Module 27: Rigid Body Dynamics: Rotation and Translation about a Fixed Axis

27.1 Introduction

We shall analyze the motion of systems of particles and rigid bodies that are undergoing translational and rotational motion. There is no longer a clearly fixed axis of rotation but we shall see that is possible to describe the motion by a translation of the center of mass and a rotation about the center of mass. By choosing a reference frame moving with the center of mass, we can analyze the rotational motion and discover that the torque about the center of mass is equal to the change in the angular momentum about the center of mass. In particular for a rigid body undergoing fixed axis rotation about the center of mass, our rotational equation of motion is similar to one we have already encountered for fixed axis rotation. We shall begin by recalling the main properties of the center of mass reference frame

27.2 Review Center of Mass Reference Frame

Denote the position vector of the i^{th} particle with respect to origin of reference frame O by $\vec{\mathbf{r}}_i$. The vector from the origin of frame O to the center of mass of the system of particles, is defined as

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m^{total}} \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i . \qquad (27.2.1)$$

The velocity of the center of mass in reference frame O is given by

$$\vec{\mathbf{V}}_{\text{cm}} = \frac{1}{m^{\text{total}}} \sum_i m_i \vec{\mathbf{v}}_i = \frac{\vec{\mathbf{p}}^{\text{sys}}}{m^{\text{total}}} . \qquad (27.2.2)$$

Define the *center of mass reference frame* O_{cm} as a reference frame moving with velocity $\vec{\mathbf{V}}_{\text{cm}}$ with respect to O. Denote the position vector of the i^{th} particle with respect to origin of reference frame O_{cm} by $\vec{\mathbf{r}}_{cm,i}$ (Figure 27.1).

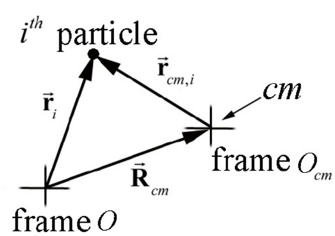


Figure 27.1 Position vector of i^{th} particle in the center of mass reference frame.

The position vector of the i^{th} particle in the two center of mass frame is given by

$$\vec{\mathbf{r}}_{\text{cm},i} = \vec{\mathbf{r}}_i - \vec{\mathbf{R}}_{cm} . \qquad (27.2.3)$$

or equivalently the position of the i^{th} particle in the reference frame O can be expressed as

$$\vec{\mathbf{r}}_i = \vec{\mathbf{r}}_{\text{cm},i} + \vec{\mathbf{R}}_{cm} . \qquad (27.2.4)$$

The displacement of the i^{th} particle in the center of mass reference frame is then given by

$$d\vec{\mathbf{r}}_{\text{cm},i} = d\vec{\mathbf{r}}_i - d\vec{\mathbf{R}}_{cm} . \qquad (27.2.5)$$

The velocity of the i^{th} particle in the center of mass reference frame is then given by

$$\vec{\mathbf{v}}_{\text{cm},i} = \vec{\mathbf{v}}_i - \vec{\mathbf{V}}_{cm} \qquad (27.2.6)$$

or equivalently the velocity of the i^{th} particle in the reference frame O can be expressed as

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}_{\text{cm},i} + \vec{\mathbf{V}}_{cm} . \qquad (27.2.7)$$

27.3 Translational Equation of Motion

We shall think about the system of particles as follows. We treat the whole system as a single point-like particle of mass m^{total} located at the center of mass moving with the velocity of the center of mass $\vec{\mathbf{V}}_{cm}$. The total external force acting on the system acts at the center of mass and from our earlier result we have that

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt} = \frac{d}{dt}(m^{\text{total}}\vec{\mathbf{V}}_{cm}) . \qquad (27.3.1)$$

27.4 Translational and Rotational Equations of Motion

Recall that that we have previous shown in Module 25 that it is always true that

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d\vec{\mathbf{L}}_S^{\text{total}}}{dt} . \qquad (27.4.1)$$

where

$$\vec{\mathbf{L}}_S^{\text{total}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}} + \sum_{i=1}^{N} (\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) . \tag{27.4.2}$$

We differentiate the LHS of Eq. (27.4.2) and find that

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d}{dt}(\vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}}) + \frac{d}{dt}\left(\sum_{i=1}^{N} (\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) \right) . \tag{27.4.3}$$

We apply the vector identity to Eq. (27.4.3)

$$\frac{d}{dt}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt} , \tag{27.4.4}$$

to Eq. (27.4.3) yielding

$$\begin{aligned} \vec{\boldsymbol{\tau}}_S^{\text{total}} = {} & \frac{d\vec{\mathbf{r}}_{S,\text{cm}}}{dt} \times \vec{\mathbf{p}}^{\text{sys}} + \vec{\mathbf{r}}_{S,\text{cm}} \times \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt} \\ & + \sum_{i=1}^{N} \left(\frac{d\vec{\mathbf{r}}_{cm,i}}{dt} \times m_i \vec{\mathbf{v}}_{cm,i} \right) + \sum_{i=1}^{N} \left(\vec{\mathbf{r}}_{cm,i} \times \frac{d}{dt}(m_i \vec{\mathbf{v}}_{cm,i}) \right) \end{aligned} . \tag{27.4.5}$$

The first and third terms in Equation (27.4.5) are eliminated by noting that

$$\begin{aligned} \frac{d\vec{\mathbf{r}}_{S,\text{cm}}}{dt} \times \vec{\mathbf{p}}^{\text{sys}} &= \vec{\mathbf{V}}_{cm} \times m^{\text{total}} \vec{\mathbf{V}}_{cm} = \vec{\mathbf{0}} \\ \sum_{i=1}^{N} \left(\frac{d\vec{\mathbf{r}}_{cm,i}}{dt} \times m_i \vec{\mathbf{v}}_{cm,i} \right) &= \sum_{i=1}^{N} (\vec{\mathbf{v}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) = \vec{\mathbf{0}}. \end{aligned} \tag{27.4.6}$$

