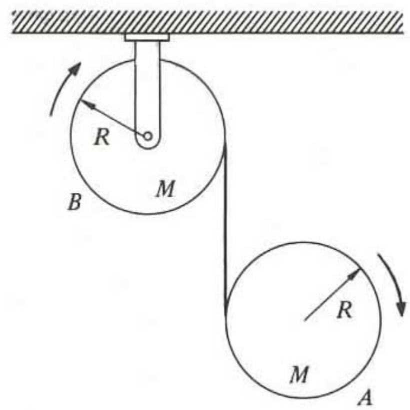
# Rotation and Translation Dynamics
## Challenge Problems

**Problem 1:**

A drum $A$ of mass $m$ and radius $R$ is suspended from a drum $B$ also of mass $m$ and radius $R$, which is free to rotate about its axis. The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind. Gravity is directed downwards. Both drums are initially at rest. Find the initial acceleration of drum $A$, assuming that it moves straight down.




**Problem 1 Solution:**

The key to solving this problem is to determine the relation between the three kinematic quantities $\alpha_A$, $\alpha_B$ and $a_A$, the angular accelerations of the two drums and the linear acceleration of drum $A$. One way to do this is to introduce the auxiliary variable $z$ for the length of the tape that is unwound from the upper drum. Then, $\alpha_B R = \frac{d^2z}{dt^2}$. The linear velocity $v_A$ may then be expressed as the sum of two terms, the rate $\frac{dz}{dt}$ at which the tape is unwinding from the upper drum and the rate $\omega_A R$ at which the falling drum is moving relative to the lower end of the tape. Taking derivatives, we obtain

$$a_A = \frac{d^2z}{dt^2} + \alpha_A R = \alpha_B R + \alpha_A R.$$

Denote the tension in the tape as (what else) $T$. The net torque on the upper drum about its center is then $\tau_B = TR$, directed clockwise in the figure, and the net torque on the falling drum about its center is also $\tau_A = TR$, also directed clockwise. Thus,

$\alpha_B = TR/I = 2T/MR$, $\alpha_A = TR/I = 2T/MR$. Where we have assumed that the moment of inertia of the drum and unwinding tape is $I = (1/2)MR^2$. Newton's Second Law, applied to the falling drum, with the positive direction downward, is $Mg - T = Ma_A$. We now have five equations,

$$\alpha_B R = \frac{d^2 z}{dt^2}, \quad a_A = \frac{d^2 z}{dt^2} + \alpha_A R, \quad \alpha_B = \frac{2T}{MR}, \quad \alpha_A = \frac{2T}{MR}, \quad Mg - T = Ma_A,$$

in the five unknowns $\alpha_A$, $\alpha_B$, $a_A$, $\frac{d^2 z}{dt^2}$ and $T$.

It's easy to see that

$$\alpha_A = \alpha_B .$$

Therefore

$$a_A = \alpha_B R + \alpha_A R = 2\alpha_A R .$$

The tension in the tape is then

$$T = \frac{\alpha_A MR}{2} = \frac{a_A MR}{4R} = \frac{Ma_A}{4}$$

Newton's Second Law then becomes

$$Mg - \frac{Ma_A}{4} = Ma_A .$$

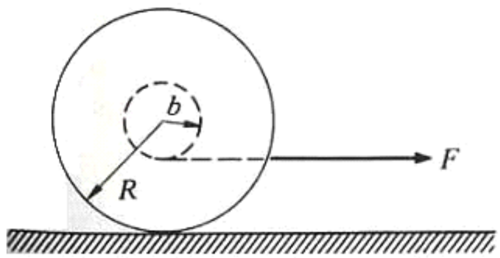
Therefore solving for the acceleration yields

$$a_A = \frac{4}{5} g$$

This result is certainly plausible. We expect $a_A < g$, and we also expect that with both drums free to rotate, the acceleration will be almost but not quite $g$.

**Problem 2:** ***Yo-Yo rolling on a plane Rotation and Translation***

A Yo-Yo of mass $m$ has an axle of radius $b$ and a spool of radius $R$. It's moment of inertia about an axis passing through the center of the Yo-Yo can be approximated by $I_0 = (1/2)mR^2$. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force $\vec{\mathbf{F}}$ to the right as shown in the figure.



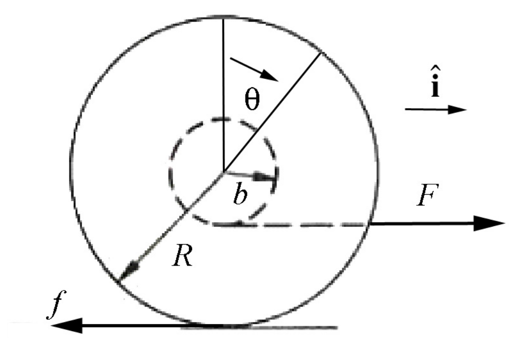

The coefficient of static friction between the Yo-Yo and the table is $\mu_s$.

a) Which way will the Yo-Yo rotate if the string is pulled very gently? If the string is jerked hard, which way will the Yo-Yo rotate?

b) What is the maximum magnitude of the pulling force, $|\vec{\mathbf{F}}|$, for which the Yo-Yo will roll without slipping?

