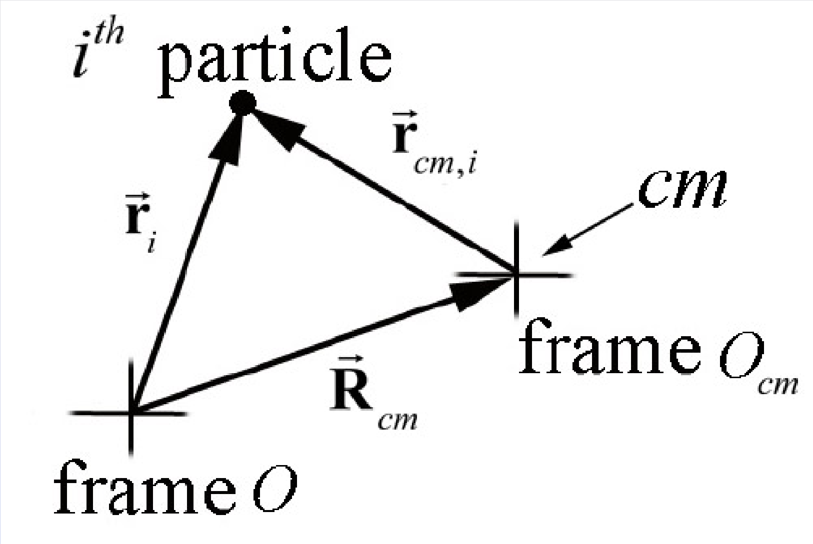
Velocity of an object in two different reference frames

$$\frac{d\vec{\mathbf{r}}_1}{dt} = \frac{\vec{\mathbf{R}}}{dt} + \frac{\vec{\mathbf{r}}_2}{dt}$$

$$\vec{\mathbf{v}}_1 = \vec{\mathbf{V}} + \vec{\mathbf{v}}_2$$

Center of Mass Reference Frame

Frame *O:* At rest with respect to ground



Frame O_{cm}: Origin located at center of mass

Position vectors in different frames:

$$\vec{\mathbf{r}}_i = \vec{\mathbf{r}}_{\mathrm{cm},i} + \vec{\mathbf{R}}_{cm} \qquad \vec{\mathbf{r}}_{\mathrm{cm},i} = \vec{\mathbf{r}}_i - \vec{\mathbf{R}}_{cm}$$

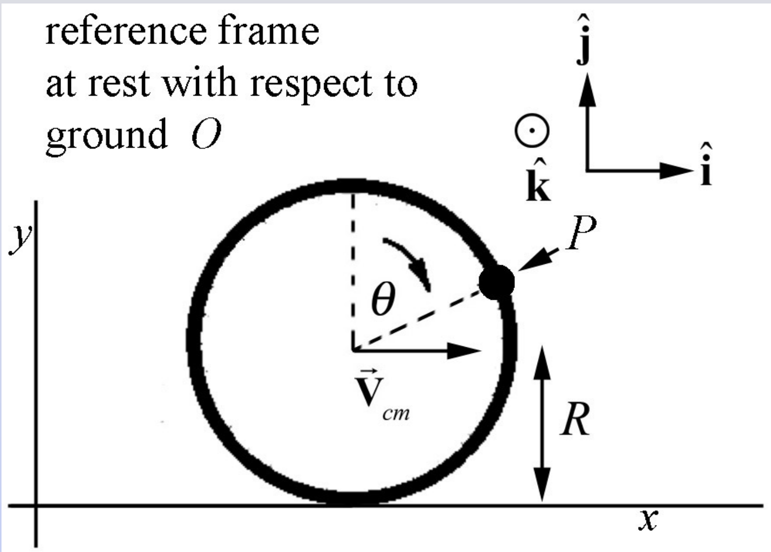
Relative velocity between the two reference frames

$$\vec{\mathbf{V}}_{cm} = d\vec{\mathbf{R}}_{cm} / dt \qquad \vec{\mathbf{A}}_{cm} = d\vec{\mathbf{V}}_{cm} / dt = \vec{\mathbf{0}}$$