Therefore the time derivative of the angular momentum about a point S, Equation (27.4.5), becomes

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt} \quad + \quad \sum_{i=1}^{N} \left(\vec{\mathbf{r}}_{cm,i} \times \frac{d}{dt}(m_i \vec{\mathbf{v}}_{cm,i}) \right) . \tag{27.4.7}$$

In the first term in Equation (27.4.7), the time derivative of the total momentum of the system is the total external force,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt} \tag{27.4.8}$$

and in the second term, the force acting on the element is

$$\vec{\mathbf{F}}_i = \frac{d}{dt}(m_i \vec{\mathbf{v}}_{cm,i}) \tag{27.4.9}$$

The expression in Equation (27.4.7) then becomes

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} + \sum_{i=1}^{N} (\vec{\mathbf{r}}_{cm,i} \times \vec{\mathbf{F}}_i) . \tag{27.4.10}$$

The first term is the contribution to the torque about the point S if all the external forces were to act at the center of mass. The second term, $\sum_{i=1}^{N} (\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i)$, is the sum of the torques on the individual particles in the center of mass reference frame. Then Equation (27.4.10) may be expressed as

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} + \vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}} . \tag{27.4.11}$$

Retracing our calculating we have two conditions. For a system of particles, when we calculate the torque about a point S, we treat the system as a point-like particle located at the center of mass of the system. All the external forces $\vec{\mathbf{F}}_{\text{ext}}^{\text{total}}$ act at the center of mass. We calculate the orbital angular momentum of the center of mass and determine it's time derivative and then apply

$$\vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{orbital}}}{dt} . \tag{27.4.12}$$

In addition, we calculate the torque about the center of mass due to all the forces acting on the particles in the center of mass reference frame. We calculate the is equal to the time derivative of the total angular momentum of the system with respect to the center of mass in the center of mass reference frame and then apply

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}} = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt} . \tag{27.4.13}$$

27.5 Translation and Rotation of a Rigid Body Undergoing Fixed Axis Rotation

For the special case of rigid body of mass m, we showed that with respect to a reference frame in which the center of mass of the rigid body is moving with velocity $\vec{\mathbf{V}}_{cm}$, all elements of the rigid body are rotating about the center of mass with the same angular velocity $\vec{\boldsymbol{\omega}}_{\text{cm}}$.

For the rigid body of mass m and momentum $\vec{\mathbf{p}} = m\vec{\mathbf{V}}_{cm}$, the translational equation of motion is still given by Eq. (27.3.1) which we repeat in the form

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = m\vec{\mathbf{A}}_{cm} . \tag{27.5.1}$$

Let's choose the z-axis as the axis of rotation that passes through the center of mass of the rigid body. We have already seen in our discussion of angular momentum of a rigid body that the angular momentum does not necessary point in the same direction as the angular velocity. However we can take the z-component of Eq. (27.4.13)

$$(\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}})_z = \frac{d(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}})_z}{dt} . \tag{27.5.2}$$

For a rigid body rotating about the center of mass with $\vec{\boldsymbol{\omega}}_{\text{cm}} = \omega_{\text{cm},z}\hat{\mathbf{k}}$, the z-component of angular momentum bout the center of mass is

$$(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}})_z = I_{\text{cm}}\omega_{\text{cm},z}\hat{\mathbf{k}} . \tag{27.5.3}$$

The our rotational equation of motion is

$$(\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}})_z = I_{\text{cm}}\frac{d\omega_{\text{cm},z}}{dt} = I_{\text{cm}}\alpha_{\text{cm},z} . \tag{27.5.4}$$

27.9 Work-Energy Theorem

For a rigid body, we can also consider the work-energy theorem separately for the translational motion and the rotational motion. Once again treat the rigid body as a point-like particle moving with velocity $\vec{\mathbf{V}}_{cm}$ in reference frame O. We can use the same technique that when treating point particles to show that the work done by the external forces is equal to the change in kinetic energy

$$\begin{aligned} W_{\text{ext}}^{\text{total}} &= \int_o^f \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} \cdot d\vec{\mathbf{r}} = \int_o^f \frac{d(m\vec{\mathbf{V}}_{cm})}{dt} \cdot d\vec{\mathbf{R}}_{cm} = m\int_o^f \frac{d(\vec{\mathbf{V}}_{cm})}{dt} \cdot \vec{\mathbf{V}}_{cm} dt \\ &= \frac{1}{2}m\int_o^f d(\vec{\mathbf{V}}_{cm} \cdot \vec{\mathbf{V}}_{cm}) = \frac{1}{2}mV_{\text{cm,f}}^2 - \frac{1}{2}mV_{\text{cm,f}}^2 = \Delta K_{\text{trans}} \end{aligned} . \tag{27.6.1}$$

Thus for the translational motion

$$\int_o^f \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} \cdot d\vec{\mathbf{r}} = \frac{1}{2} m V_{\text{cm,f}}^2 - \frac{1}{2} m V_{\text{cm,f}}^2 . \qquad (27.6.2)$$

For the rotational motion we go the center of mass reference frame and the rotational work done i.e. the integral of the z-component of the torque about the center of mass with respect to $d\theta$ as we did for fixed axis rotational work. Then

$$\begin{aligned}\int_o^f (\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}})_z \, d\theta &= \int_o^f I_{\text{cm}} \frac{d\omega_{cm}}{dt} d\theta = \int_o^f I_{\text{cm}} d\omega_{cm} \frac{d\theta}{dt} = \int_o^f I_{\text{cm}} d\omega_{cm} \omega_{cm} \\ &= \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 - \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},i}^2 = \Delta K_{\text{rot}} \end{aligned} . \qquad (27.6.3)$$

Therefore we can combine these two separate results Eqs. (27.6.2) and (27.6.3) and get the work-energy theorem for a rotating and translating rigid body that undergoes fixed axis rotation about the center of mass.