**Problem 2 Solutions:**

The force of friction acting opposite to the applied force is shown in the figure below.




Torque equation:

$$\tau = I\alpha = Rf - bF \tag{2.1}$$

Force equation:

$$F - f = ma \tag{2.2}$$

For rolling without slipping motion along the floor,

$$a = +R\alpha \tag{2.3}$$

(Note that the positive sign is a result of our two choices of sign convention for positive rotation and linear acceleration.)

From (1) and (3),

$$Rf - bF = I\alpha = \frac{Ia}{R} = \frac{1}{2}mR^2\frac{a}{R} = \frac{1}{2}maR \tag{2.4}$$

implies that

$$f - \frac{b}{R}F = \frac{1}{2}ma \tag{2.5}$$

From (2) and (4),

$$F\left(1 - \frac{b}{R}\right) = \frac{3}{2}ma \tag{2.6}$$

implies that

$$a = \frac{2}{3}\frac{F}{m}\left(1 - \frac{b}{R}\right) \tag{2.7}$$

From (2) and (5),

$$f = \frac{F}{3}\left(1 + \frac{2b}{R}\right) \tag{2.8}$$

The frictional force, $f$, is maximum when $b = R$, and is given by $f = F$ (i.e it equals the applied force). We also know that the maximum possible value of $f$ is given by, $f = \mu_s mg$. So, our assumption of pure clockwise rotation breaks down for $F > \mu_s mg$ and slipping occurs.

a) If the string is pulled very gently, our assumption of pure clockwise rotation holds. And, the Yo-Yo rotates in the forward (clockwise) direction without any slipping.

If the string is jerked hard, our assumption of pure rotation in the clockwise direction fails, and slippage occurs. The Yo-Yo rotates in the anti-clockwise direction but still moves forward (by slipping).

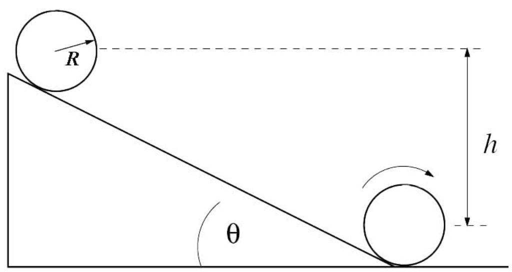
b) From the results above Eq. (2.8)

$$F = \frac{3f}{\left(1+\frac{2b}{R}\right)} \tag{2.9}$$

$$F_{\max} = \frac{3f_{\max}}{\left(1+\frac{2b}{R}\right)} = \frac{3\mu mg}{\left(1+\frac{2b}{R}\right)} \tag{2.10}$$

**Problem 3**

A hollow cylinder of outer radius $R$ and mass $M$ with moment of inertia about the center of mass $I_{\text{cm}} = M R^2$ starts from rest and moves down an incline tilted at an angle $\theta$ from the horizontal. The center of mass of the cylinder has dropped a vertical distance $h$ when it reaches the bottom of the incline. Let $g$ denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is $\mu_{\text{s}}$. The cylinder rolls without slipping down the incline. The goal of this problem is to find an expression for the smallest possible value of $\mu_{\text{s}}$ such that the cylinder rolls without slipping down the incline plane and the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.



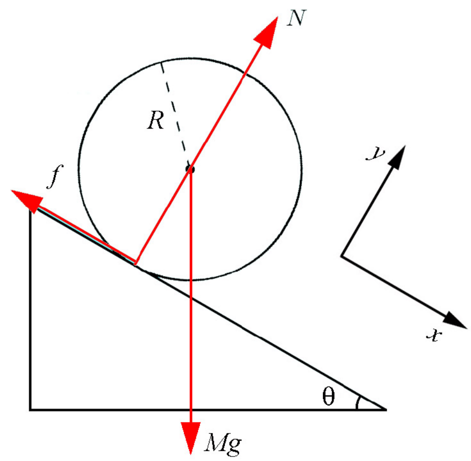

a) Write down a plan for solving this problem. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any diagrams or sketches that you plan to use.

b) What is the minimum value for the coefficient of static friction $\mu_{\text{s}}$ such that the cylinder rolls without slipping down the incline plane? Express your answer in terms of $M$, $R$, $g$, $\theta$ and $h$ as needed.

c) What is the magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline? Express your answer in terms of $M$, $R$, $g$, $\theta$ and $h$ as needed.