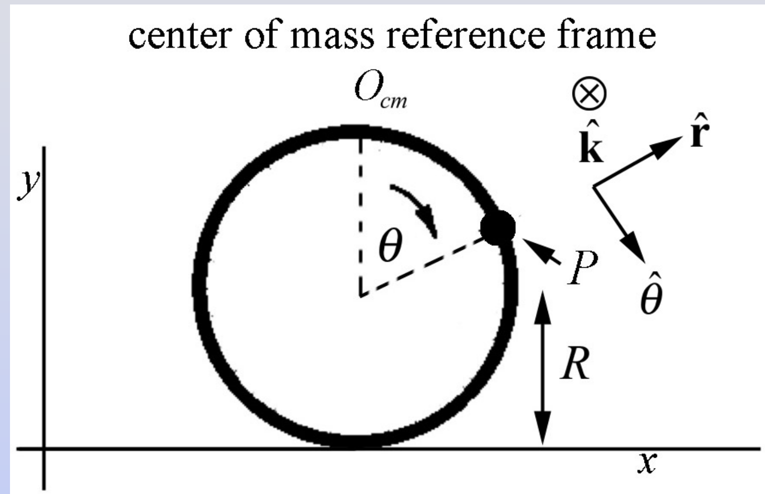
Law of addition of velocities:

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}_{\mathrm{cm},i} + \vec{\mathbf{V}}_{cm} \qquad \vec{\mathbf{v}}_{\mathrm{cm},i} = \vec{\mathbf{v}}_i - \vec{\mathbf{V}}_{cm}$$

Rolling Bicycle Wheel



Reference frame fixed to ground



Center of mass reference frame

Motion of point *P* on rim of rolling bicycle wheel

Relative velocity of point *P* on rim: $\vec{\mathbf{v}}_P = \vec{\mathbf{v}}_{\mathrm{cm},P} + \vec{\mathbf{V}}_{cm}$