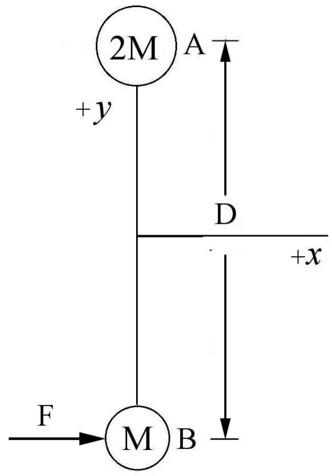
$$\begin{aligned} W_{o,f}^{\text{total}} &= \int_o^f \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} \cdot d\vec{\mathbf{r}} + \int_o^f (\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}})_z \, d\theta \\ &= \left(\frac{1}{2} m V_{\text{cm,f}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 \right) - \left(\frac{1}{2} m V_{\text{cm,f}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},i}^2 \right) . \\ &= \Delta K_{\text{trans}} + \Delta K_{\text{rot}} = \Delta K_{\text{total}} \end{aligned} \qquad (27.6.4)$$

So Eqs. (27.5.1), (27.5.4), and (27.6.4) are principles that we shall employ to analyze the motion of a rigid bodies undergoing translation and fixed axis rotation about the center of mass. (We shall call these our "Rules to Live By".)

27.7 Worked Examples

27.7.1 Example *Angular Impulse*

Two point-like objects are located at the points A, and B, of respective masses $M_A = 2M$, and $M_B = M$, as shown in the figure below. The two objects are initially oriented along the y-axis and connected by a rod of negligible mass of length D, forming a rigid body. A force of magnitude $F = \left|\vec{\mathbf{F}}\right|$ along the x direction is applied to the object at B at $t = 0$ for a short time interval Δt. Neglect gravity. Give all your answers in terms of M and D as needed.



a. Describe qualitatively in words how the system moves after the force is applied: direction, translation and rotation.

b. How far is the center of mass of the system from the object at point B?

c. What is the direction and magnitude of the linear velocity of the center-of-mass after the collision?

d. What is the magnitude of the angular velocity of the system after the collision?

e. Is it possible to apply another force of magnitude F along the positive x direction to prevent the system from rotating? Does it matter where the force is applied?

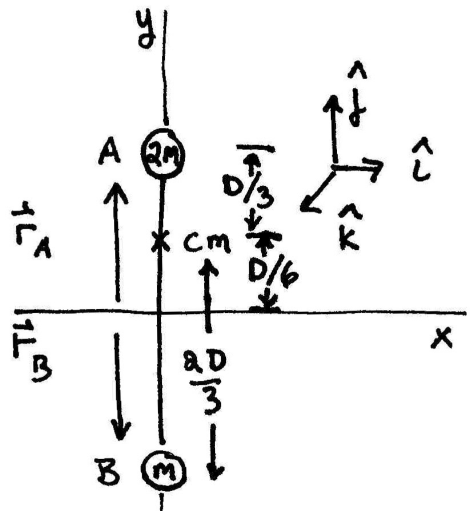
f. Is it possible to apply another force of magnitude F in some direction to prevent the center of mass from translating? Does it matter where the force is applied?

Solutions:

a) An impulse of magnitude $F\,\Delta t$ is applied in the $+x$ direction, and the center of mass of the system will move in this direction. The two masses will rotate about the center of mass, counterclockwise in the figure.

b) Before the force is applied we can calculate the position of the center of mass

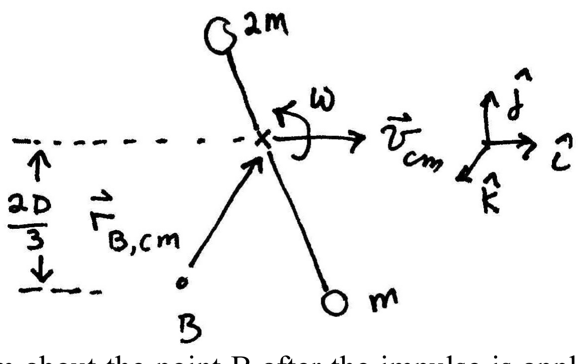
$$\vec{R}_{cm} = \frac{M_A\vec{r}_A + M_B\vec{r}_B}{M_A + M_B} = \frac{2M(D/2)\hat{\mathbf{j}} + M(D/2)(-\hat{\mathbf{j}})}{3M} = (D/6)\hat{\mathbf{j}}\,.$$



The center of mass is a distance $(2/3)\text{D}$ from the object at B and is a distance $(1/3)\text{D}$ from the object at A.

c) Because $F\Delta t\hat{i} = 3M\vec{V}_{cm}$, the magnitude of the velocity of the center of mass is then $(F\Delta t)/(3\text{M})$ and the direction is in the positive-$\hat{i}$ direction (to the right).

d) Because the force is applied at the point B, there is no torque about the point B, hence the angular momentum is constant about the point B. The initial angular momentum about the point B is zero.