**Problem 3 Solutions:**

a) The two parts of the problem are seen to be distinct; find the minimum value of $\mu_{\text{s}}$, and from the resulting acceleration find the speed of the cylinder when it reaches the bottom of the incline. To find the minimum value of $\mu_{\text{s}}$, we will need to know something about the forces and torques, specifically the relation between the friction force on the normal force; that is, the components of the contact force. As a result of this determination, we will find the acceleration and hence the speed at the bottom of the incline. A figure showing the forces is shown below.




b) With the coordinates system shown, Newton's Second Law, applied in the $x$- and $y$-directions in turn, yields

$$\begin{aligned} Mg\sin\theta - f &= Ma \\ N - Mg\cos\theta &= 0. \end{aligned} \qquad \textbf{(3.1)}$$

The equations in (3.1) represent two equations in three unknowns, and so we need one more relation. As described in part (a), this will come from torque considerations.

Choose the center of the cylinder to compute the torque about. Then, the only force exerting a torque is the friction force, and so we have

$$f\,R = I_{\mathrm{cm}}\alpha = M\,R^2\left(a\,/\,R\right) = M\,R\,a\,, \qquad (3.2)$$

where $I_{\mathrm{cm}} = M\,R^2$ and the kinematic constraint for the no-slipping condition $\alpha = a\,/\,R$ have been used. Equation (3.2) leads to $f = M\,a$, and inserting this into the first expression in (3.1) gives the two relations

$$f = \frac{1}{2}Mg\sin\theta. \qquad (3.3)$$

$$a = \frac{1}{2}g\sin\theta \qquad (3.4)$$

We're still not done. For rolling without slipping, we need $f < \mu_s N$, so we need, using the second expression in (3.1),

$$\mu_s > \frac{1}{2}\tan\theta\,. \qquad (3.5)$$

c) We shall use the fact that the energy of the cylinder-earth system is constant since the static friction force does no work. Choose a zero reference point for potential energy the bottom of the incline plane. Then the initial potential energy is

$$U_i = Mgh. \tag{3.6}$$

For the given moment of inertia, the final kinetic energy is

$$\begin{aligned} K_{\mathrm{f}} &= \frac{1}{2} M v_{\mathrm{f}}^2 + \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{f}}^2 \\ &= \frac{1}{2} M v_{\mathrm{f}}^2 + \frac{1}{2} M R^2 \left( v_{\mathrm{f}} / R \right)^2 \\ &= M v_{\mathrm{f}}^2. \end{aligned} \tag{3.7}$$

Set the final kinetic energy equal to the initial gravitational potential energy leads to

$$Mgh = M v_{\mathrm{f}}^2. \tag{3.8}$$

The speed $v_{\mathrm{f}}$ at the bottom is

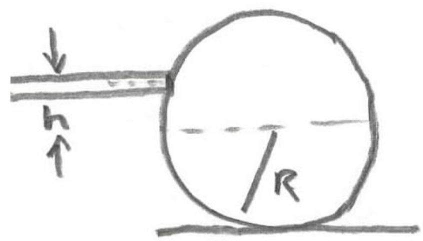
$$v_f = \sqrt{gh}. \tag{3.9}$$

We also could have used our acceleration result from part b). The cylinder rolls a distance $L = h / \sin\theta$ down the incline, and the speed $v_{\mathrm{f}}$ at the bottom is related to the acceleration (Eq. (3.4) by

$$\begin{aligned} v_{\mathrm{f}}^2 &= 2aL = 2\left( \frac{1}{2} g \sin\theta \right) \left( h / \sin\theta \right) \\ &= gh. \end{aligned} \tag{3.10}$$

**Problem 4:** ***Billiards Challenge***

A spherical billiard ball of uniform density has mass $m$ and radius $R$ and moment of inertia about the center of mass $I_{cm} = (2/5)mR^2$. The ball, initially at rest on a table, is given a sharp horizontal impulse by a cue stick that is held an unknown distance $h$ above the centerline (see diagram below). The coefficient of sliding fiction between the ball and the table is $\mu_k$. You may ignore the friction during the impulse. The ball leaves the cue with a given speed $v_0$ and an unknown angular velocity $\omega_0$. Because of its initial rotation, the ball eventually acquires a maximum speed of $(9/7)v_0$. The point of the problem is to find the ratio $h/R$.




a) Write up your plan for solving this problem. You may find some of the following concepts useful: angular impulse is equal to the change in angular momentum; linear impulse is equal to the change in momentum; Newton's Second Law; torque about the center-of-mass is proportional to the angular acceleration; if the torque about a point is zero, then the angular momentum about that point is constant; etc.

b) Find the ratio $h/R$.

**Problem 4 Solutions:**

a) There are several ways to approach this problem. The first method presented here avoids any calculation of the force or torque provided by friction, or the details of the force between the cue and the ball. This method will first consider the "collision" between the cue and the ball by taking the collision point as the origin for finding the angular momentum, as the force between the cue and the ball exerts no torque about this point, and we are given that the friction may be ignored during this interaction. After this collision, the angular momentum will be taken about the initial contact point between the ball and the felt. It will be helpful to infer, either from the figure and from the fact that $v_f > v_0$, that the ball is given overspin.