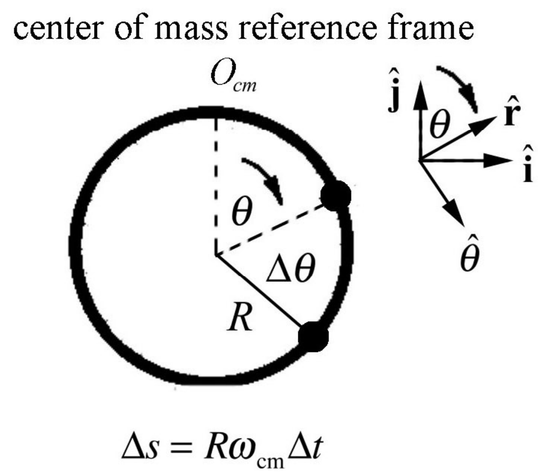
Rolling Bicycle Wheel

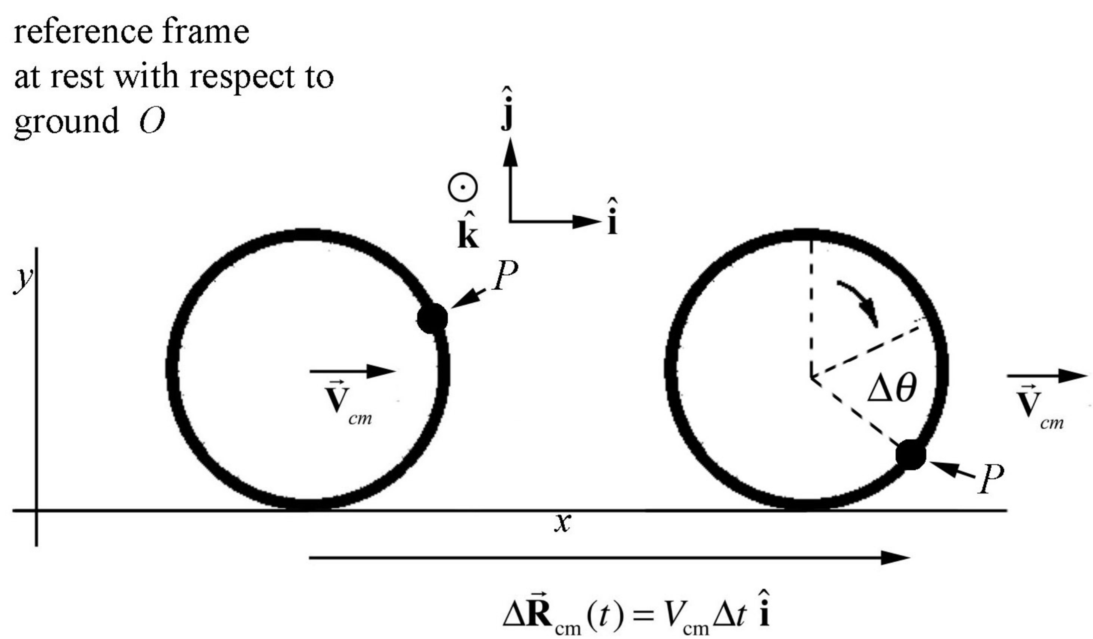
Distance traveled in center of mass reference frame of point *P* on rim in time Δt:

$$\Delta s = R\Delta\theta = R\omega_{\text{cm}}\Delta t$$

Distance traveled in ground fixed reference frame of point *P* on rim in time Δt:

$$\Delta X_{\text{cm}} = V_{\text{cm}}\Delta t$$





Rolling Bicycle Wheel: Constraint relations

Rolling without slipping:

$$\Delta s = \Delta X_{\text{cm}}$$

$$R\omega_{\text{cm}} = V_{\text{cm}}$$

Rolling and Skidding

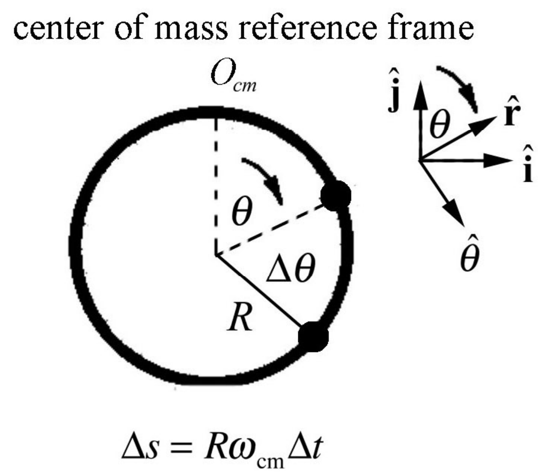
$$\Delta s < \Delta X_{\text{cm}}$$

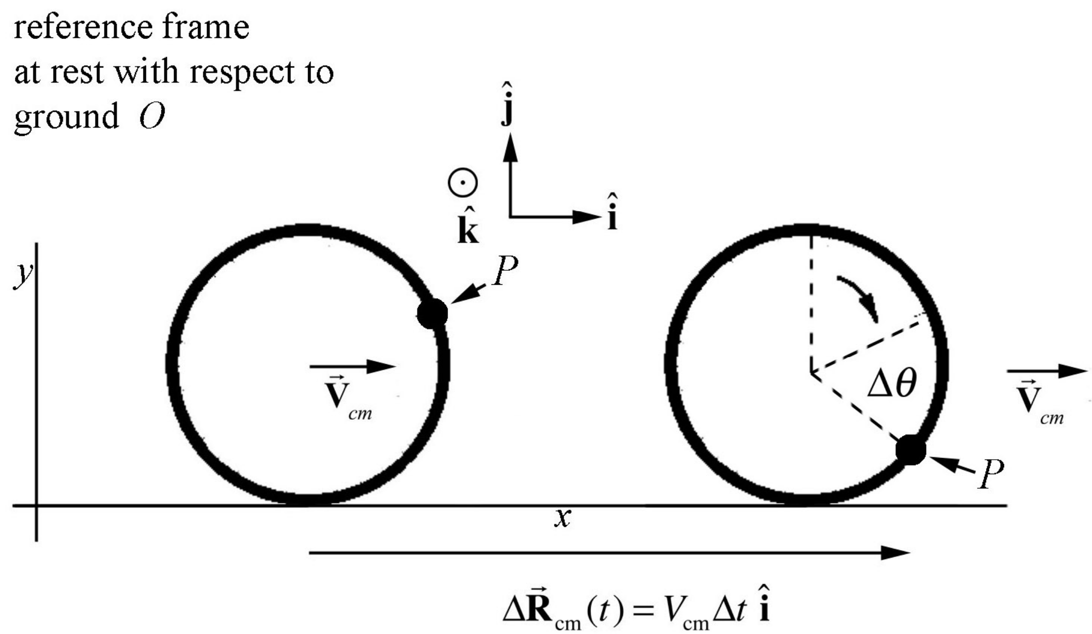
$$R\omega_{\text{cm}} < V_{\text{cm}}$$

Rolling and Slipping

$$\Delta s > \Delta X_{\text{cm}}$$

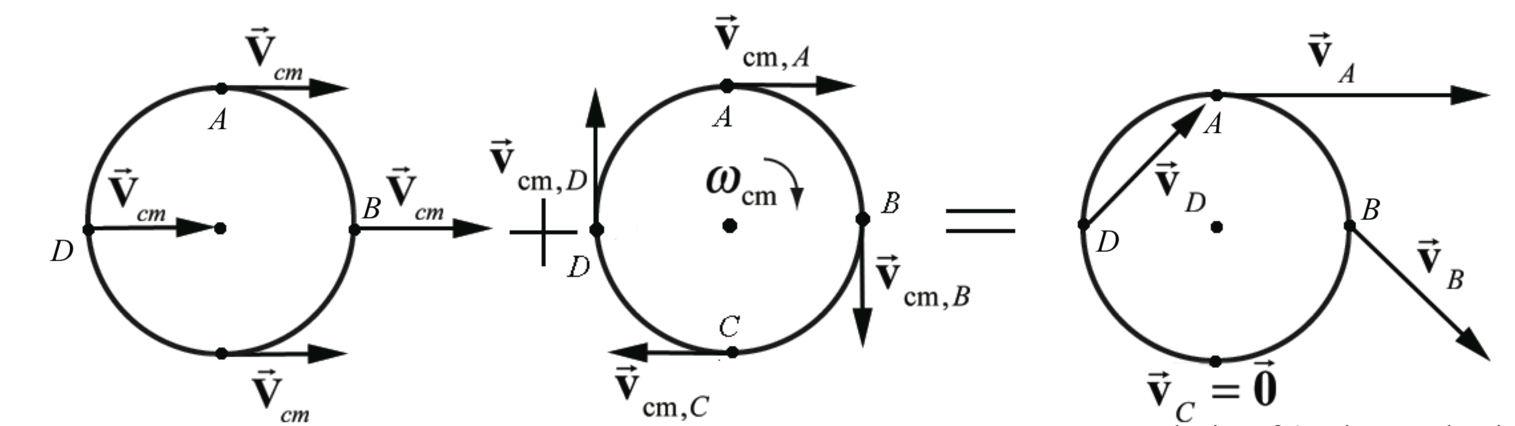
$$R\omega_{\text{cm}} > V_{\text{cm}}$$





Rolling Without Slipping: velocity of points on the rim in reference frame fixed to ground

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}_{\mathrm{cm},i} + \vec{\mathbf{V}}_{cm}$$



translational motion of center of mass of wheel

rotational motion of wheel in center of mass reference frame

velocity of 4 points on the rim in reference frame fixed to ground

The velocity of the point on the rim that is in contact with the ground is zero in the reference frame fixed to the ground.