The angular momentum about the point B after the impulse is applied is the sum of two terms,

$$\vec{0} = \vec{L}_{B,f} = \vec{r}_{B,f} \times 3M\vec{V}_{cm} + \vec{L}_{cm} = (2D/3)\hat{j} \times F\Delta t\hat{i} + \vec{L}_{cm}$$
$$\vec{0} = (2DF\Delta t/3)(-\hat{k}) + \vec{L}_{cm}$$

The angular momentum about the center of mass is given by

$$\vec{L}_{cm} = I_{cm}\omega\,\hat{k} = (2M(D/3)^2 + M(2D/3)^2)\omega\,\hat{k} = (2/3)MD^2\omega\,\hat{k}$$

Thus the angular about the point B after the impulse is applied is

$$\vec{0} = (2DF\Delta t/3)(-\hat{k}) + (2/3)MD^2\omega\,\hat{k}\,.$$

We can solve this equation for the angular speed

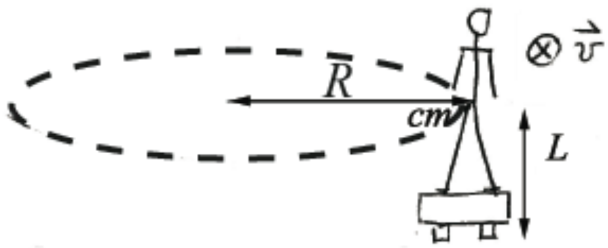
$$\omega = \frac{F\Delta t}{MD}$$

e) No. The force additional force would have to be applied at a distance $2\mathrm{D}/3$ above the center of mass, which is not a physical point of the system.

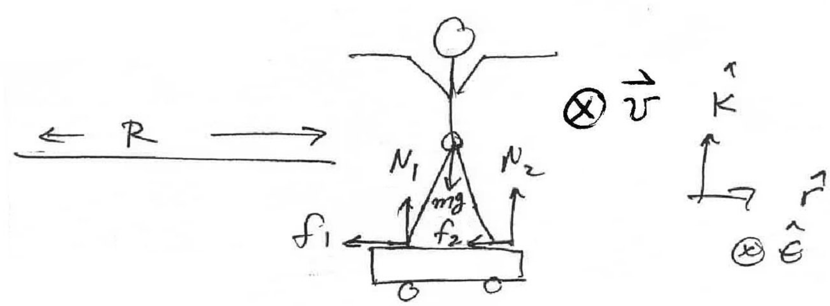
f) An additional force of the same magnitude, in the negative x direction, would result in no net force and hence no acceleration of the center of mass.

27.7.2 Example Person on a railroad car moving in a circle

A person of mass M is standing on a railroad car, which is rounding an unbanked turn of radius R at a speed v. His center of mass is at a height of L above the car midway between his feet, which are separated by a distance of d. The man is facing the direction of motion. What is the magnitude of the normal force on each foot?



Solution: We begin by choosing a cylindrical coordinate system and drawing a free-body force diagram, shown below.



We decompose the contact force between the foot closest to the center of the circular motion and the ground into a tangential component corresponding to static friction $\vec{\mathbf{f}}_1$ and a perpendicular component, $\vec{\mathbf{N}}_1$. In a similar fashion we decompose the contact force between the foot furthest from the center of the circular motion and the ground into a tangential component corresponding to static friction $\vec{\mathbf{f}}_2$ and a perpendicular component, $\vec{\mathbf{N}}_2$. We do not assume that the static friction has its maximum magnitude nor due we assume that $\vec{\mathbf{f}}_1 = \vec{\mathbf{f}}_2$ or $\vec{\mathbf{N}}_1 = \vec{\mathbf{N}}_2$. The gravitational force acts at the center of mass.

We shall use our two dynamical equations of motion, Eq. (27.5.1) for translational motion and Eq. (27.5.4) for the rotational motion about the center of mass noting that we are considering the special case that $\vec{\boldsymbol{\alpha}}_{cm} = 0$ because the object is not rotating about the center of mass.

In order to apply Eq. (27.5.1), we treat the person as a point-like particle located at the center of mass and all the eternal forces act at this point. The radial component of Newton's Second Law (Eq. (27.5.1) is given by

$$\hat{\mathbf{r}} : -f_1 - f_2 = -m\frac{v^2}{R} . \qquad (27.7.1)$$

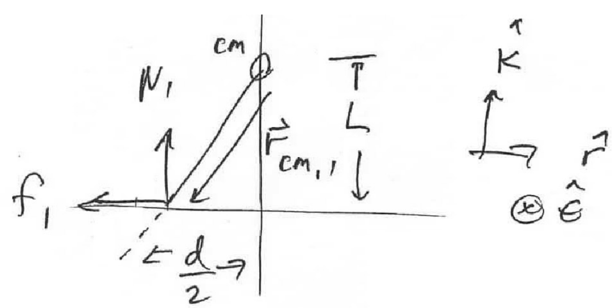
The vertical component of Newton's Second Law is given by

$$\hat{\mathbf{k}} : N_1 + N_2 - mg = 0 . \qquad (27.7.2)$$

The rotational equation of motion (Eq. (27.5.4) is

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}} = 0 . \qquad (27.7.3)$$

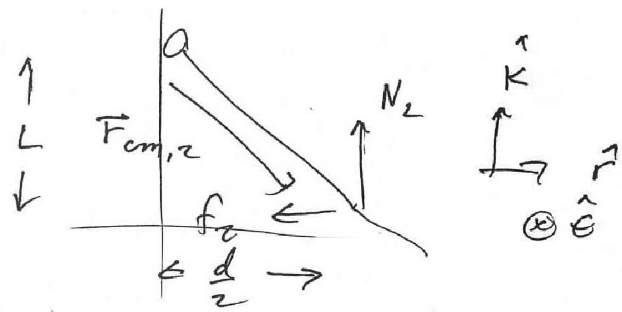
We begin our calculation of the torques about the center of mass by noting that the gravitational force does not contribute to the torque because it is acting at the center of mass. We draw a torque diagram in the figure below showing the location of the point of application of the forces, the point we are computing the torque about (which in this case is the center of mass), and the vector $\vec{\mathbf{r}}_{\text{cm},1}$ from the point we are computing the torque about to the point of application of the forces.