**First Solution:** With respect to the point where the cue is in contact with the ball, note that the rotational angular momentum and the angular momentum due to the motion of the center of mass have different signs; the former is clockwise and the latter is counterclockwise. The sum of these contributions to the angular momenta must sum to zero, and hence have the same magnitude;

$$I_{\text{cm}}\omega_0 = mv_0 h\ . \tag{4.1}$$

While the ball is rolling and slipping, angular momentum is conserved about the contact between the ball and the felt. The initial and final angular momenta are

$$\begin{aligned}
L_{\text{initial}} &= mv_0 R + I_{\text{cm}}\omega_0 \\
&= mv_0(R+h) \\
L_{\text{final}} &= mv_{\text{f}} R + I_{\text{cm}}\omega_{\text{f}} \\
&= mv_{\text{f}} R + (2/5)(mR^2)(v_{\text{f}}/R) \\
&= (7/5)mv_{\text{f}} R \\
&= (9/5)mv_0 R,
\end{aligned} \tag{4.2}$$

Setting the initial and final angular momenta equal and solving for $h/R$ gives

$$\frac{h}{R} = \frac{4}{5} \tag{4.3}$$

(note that the figure is not quite to scale).

**Second Solution:** As an alternative, taking the angular momentum after the collision about the center of the ball, note that the time $\Delta t$ between the moments the ball is struck and when it begins to roll without slipping is $\Delta v/(\mu_{\text{k}} g)$. But, if the angular momentum is taken about the center of the ball, after the ball is struck the angular impulse delivered to the ball by the friction force is

$$(\mu_{\text{k}} mg)R\,\Delta t = I_{\text{cm}}(\omega_f - \omega_0)\ . \tag{4.4}$$

**Third Solution:**

When the cue stick strikes the ball with a moment arm $h$ from the center of the ball, the ball will eventually roll without slipping. The final velocity should depend on this parameter $h$ and in fact should depend on the ratio $h/R$. If we measure the final velocity we should be able to determine this ratio.

As the cue stick strikes the ball, it imparts both a linear impulse and an angular impulse to the ball resulting in changing the momentum associated with the center of mass translation of the ball and the angular momentum associated with the rotation about the center of mass. (We assume the impulse due to kinetic friction while the ball is being struck is negligible.) Both impulses (linear and angular) involve the integral of the force during the time interval that the force is applied and so we must have a constraint between the change in the momentum and the angular momentum. We can use this constraint to find a relation between the angular velocity about the center of mass and the velocity of the center of mass immediately after the ball is struck. We expect that the constant of proportionality will depend in some manner on the parameters $h$ and $R$. Applying this constraint condition, we can determine the total angular momentum immediately after the ball has been stuck about any point.

Once the ball rolls without slipping the final angular velocity doesn't change and can be expressed in terms of the final velocity by the rolling without slipping condition. Since we are also given a relation between the final velocity and the initial velocity after the ball has been struck (in principle we can measure this), we can then calculate the angular momentum about the any point when the ball is rolling without slipping (using the rolling without slipping condition) entirely in terms of the initial velocity.

In fact there is a particular set of points about which angular momentum is constant. We note that the friction force acts at the contact point. If we choose any point S along the contact line between the ball and the ground, the frictional torque is zero about that point since $\vec{\mathbf{r}}_{S,f} \times \vec{\mathbf{f}}_k = \vec{\mathbf{0}}$. The torques due to the normal force and the gravitational force have the same moment arm, are equal in magnitude (since the two forces are equal in magnitude), but are opposite in direction, and hence cancel. Therefore the angular momentum is constant about any point S along the contact line between the ball and the ground. We can then apply conservation of angular momentum to eliminate the initial velocity and find the constraint condition for the ratio $h/R$.

**Carrying out the Plan:** The magnitude of the linear impulse due to the force $\vec{\mathbf{F}}$ of the cue stick hitting the billiard ball is equal to the change in momentum of the center of mass of the ball according to

$$\int F dt = m v_0 . \tag{4.5}$$

The force $\vec{\mathbf{F}}$ exerts a torque about the center of mass that has a magnitude $\tau = hF$. The magnitude of the angular impulse is equal to the change in angular momentum about the center of mass according to

$$\int \tau dt = \int hF dt = h \int F dt = I_{cm} \omega_0 . \tag{4.6}$$

Substituting the Equation (4.5) into the Equation (4.6) yields

$$hmv_0 = I_{cm}\omega_0 . \tag{4.7}$$

The moment of inertia about the center of mass for a sphere is

$$I_{cm} = (2/5)mR^2 \tag{4.8}$$

We can substitute Equation (4.8) into Equation (4.7) and solve for the angular velocity of the ball immediately after it was struck,

$$\omega_0 = hmv_0 /(2/5)mR^2 = 5hv_0 / 2R^2 . \tag{4.9}$$

Since the angular momentum is constant about any point S along the contact line between the ball and the ground contact friction force acts at the contact point, we now compute the initial and final angular momentums about S.

There are two contributions to the angular momentum of the billiard ball about the point S, the orbital angular momentum due to the momentum of the center of mass, and the spin angular momentum about the center of mass.