Rotational Work-Kinetic Energy Theorem

Change in kinetic energy of rotation about center-of-mass

$$\Delta K_{\text{rot}} \equiv K_{\text{rot},f} - K_{\text{rot},i} = \frac{1}{2} I_{\text{cm}} \omega^2_{\text{cm},f} - \frac{1}{2} I_{\text{cm}} \omega^2_{\text{cm},i}$$

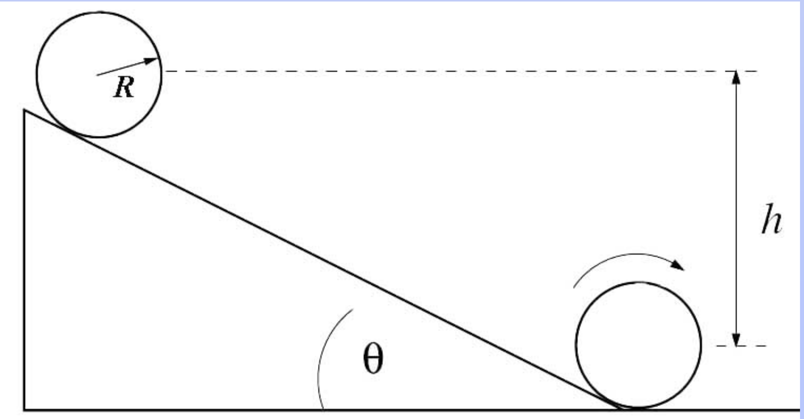
Change in rotational and translational kinetic energy

$$\Delta K = \Delta K_{\text{trans}} + \Delta K_{\text{rot}}$$

$$\Delta K = \Delta K_{\text{trans}} + \Delta K_{\text{rot}} = \left(\frac{1}{2} m v^2_{\text{cm},f} - \frac{1}{2} m v^2_{\text{cm},i} \right) + \left(\frac{1}{2} I_{\text{cm}} \omega^2_{cm,f} - \frac{1}{2} I_{\text{cm}} \omega^2_{cm,i} \right)$$

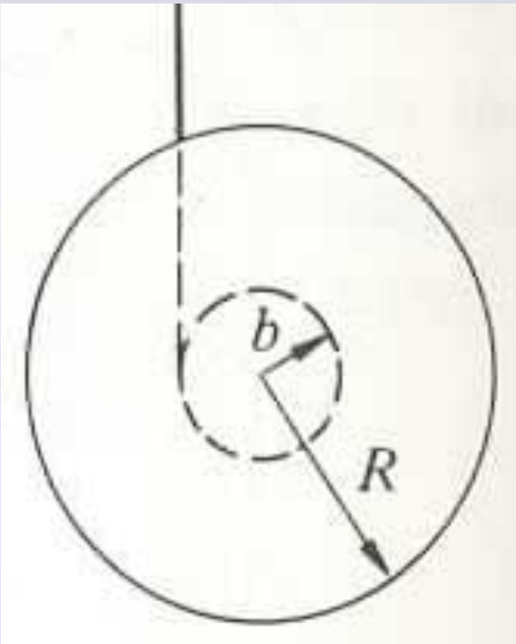
Checkpoint Problem: Cylinder on Inclined Plane Energy Method

A hollow cylinder of outer radius R and mass m with moment of inertia I_{cm} about the center of mass starts from rest and moves down an incline tilted at an angle θ from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ_s. The cylinder rolls without slipping down the incline. Using energy techniques calculate the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.



Checkpoint Problem: Descending Yo-Yo

A Yo-Yo of mass m has an axle of radius b and a spool of radius R. It's moment of inertia about the center of mass can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest. What is the angular speed of the Yo-Yo at the bottom of its descent.