The torque on the inner foot is given by

$$\vec{\boldsymbol{\tau}}_{\text{cm},1} = \vec{\mathbf{r}}_{\text{cm},1} \times (\vec{\mathbf{f}}_1 + \vec{\mathbf{N}}_1) = \left(-\frac{d}{2}\hat{\mathbf{r}} - L\hat{\mathbf{k}}\right) \times (-f_1\hat{\mathbf{r}} + N_1\hat{\mathbf{k}}) = \left(\frac{d}{2}N_1 + Lf_1\right)\hat{\boldsymbol{\theta}} . \qquad (27.7.4)$$

We draw a similar torque diagram for the forces applied to the outer foot.



The torque on the outer foot is given by

$$\vec{\boldsymbol{\tau}}_{\text{cm},2} = \vec{\mathbf{r}}_{\text{cm},2} \times (\vec{\mathbf{f}}_2 + \vec{\mathbf{N}}_2) = \left(+\frac{d}{2}\hat{\mathbf{r}} - L\hat{\mathbf{k}} \right) \times (-f_2\hat{\mathbf{r}} + N_2\hat{\mathbf{k}}) = \left(-\frac{d}{2}N_2 + Lf_2 \right)\hat{\boldsymbol{\theta}} . \quad (27.7.5)$$

Notice that the forces $\vec{\mathbf{f}}_1$, $\vec{\mathbf{N}}_1$, and $\vec{\mathbf{f}}_2$ all contribute torques about the center of mass in the positive $\hat{\boldsymbol{\theta}}$-direction while $\vec{\mathbf{N}}_2$ contribute torques about the center of mass in the negative $\hat{\boldsymbol{\theta}}$-direction while $\vec{\mathbf{N}}_2$. According to Eq. (27.7.3) the sum of these torques about the center of mass must be zero. Therefore

$$\begin{aligned} \vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}} &= \vec{\boldsymbol{\tau}}_{\text{cm},1} + \vec{\boldsymbol{\tau}}_{\text{cm},2} = \left(\frac{d}{2}N_1 + Lf_1 \right)\hat{\boldsymbol{\theta}} + \left(-\frac{d}{2}N_2 + Lf_2 \right)\hat{\boldsymbol{\theta}} \\ &= \left(\frac{d}{2}(N_1 - N_2) + L(f_1 + f_2) \right)\hat{\boldsymbol{\theta}} = 0 \end{aligned} . \quad (27.7.6)$$

Notice that the magnitudes of the two friction forces appear together as a sum in Eqs. (27.7.6) and (27.7.1). We now can solve Eq. (27.7.1) for $f_1 + f_2$ and substitute the result into Eq. (27.7.6) yielding the condition that

$$\frac{d}{2}(N_1 - N_2) + Lm\frac{v^2}{R} = 0 . \quad (27.7.7)$$

We can rewrite this equation as

$$N_2 - N_1 = \frac{2Lmv^2}{dR} . \quad (27.7.8)$$

We also rewrite the vertical equation of motion (Eq. (27.7.2) in the form

$$N_2 + N_1 = mg . \quad (27.7.9)$$

We now can solve for N_2 by adding together Eqs. (27.7.8) and (27.7.9) and then divide by two,

$$N_2 = \frac{1}{2}\left(\frac{2Lmv^2}{dR} + mg\right). \qquad (27.7.10)$$

We now can solve for N_1 by subtracting Eqs. (27.7.8) from (27.7.9) and then divide by two,

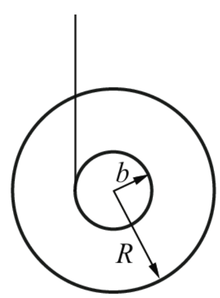
$$N_1 = \frac{1}{2}\left(mg - \frac{2Lmv^2}{dR}\right). \qquad (27.7.11)$$

Check your result: We see that the normal force acting on the outer foot is greater in magnitude than the normal force acting on the inner foot. We expect this result because as we increase the speed v, we find that at a maximum speed $v_{\max}$, the normal force on the inner foot goes to zero and we start to rotate in the positive $\hat{\boldsymbol{\theta}}$-direction, tipping outward. We can find this maximum speed by setting $N_1 = 0$ in Eq. (27.7.11) resulting in

$$v_{\max} = \sqrt{\frac{gdR}{2L}}. \qquad (27.7.12)$$

27.7.3 Example Torque, Rotation and Translation: Yo-Yo

A Yo-Yo of mass m has an axle of radius b and a spool of radius R. Its moment of inertia about the center can be taken to be $I_{cm} = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest. You will need to assume that the center of mass of the Yo-Yo descends vertically, and that the string is vertical as it unwinds.

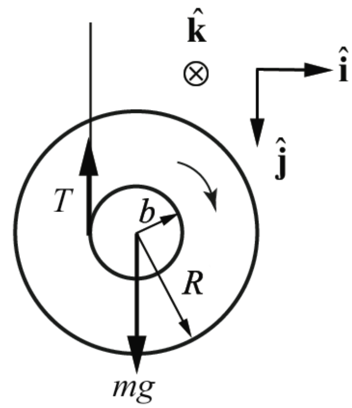


a) What is the tension in the cord as the Yo-Yo descends?
b) What is the magnitude of the angular acceleration as the yo-yo descends and the magnitude of the linear acceleration?

c) Find the angular velocity of the Yo-Yo when it reaches the bottom of the string, when a length l of the string has unwound.