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_{S,cm} \times m_{total}\vec{\mathbf{v}}_{cm} + I_{cm}\omega\,\hat{\mathbf{k}} = r_{s,m}mv_{cm}\,\hat{\mathbf{k}} + I_{cm}\omega\,\hat{\mathbf{k}} , \tag{4.10}$$

where $\hat{\mathbf{k}}$ points into the page in Figure 16.4. The initial angular momentum about S is therefore

$$\vec{\mathbf{L}}_{S,0} = \vec{\mathbf{r}}_{S,cm} \times m_{total}\vec{\mathbf{v}}_{cm,0} + I_{cm}\omega_0\,\hat{\mathbf{k}} = (mv_0R + I_{cm}\omega_0)\hat{\mathbf{k}} \tag{4.11}$$

Applying Equations (4.8) and (4.9) to Equation (4.11) here yields

$$\vec{\mathbf{L}}_{S,0} = (mv_0R + mhv_0)\hat{\mathbf{k}} \tag{4.12}$$

The final angular momentum about S is given by

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{r}}_{S,cm} \times m_{total}\vec{\mathbf{v}}_{cm,f} + I_{cm}\omega_f\,\hat{\mathbf{k}} = (mv_fR + I_{cm}\omega_f)\hat{\mathbf{k}} \tag{4.13}$$

The ball is rolling without slipping so $v_f = \omega_f R$. The final velocity was given to be $v_f = (9/7)v_0$. So Equation (4.12) becomes

$$\vec{\mathbf{L}}_{S,f} = \frac{9}{5}mRv_0\hat{\mathbf{k}} \tag{4.14}$$

Using conservation of angular momentum about S, we then set Equation (4.12) equal to Equation (4.14) and find that

$$mv_0R + mhv_0 = \frac{9}{5}mRv_0\,. \tag{4.15}$$

We can solve Equation (4.15) for the ratio

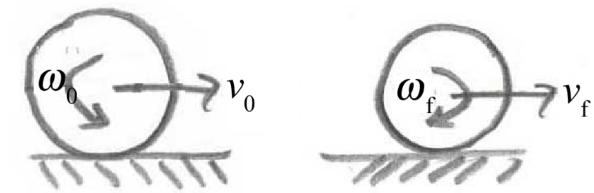
$$\frac{h}{R} = \frac{4}{5}\,. \tag{4.16}$$

**Problem 5: Bowling Ball**

A bowling ball of mass $m$ and radius $R$ is initially thrown down an alley with an initial speed $v_0$ and backspin with angular speed $\omega_0$, such that $v_0 > R\omega_0$. The moment of inertia of the ball about its center of mass is $I_{\text{cm}} = (2/5)mR^2$. Your goal is to determine the speed $v_{\text{f}}$ of the bowling ball when it just starts to roll without slipping.

a) Write up your plan for solving this problem. You may find some of the following concepts useful: angular impulse is equal to the change in angular momentum; linear impulse is equal to the change in momentum; Newton's Second Law; torque about the center-of-mass is proportional to the angular acceleration; if the torque about a point is zero, then the angular momentum about that point is constant; etc.

b) What is the speed $v_{\text{f}}$ of the bowling ball when it just starts to roll without slipping?




**Problem 5 Solutions:**

a) The easiest approach to solve this problem is to find a fixed point about which the torque is zero and then use the fact that angular momentum is constant about that point. So, if we take the point for determining torques and angular momenta about a point where the friction force exerts no torque, we shouldn't need to know about the nature of the friction force. Accordingly, choose the point to be the original point of contact of the ball with the lane surface. Subsequently, even though the ball has moved, friction will still exert no torque.

b) With respect to the contact point on the ground, the initial and final angular momenta are both the sum of two terms, one representing the motion of the center of mass and the other the rotation ("spin") of the ball;

$$\begin{aligned} L_{\text{initial}} &= mv_0 R - I_{\text{cm}}\omega_0 \\ L_{\text{final}} &= mv_{\text{f}} R + I_{\text{cm}}\omega_{\text{f}}. \end{aligned} \tag{5.1}$$

The problem is now one of basic algebra. For rolling without slipping, $\omega_{\mathrm{f}} = v_{\mathrm{f}} / R$, and the given $I_{\mathrm{cm}} = (2/5)mR^2$ gives

$$v_{\mathrm{f}} = \left(5v_0 - 2\omega_0 R\right)/7 . \tag{5.2}$$

It's important to note the signs in the expressions in (5.1). We are given (and the figure certainly implies) that the scalar quantity $\omega_0$, representing backspin, is positive, and so with positive direction for angular momenta being clockwise, the $\omega_0$ term in the initial angular momentum is negative.