Demo B107: Descending and Ascending Yo-Yo

$$M_{\text{wheel+axle}} = 435 \text{ g}$$

$$R_{\text{outer}} \cong 6.3 \text{ cm}$$

$$R_{\text{inner}} \cong 4.9 \text{ cm}$$

$$I_{\text{cm}} \cong \frac{1}{2} M \left(R_{\text{outer}}{}^2 + R_{\text{inner}}{}^2 \right)$$

$$= 1.385 \times 10^4 \text{ g} \cdot \text{cm}^2$$

Angular Momentum for Rotation and Translation

Angular Momentum for 2-Dim Rotation and Translation

The angular momentum for a rotating and translating object is given by (see next two slides for details of derivation)

$$\vec{\mathbf{L}}_S = \vec{\mathbf{R}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\text{cm,i}} \times m_i \vec{\mathbf{v}}_{\text{cm,i}}$$

The first term in the expression for angular momentum about *S* arises from treating the body as a point mass located at the center-of-mass moving with a velocity equal to the center-of-mass velocity,

$$\vec{\mathbf{L}}_{S,\text{cm}} = \vec{\mathbf{R}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}}$$

The second term is the angular momentum about the center-of mass,

$$\vec{\mathbf{L}}_{\text{cm}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\text{cm,i}} \times m_i \vec{\mathbf{v}}_{\text{cm,i}}$$

Derivation: Angular Momentum for 2-Dim Rotation and Translation

The angular momentum for a rotating and translating object is given by

$$\vec{\mathbf{L}}_S = \left(\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_{\mathrm{i}} \times m_i \vec{\mathbf{v}}_{\mathrm{i}} \right)$$

The position and velocity with respect to the center-of-mass reference frame of each mass element is given by

$$\vec{\mathbf{r}}_{\mathrm{i}} = \vec{\mathbf{R}}_{S,\mathrm{cm}} + \vec{\mathbf{r}}_{\mathrm{cm,i}} \qquad \vec{\mathbf{v}}_{\mathrm{i}} = \vec{\mathbf{V}}_{\mathrm{cm}} + \vec{\mathbf{v}}_{\mathrm{cm,i}}$$

So the angular momentum can be expressed as

$$\vec{\mathbf{L}}_S = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \left(\sum_{i=1}^{i=N} m_i \right) \vec{\mathbf{V}}_{\mathrm{cm}} + \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \sum_{i=1}^{i=N} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} + \left(\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_{\mathrm{cm,i}} \right) \times \vec{\mathbf{V}}_{\mathrm{cm}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}}$$

Derivation: Angular Momentum for 2-Dim Rotation and Translation

$$\vec{\mathbf{L}}_S = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \left(\sum_{i=1}^{i=N} m_i\right)\vec{\mathbf{V}}_{\mathrm{cm}} + \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \sum_{i=1}^{i=N} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} + \left(\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_{\mathrm{cm,i}}\right)\times \vec{\mathbf{V}}_{\mathrm{cm}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}}$$

The two middle terms in the above expression vanish because in the center-of-mass frame, the position of the center-of-mass is at the origin, and the total momentum in the center-of-mass frame is zero,

$$\frac{1}{m^{\mathrm{total}}}\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_{\mathrm{cm,i}} = \vec{\mathbf{0}} \qquad\qquad \sum_{i=1}^{i=N} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} = \vec{\mathbf{0}}$$