Solutions:

a) As the Yo-Yo descends it rotates clockwise in the above diagram. The torque about the center of mass of the Yo-Yo is due to the tension and increases the magnitude of the angular velocity (see the figure below).



The direction of the torque is into the page in the figure above (positive z-direction). Use the right-hand rule to check this, or use the cross-product definition of torque:

$$\vec{\tau}_{\text{cm}} = \vec{\mathbf{r}}_{\text{cm},T} \times \vec{\mathbf{T}} . \qquad (27.7.13)$$

About the center of mass, $\vec{\mathbf{r}}_{\text{cm},T} = -b\,\hat{\mathbf{i}}$ and $\vec{\mathbf{T}} = -T\,\hat{\mathbf{j}}$, so the torque is

$$\vec{\tau}_{\text{cm}} = \vec{\mathbf{r}}_{\text{cm},T} \times \vec{\mathbf{T}} = (-b\,\hat{\mathbf{i}}) \times (-T\,\hat{\mathbf{j}}) = bT\,\hat{\mathbf{k}} . \qquad (27.7.14)$$

Applying Newton's Second Law in the $\hat{\mathbf{j}}$-direction,

$$mg - T = ma_y \qquad (27.7.15)$$

Applying the torque equation for the Yo-Yo:

$$bT = I_{\text{cm}}\alpha_z \qquad (27.7.16)$$

where α is the z-component of the angular acceleration.

The z-component of the angular acceleration and the y-component of the linear acceleration are related by the constraint condition

$$a_y = b\alpha_z \qquad (27.7.17)$$

where b is the axle radius of the Yo-Yo. Substitute Eq. (27.7.17) into (27.7.15) yielding

$$mg - T = mb\alpha_z \tag{27.7.18}$$

Now solve Eq. (27.7.16) for α_z and substitute the result into Eq.(27.7.18),

$$mg - T = \frac{mb^2 T}{I_{\text{cm}}} \tag{27.7.19}$$

Solve Eq. (27.7.19)for the tension T,

$$T = \frac{mg}{\left(1 + \dfrac{mb^2}{I_{\text{cm}}}\right)} = \frac{mg}{\left(1 + \dfrac{mb^2}{(1/2)mR^2}\right)} = \frac{mg}{\left(1 + \dfrac{2b^2}{R^2}\right)} \tag{27.7.20}$$

b) Substitute Eq. (27.7.20) into Eq. (27.7.16) to determine the z-component of the angular acceleration,

$$\alpha_z = \frac{bT}{I_{\text{cm}}} = \frac{2bg}{(R^2 + 2b^2)}. \tag{27.7.21}$$

Using the constraint condition Eq. (27.7.17), we can find the y-component of linear acceleration is then

$$a_y = b\alpha_z = \frac{2b^2 g}{(R^2 + 2b^2)} = \frac{g}{1 + R^2 / 2b^2}; \tag{27.7.22}$$

Note that both quantities $\alpha_z > 0$ and so Eqs. (27.7.21) and (27.7.22) are the magnitudes of the respective quantities. For a typical Yo-Yo, the acceleration is much less than that of an object in free fall.

b) Use conservation of energy to determine the angular velocity of the Yo-Yo when it reaches the bottom of the string. As in the above diagram, choose the downward vertical direction as the positive $\hat{\mathbf{j}}$-direction and let $y = 0$ designate the location of the center of mass of the Yo-Yo when the string is completely wound. Choose $U(y = 0) = 0$ for the zero reference potential energy. This choice of direction and reference means that the gravitational potential energy will be negative but increasing while the Yo-Yo descends. For this case, the gravitational potential energy is

$$U = -mg\,y. \tag{27.7.23}$$

Mechanical energy in the initial state (Yo-Yo is completely wound): the Yo-Yo is not yet moving downward or rotating, and the center of mass is located at $y = 0$ so the mechanical energy is zero

$$E_i = 0\,. \tag{27.7.24}$$

Mechanical energy in the final state (Yo-Yo is completely unwound): Denote the linear speed of the Yo-Yo as v_f and its angular speed as ω_f (at the point $y = l$). The constraint condition between v_f and ω_f is given by

$$v_f = b\omega_f\,, \tag{27.7.25}$$

consistent with Eq. (27.7.17). The kinetic energy is the sum of translational and rotational kinetic energy, where we have used $I_{\text{cm}} = (1/2)mR^2$,

$$\begin{aligned} E_f = K_f + U_f &= \frac{1}{2}mv_f^2 + \frac{1}{2}I_{\text{cm}}\omega_f^2 - mgl \\ &= \frac{1}{2}mb^2\omega_f^2 + \frac{1}{4}mR^2\omega_f^2 - mgl \end{aligned}\,. \tag{27.7.26}$$

There are no external forces doing work on the system, so

$$0 = E_f = E_i \tag{27.7.27}$$

Thus

$$\left(\frac{1}{2}mb^2 + \frac{1}{4}mR^2\right)\omega_f^2 = mgl\,. \tag{27.7.28}$$

Solving for ω_f,

$$\omega_f = \sqrt{\frac{4gl}{(2b^2 + R^2)}}\,. \tag{27.7.29}$$

Note: We could also use kinematics to determine the final angular velocity by solving for the time interval Δt that it takes for the Yo-Yo to travel a distance l at the constant acceleration found in Eq. (27.7.22)),

$$\Delta t = \sqrt{2l / a_y} = \sqrt{\frac{l(R^2 + 2b^2)}{b^2 g}} \tag{27.7.30}$$