This problem may of course be done by considering torques and angular momenta about the center of the ball. The change in linear momentum (watch the signs again) is the impulse

$$\Delta p = m\left(v_{\mathrm{f}} - v_0\right) = -\int f\, dt \tag{5.3}$$

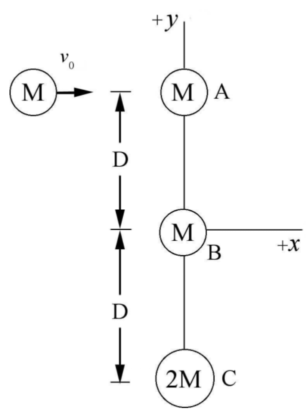
and the change in angular momentum is the angular impulse (the signs are still important)

$$\Delta L = I_{\mathrm{cm}}\left(\omega_f + \omega_0\right) = \int Rf\, dt . \tag{5.4}$$

Eliminating the linear impulse $-\int f\, dt$ between Equations (5.3) and (5.4), and using the given $I_{\mathrm{cm}} = (2/5)mR^2$ yields the same result as that in Equation (5.2).

**Problem 6: Rotational Collision**

Three point-like objects located at the points A, B and C of respective masses $M_A = M$, $M_B = M$ and $M_C = 2M$, as shown in the figure below. The three objects are initially oriented along the $y$-axis and connected by rods of negligible mass each of length $D$, forming a rigid body. A fourth object of mass $M$ moving with velocity $v_0\,\hat{\mathbf{i}}$ collides and sticks to the object at rest at point A. Neglect gravity. Give all your answers in terms of $M$, $v_0$ and $D$ as needed. The $z$-axis points out of the page.




a) Describe qualitatively in words how the system moves after the collision: direction, translation and rotation.

b) What is the direction and magnitude of the linear velocity of the center of mass after the collision?

c) What is the magnitude of the angular velocity of the system after the collision?

d) What is the direction and magnitudes of the velocity $\vec{\mathbf{v}}_C$ and acceleration $\vec{\mathbf{a}}_C$ of the object located at the point C immediately after the collision?

**Problem 6 Solutions**

a) From conservation of linear momentum, the system will move to the right (in the positive $x$-direction). The system will rotate about its center of mass, clockwise (in the negative $z$-direction from the right-hand rule).

b) At the instant just before the collision, the position of the center of mass of the system is at the initial position of object B. The velocity of the center of mass is

$$\vec{\mathbf{v}}_{cm} = \frac{M v_0 \hat{\mathbf{i}}}{M + (M + M + 2M)} = \frac{1}{5} v_0 \hat{\mathbf{i}} \qquad (6.1)$$

and this will be the velocity of the center of mass after the collision.

c) We could of course choose any point about which to calculate the angular momentum. Since the system after the collision is symmetric about the object at point B, choosing this point as the origin will simplify calculations. (The choice of this point as the origin is strongly suggested by the diagram.) The initial and final angular momenta about this point are

$$\begin{aligned}\vec{\mathbf{L}}_{\text{initial, B}} &= \mathrm{M}v_0\mathrm{D}(-\hat{\mathbf{k}}) \\ \vec{\mathbf{L}}_{\text{final, B}} &= I_{\text{cm, B}}\omega_{\text{f}}(-\hat{\mathbf{k}})\end{aligned} \tag{6.2}$$

where $I_{\text{cm, B}} = 2(2\mathrm{MD}^2)$ is the moment of inertia of the system about point B. Equating initial and final angular momenta yields $\omega_{\text{f}} = v_0 / (4\mathrm{D})$.

d) The velocity is found by adding the center of mass velocity of the system to the velocity of the mass at point C relative to the center of mass. The velocity of the center of mass is that found in part (b) and the velocity with respect to the center of mass is given by the cross product of the vector angular velocity and the vector displacement of point C from the center, or

$$\vec{\mathbf{v}}_{\text{C}} = \left(\frac{1}{5}v_0\hat{\mathbf{i}}\right) + \left(-\frac{v_0}{4\mathrm{D}}\hat{\mathbf{k}}\right) \times (-\mathrm{D}\hat{\mathbf{j}}) = -\frac{1}{20}v_0\hat{\mathbf{i}}\,. \tag{6.3}$$

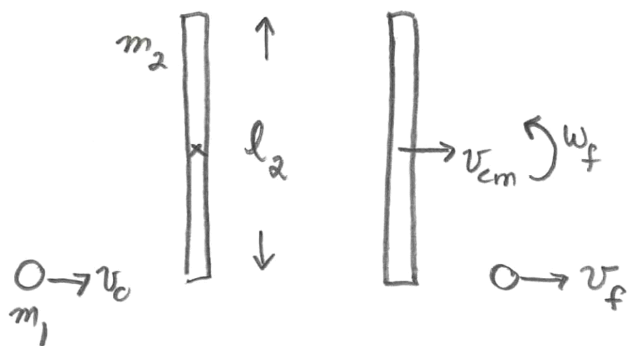
In the limit of the collision being instantaneous, immediately after the collision the rod attaching the object at point C to the center of mass is parallel to the $\hat{\mathbf{j}}$-direction. Viewed from the center of mass, this must be the direction of the acceleration, and so the vector acceleration is

$$\vec{\mathbf{a}}_{\text{C}} = \frac{v_{\text{C}}^2}{\mathrm{D}}\hat{\mathbf{j}} = \frac{1}{400}\frac{v_0^2}{\mathrm{D}}\hat{\mathbf{j}}. \tag{6.4}$$

This acceleration will be the same in both the “lab” frame and the center of mass frame.