Then then angular momentum about *S* becomes

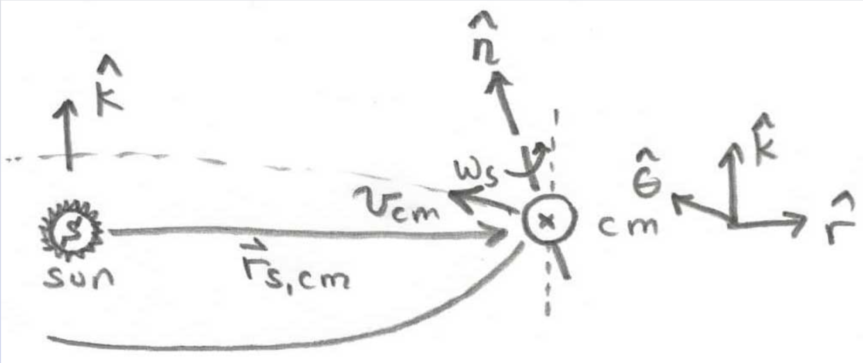
$$\vec{\mathbf{L}}_S = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \left(\sum_{i=1}^{i=N} m_i\right)\vec{\mathbf{V}}_{\mathrm{cm}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}}$$

The momentum of system is $\vec{\mathbf{p}}^{\mathrm{sys}} = \left(\sum_{i=1}^{i=N} m_i\right)\vec{\mathbf{V}}_{\mathrm{cm}}$

So the angular momentum about *S* is

$$\vec{\mathbf{L}}_S = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \vec{\mathbf{p}}^{\mathrm{sys}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}}$$

Earth's Motion about Sun: Orbital Angular Momentum



For a body undergoing orbital motion like the earth orbiting the sun, the two terms can be thought of as an orbital angular momentum about the center-of-mass of the earth-sun system, denoted by *S*,

$$\vec{\mathbf{L}}_{S,\mathrm{cm}} = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \vec{\mathbf{p}}^{\mathrm{sys}} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}}$$

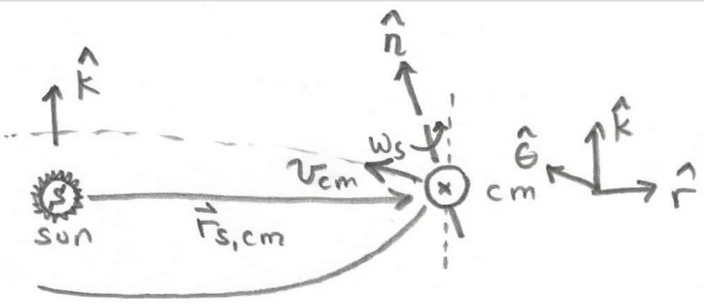
Spin angular momentum about center-of-mass of earth

$$\vec{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}} = I_{\mathrm{cm}} \vec{\boldsymbol{\omega}}_{\mathrm{spin}} = \frac{2}{5} m_e R_e^{\,2} \omega_{\mathrm{spin}} \hat{\mathbf{n}}$$

Total angular momentum about *S*

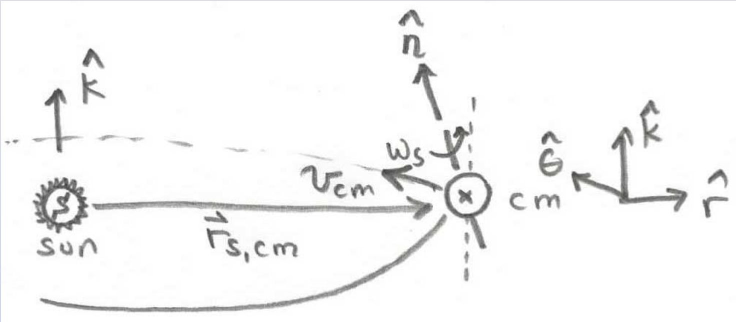
$$\vec{\mathbf{L}}_{S}^{\mathrm{total}} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}} + \frac{2}{5} m_e R_e^{\,2} \omega_{\mathrm{spin}} \hat{\mathbf{n}} s$$