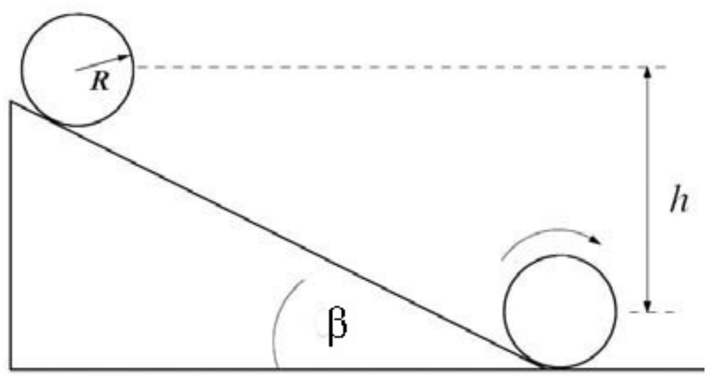
The final angular velocity of the Yo-Yo is then (using Eq. (27.7.21) for the z-component of the angular acceleration),

$$\omega_f = \alpha_z \Delta t = \sqrt{\frac{4gl}{(R^2 + 2b^2)}}, \qquad (27.7.31)$$

in agreement with Eq. (27.7.29).

27.7.4 Example Cylinder rolling down an inclined plane

A uniform cylinder of outer radius R and mass M with moment of inertia about the center of mass $I_{\text{cm}} = (1/2)M R^2$ starts from rest and moves down an incline tilted at an angle β from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ_s. The cylinder rolls without slipping down the incline. The goal of this problem is to find the magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.



Solution: We shall solve this problem three different ways.

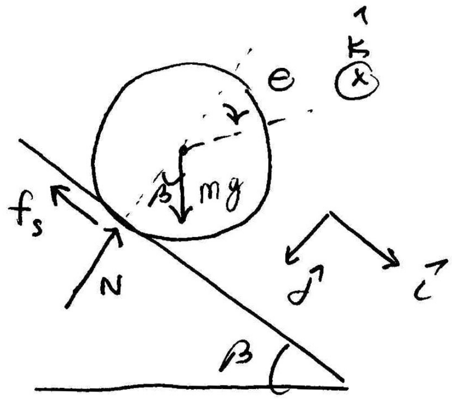
1. Applying the torque equation about the center of mass and the force equation for the center of mass motion.

2. Applying the energy equation.

3. Using torque about a fixed point that lies along the line of contact between the cylinder and the surface,

Applying the torque equation about the center of mass and the force equation for the center of mass motion

We will first find the acceleration and hence the speed at the bottom of the incline using kinematics. A figure showing the forces is shown below.

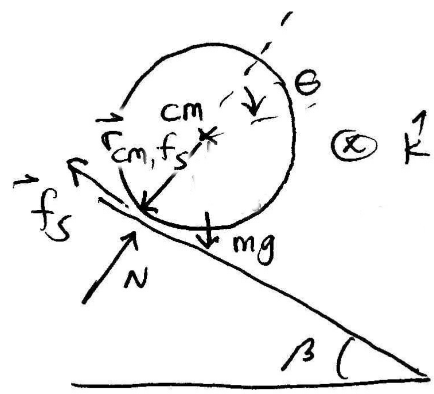


Choose $x = 0$ as the point where the cylinder just starts to roll. With the unit vectors shown in the figure above, Newton's Second Law, applied in the x- and y-directions in turn, yields

$$Mg\sin\beta - f_s = Ma_x , \qquad (27.7.32)$$

$$-N + Mg\cos\beta = 0 . \qquad (27.7.33)$$

Choose the center of the cylinder to compute the torque about (see figure below).



Then, the only force exerting a torque about the center of mass is the friction force, and so we have

$$f_s R = I_{\text{cm}}\alpha_z . \qquad (27.7.34)$$

Use $I_{\text{cm}} = (1/2)M R^2$ and the kinematic constraint for the no-slipping condition $\alpha_z = a_x / R$ in Eq. (27.7.34) to solve for the magnitude of the static friction force yielding

$$f_s = (1/2)Ma_x \ . \qquad (27.7.35)$$

Substituting Eq. (27.7.35) into Eq. (27.7.32)

$$Mg\sin\theta - (1/2)Ma_x = Ma_x \quad (27.7.36)$$

which we can solve for the acceleration

$$a_x = \frac{2}{3}g\sin\beta\,. \quad (27.7.37)$$

The displacement of the cylinder is $x_f = h/\sin\beta$ in the time it takes to reach the bottom, t_f. The x-component of the velocity v_x at the bottom is $v_{x,f} = a_x t_f$. The displacement in the time interval t_f satisfies $x_f = (1/2)a_x t_f^{\,2}$. After eliminating t_f, we have $x_f = v_{x,f}^{\,2}/2a_x$, so the magnitude of the velocity when the cylinder reaches the bottom of the inclined plane is

$$v_{x,f} = \sqrt{2a_x x_f} = \sqrt{2((2/3)g\sin\beta)(h/\sin\beta)} = \sqrt{(4/3)gh}\,. \quad (27.7.38)$$

Note that if we substitute Eq. (27.7.37) into Eq. (27.7.35) the magnitude of the friction force is

$$f_s = (1/3)Mg\sin\beta\,. \quad (27.7.39)$$

In order for the cylinder to roll without slipping

$$f_s \leq \mu_s Mg\cos\beta\,. \quad (27.7.40)$$

So combining Eq. (27.7.39) and Eq. (27.7.40) we have the condition that

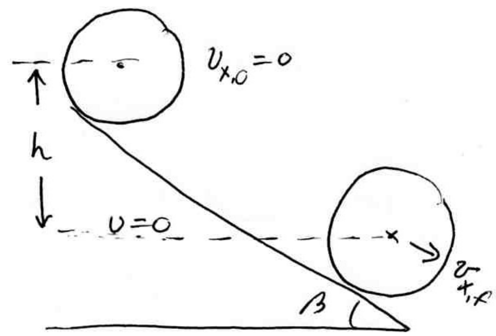
$$(1/3)Mg\sin\beta \leq \mu_s Mg\cos\beta \quad (27.7.41)$$

Thus in order to roll without slipping, the coefficient of static friction must satisfy

$$\mu_s \geq \frac{1}{3}\tan\beta\,. \quad (27.7.42)$$

Applying the energy equation

We shall use the fact that the energy of the cylinder-earth system is constant since the static friction force does no work. Choose a zero reference point for potential energy at the center of mass when the cylinder reaches the bottom of the incline plane.