**Problem 7: Hockey Puck Collision**

A hockey puck of mass $m_1$ slides along ice with a velocity $v_0$ and strikes one end of a stick lying on the ice of length $l_2$ and mass $m_2$. The center of mass of the stick moves with an unknown magnitude $v_{cm}$. The stick also rotates about the center of mass with unknown angular velocity $\omega_f$. The puck continues to move in the same straight line as before it hit the stick with velocity $v_f$. Assume the ice is frictionless and there is no loss of mechanical energy during the collision.




a) Write down the equation for conservation of momentum.

b) Write down the equation for conservation of energy.

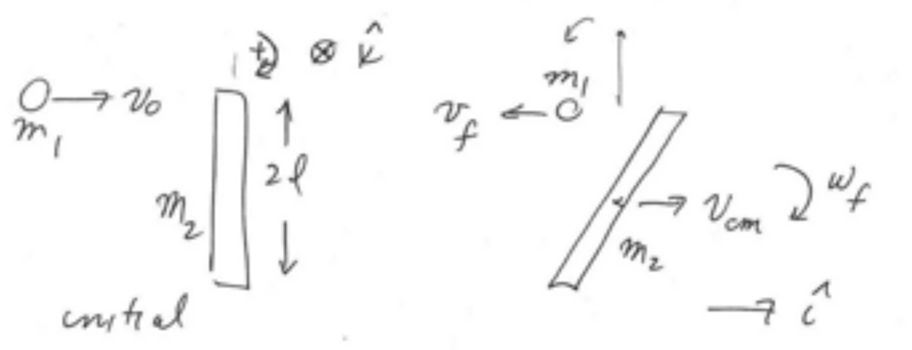
c) Is there any external torques acting on the system consisting of the puck and the stick? Write down the equation for conservation of angular momentum about a convenient point.

d) Find the velocity of the center of mass of the stick.

e) Find the velocity of the puck after the collision.

f) Find the angular velocity of the stick after the collision.

**Problem 7 Solutions:**




Conservation of Energy:

$$\frac{1}{2}m_1v_0^2 = \frac{1}{2}m_1v_f^2 + \frac{1}{2}I_{cm}w_f^2 \tag{7.1}$$

Conservation of Momentum

$$\hat{i} : m_1v_0 = -m_1v_f + m_2v_{cm} \tag{7.2}$$

$$I_{cm} = \frac{1}{12}m_2\left(2\ell\right)^2 = \frac{1}{3}m_2\ell^2 \tag{7.3}$$

Solve equation (7.3) for $v_{cm}$:

$$v_{cm} = \frac{m_1}{m_2}\left(v_0 + v_f\right) \tag{7.4}$$

Substitute into equation (7.1):

$$\frac{1}{2}m_1v_0^2 = \frac{1}{2}m_1v_f^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_2}\left(v_0 + v_f\right)\right)^2 + \frac{1}{2}\left(\frac{1}{3}m_2\ell^2\right)w_f \tag{7.5}$$

Conservation of Angular Momentum about cm:

$$\overline{L}_{0,cm} = \ell m_1 v_0 \hat{k}$$
$$\overline{L}_{f,cm} = -\ell m_1 v_f + I_{cm} w_f \hat{k}$$
$$\overline{L}_{0,cm} = \overline{L}_{f,cm}$$

$$\hat{k}: \ \ell m_1 v_0 = -\ell m_1 v_f + I_{cm} w_f$$

$$\Rightarrow \ w_f = \frac{\ell m_1 \left(v_0 + v_f\right)}{\frac{1}{3} m_2 \ell^2} = 3\frac{m_1}{m_2}\left(v_0 + v_f\right)$$

$$\frac{1}{2} I_{cm} w_f^2 = \frac{1}{2} \frac{3\ell^2 m_1^2 \left(v_0 + v_f\right)^2}{m_2 \ell^2}$$

$$\frac{1}{2} I_{cm} w_f^2 = \frac{3}{2}\frac{m_1^2}{m_2}\left(v_0 - v_f\right)^2$$

Substitute this into equation (7.5):

$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_f^2 + \frac{1}{2} m_2 \left(\frac{m_1^2}{m_2^2}\left(v_0 + v_f\right)^2\right) + \frac{3}{2}\frac{m_1^2}{m_2}\left(v_0 + v_f\right)^2$$

$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_f^2 + \frac{1}{2}\left(4\frac{m_1^2}{m_2}\left(v_0 + v_f\right)^2\right)$$

$$\Rightarrow v_0^2 = v_f^2 + 4\frac{m_1}{m_2}\left(v_0 + v_f\right)^2$$

$$v_0^2 = v_f^2 + 4\frac{m_1}{m_2}\left(v_0^2 + 2 v_0 v_f + v_f^2\right)$$

$$0 = v_0^2\left(-1 + 4\frac{m_1}{m_2}\right) + 8\frac{m_1}{m_2} v_0 v_f + v_f^2\left(1 + 4\frac{m_1}{m_2}\right)$$

$$0 = v_f^2 + \left(\frac{8\frac{m_1}{m_2}}{1 + 4\frac{m_1}{m_2}}\right) v_0 v_f + v_0^2 \frac{\left(-1 + 4\frac{m_1}{m_2}\right)}{\left(1 + 4\frac{m_1}{m_2}\right)}$$