Earth's Motion Orbital Angular Momentum about Sun



- Orbital angular momentum about center of sun $\vec{\mathbf{L}}_S^{orbital} = \vec{\mathbf{r}}_{S,cm} \times \vec{\mathbf{p}}^{total} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}}$
- Center of mass velocity and angular velocity $v_{cm} = r_{s,e}\omega_{orbit}$
- Period and angular velocity $\omega_{orbit} = \dfrac{2\pi}{T_{orbit}} = 2.0 \times 10^{-7}\, rad \cdot s^{-1}$
- Magnitude $\vec{\mathbf{L}}_S^{orbital} = m_e r_{s,e}{}^2 \omega_{orbit} \hat{\mathbf{k}} = \dfrac{m_e r_{s,e}{}^2 2\pi}{T_{orbit}} \hat{\mathbf{k}}$ $\quad \vec{\mathbf{L}}_S^{orbital} = 2.67 \times 10^{40}\, kg \cdot m^2 \cdot s^{-1} \hat{\mathbf{k}}$

Earth's Motion
Spin Angular Momentum



- Spin angular momentum about center of mass of earth

$$\vec{\mathbf{L}}_{cm}^{spin} = I_{cm}\vec{\boldsymbol{\omega}}_{spin} = \frac{2}{5}m_e R_e^{\,2}\omega_{spin}\hat{\mathbf{n}}$$

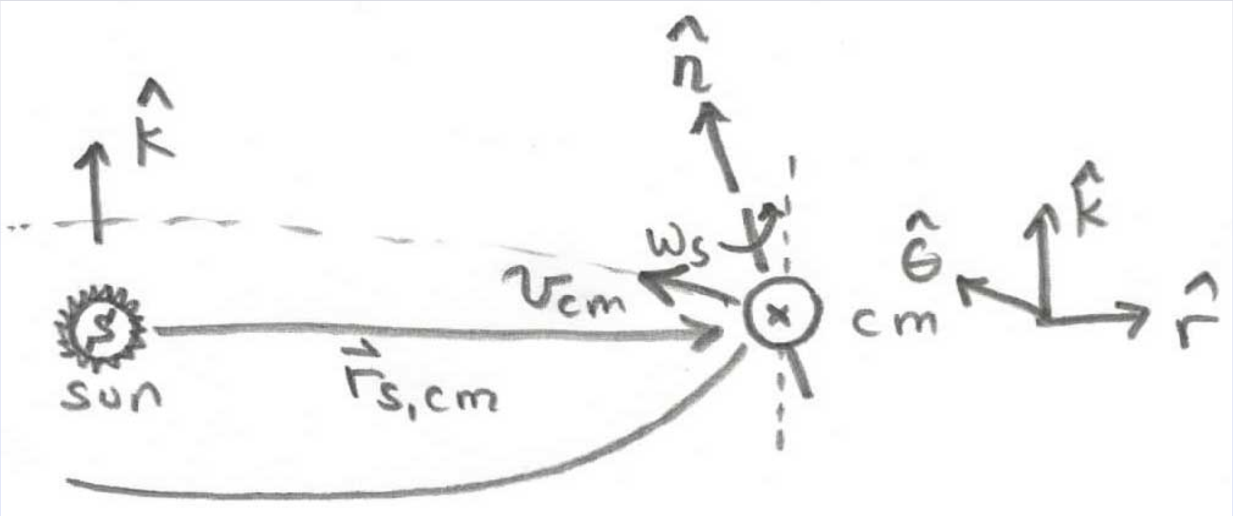
- Period and angular velocity

$$\omega_{spin} = \frac{2\pi}{T_{spin}} = 7.29\times 10^{-5}\,rad\cdot s^{-1}$$

- Magnitude

$$\vec{\mathbf{L}}_{cm}^{spin} = 7.09\times 10^{33}\,kg\cdot m^2\cdot s^{-1}\hat{\mathbf{n}}$$

Checkpoint Problem: Angular Momentum for Earth



- What is the ratio of the spin angular momentum to the orbital angular momentum of the Earth?
- What is the vector expression for the total angular momentum of the Earth about the center of its orbit around the sun (you may assume the orbit is circular and centered at the sun)?