Let $\beta = 4 m_1 / m_2$

$$0 = v_f^2 + \frac{2\beta}{1+\beta} v_0 v_f + v_0^2 \frac{(-1+\beta)}{(1+\beta)}$$

$$v_f = \frac{-2\beta}{1+\beta} v_0 \pm \left( \frac{4\beta^2 v_0^2}{(1+\beta)^2} - 4v_0^2 \left( \frac{-1+\beta}{1+\beta} \right) \right)^{\frac{1}{2}}$$

$$= -\frac{\beta}{1+\beta} v_0 \pm \frac{v_0}{2} \left( \frac{4\beta^2 - 4(-1+\beta)(1+\beta)}{(1+\beta)^2} \right)^{\frac{1}{2}}$$

$$= -\frac{\beta}{1+\beta} v_0 \pm \frac{\left(\beta^2 - (-1+\beta^2)\right)^{\frac{1}{2}}}{1+\beta} v_0$$

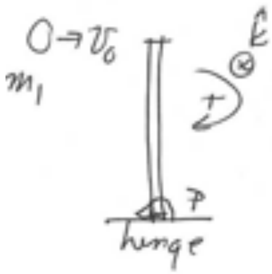
$$= -\frac{\beta}{1+\beta} v_0 \pm \frac{1}{1+\beta} v_0$$

Positive root: $\boxed{v_f = \frac{(-\beta+1)}{(1+\beta)} v_0}$

Negative root: $v_f = \frac{-\beta-1}{1+\beta} v_0 = -v_0$. This is just the initial state.

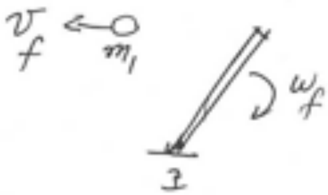
$$v_f = \frac{\left(-4\frac{m_1}{m_2} + 1\right)}{\left(4\frac{m_1}{m_2} + 1\right)} v_0$$

when $m_1 = m_2 \Rightarrow v_f = \frac{3}{5} v_0$




Angular momentum about the pivot point is conserved.

$$L_{P,0} = m_1 v_0 2\ell \hat{k}$$



$$\bar{L}_{p,f} = -m_1 v_f 2\ell \hat{k} + I_p w_f \hat{k}$$

$$I_p = \frac{1}{3} m_2 (2\ell)^2 = \frac{4}{3} m_2 \ell^2$$

$$m_1 v_0 2\ell = -m_1 v_f 2\ell + I_p w_f \tag{7.6}$$

Assume energy is still conserved.

$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_f^2 + \frac{1}{2} I_p w_f^2 \tag{7.7}$$

Solve equation (7.6) for $I_p w_f = m_1 2\ell (v_0 + v_f)$ and substitute into equation (7.7) using:

$$\frac{1}{2} I_p w_f^2 = \frac{1}{2} \frac{m_1^2 4\ell^2 (v_0 + v_f)^2}{I_p} = \frac{\frac{1}{2} m_1^2 4\ell^2 (v_0 + v_f)^2}{\frac{4}{3} m_2 \ell^2}$$

$$\frac{1}{2} I_p w_f^2 = \frac{1}{2} \cdot 3 \cdot \frac{m_1^2}{m_2} (v_0 + v_f)^2$$

Equation (7.7) becomes:

$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_f^2 + \frac{1}{2} \cdot 3 \cdot \frac{m_1^2}{m_2} (v_0 + v_f)^2$$

$$v_0^2 = v_f^2 + 3 \frac{m_1}{m_2} (v_0 + v_f)^2$$

$$v_0^2 = v_f^2 + 3 \frac{m_1}{m_2} (v_0^2 + 2 v_0 v_f + v_f^2)$$

$$\Rightarrow 0 = v_f^2 \left(1 + 3 \frac{m_1}{m_2}\right) + 3 \frac{m_1}{m_2} \cdot 2 v_0 v_f + v_0^2 \left(-1 + 3 \frac{m_1}{m_2}\right)$$

let $\beta = 3m_1 / m_2$ then:

$$0 = v_f^2 (1 + \beta) + 2\beta v_0 v_f + v_0^2 (1 - \beta)$$

This equation has exactly the same form as part a) so the solution is identical

$$v_f = \frac{1 - \beta}{1 + \beta} v_0 = \frac{1 - 3\frac{m_1}{m_2}}{1 + 3\frac{m_1}{m_2}} v_0$$

when $m_1 = m_2 \;\Rightarrow\; v_f = \frac{1}{2} v_0$