Then the initial potential energy is

$$U_i = Mgh. \tag{27.7.43}$$

For the given moment of inertia, the final kinetic energy is

$$\begin{aligned} K_{\text{f}} &= \frac{1}{2} M\, v_{x,f}^{\;2} + \frac{1}{2} I_{\text{cm}} \omega_{z,f}^{\;2} \\ &= \frac{1}{2} M\, v_{x,f}^{\;2} + \frac{1}{2}(1/2) M R^2 (v_{x,f} / R)^2 \,. \\ &= \frac{3}{4} M\, v_{x,f}^{\;2} \end{aligned} \tag{27.7.44}$$

Setting the final kinetic energy equal to the initial gravitational potential energy leads to

$$Mgh = \frac{3}{4} M\, v_{x,f}^{\;2}. \tag{27.7.45}$$

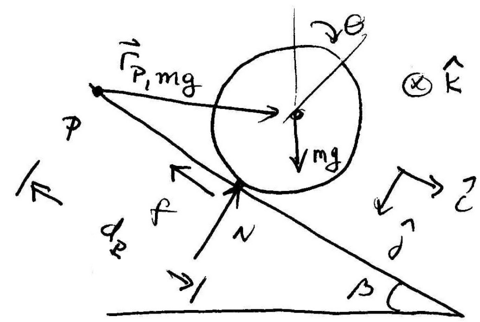
The magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline is

$$v_{x,f} = \sqrt{(4/3)gh}\,, \tag{27.7.46}$$

in agreement with Eq. (27.7.38).

Using torque about a fixed point that lies along the line of contact between the cylinder and the surface

Choose a fixed point that lies along the line of contact between the cylinder and the surface. Then the torque diagram is shown below.



The gravitational force $M\vec{\mathbf{g}} = Mg\sin\beta\,\hat{\mathbf{i}} + Mg\cos\beta\,\hat{\mathbf{j}}$ acts at the center of mass. The vector from the point P to the center of mass is given by $\vec{\mathbf{r}}_{P,mg} = d_P\,\hat{\mathbf{i}} - R\,\hat{\mathbf{j}}$, so the torque due to the gravitational force about the point P is given by

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{P,Mg} &= \vec{\mathbf{r}}_{P,Mg} \times M\vec{\mathbf{g}} = (d_P\,\hat{\mathbf{i}} - R\,\hat{\mathbf{j}}) \times (Mg\sin\beta\,\hat{\mathbf{i}} + Mg\cos\beta\,\hat{\mathbf{j}}) \\ &= (d_P Mg\cos\beta + RMg\sin\beta)\hat{\mathbf{k}}\end{aligned} . \qquad (27.7.47)$$

The normal force acts at the point of contact between the cylinder and the surface and is given by $\vec{\mathbf{N}} = -N\,\hat{\mathbf{j}}$. The vector from the point P to the point of contact between the cylinder and the surface is $\vec{\mathbf{r}}_{P,N} = d_P\,\hat{\mathbf{i}}$. So the torque due to the normal force about the point P is given by

$$\vec{\boldsymbol{\tau}}_{P,N} = \vec{\mathbf{r}}_{P,N} \times \vec{\mathbf{N}} = (d_P\,\hat{\mathbf{i}}) \times (-N\,\hat{\mathbf{j}}) = -d_P N\,\hat{\mathbf{k}}\,. \qquad (27.7.48)$$

Substituting Eq. (27.7.33) for the normal force into Eq. (27.7.48) yields

$$\vec{\boldsymbol{\tau}}_{P,N} = -d_P Mg\cos\beta\hat{\mathbf{k}}\,. \qquad (27.7.49)$$

Therefore the sum of the torques about the point P is

$$\vec{\boldsymbol{\tau}}_P = \vec{\boldsymbol{\tau}}_{P,Mg} + \vec{\boldsymbol{\tau}}_{P,N} = (d_P Mg\cos\beta + RMg\sin\beta)\hat{\mathbf{k}} - d_P Mg\cos\beta\hat{\mathbf{k}} = Rmg\sin\beta\hat{\mathbf{k}}\,. (27.7.50)$$

The angular momentum about the point P is given by

$$\begin{aligned}\vec{\mathbf{L}}_P &= \vec{\mathbf{L}}_{\text{cm}} + \vec{\mathbf{r}}_{P,\text{cm}} \times M\vec{\mathbf{V}}_{cm} = I_{\text{cm}}\omega_z\hat{\mathbf{k}} + (d_P\,\hat{\mathbf{i}} - R\,\hat{\mathbf{j}}) \times (Mv_x)\,\hat{\mathbf{i}} \\ &= (I_{\text{cm}}\omega_z + RMv_x)\,\hat{\mathbf{k}}\end{aligned} . \qquad (27.7.51)$$

The time derivative of the angular momentum about the point P is then

$$\frac{d\vec{\mathbf{L}}_P}{dt} = (I_{\text{cm}}\alpha_z + RMa_x)\,\hat{\mathbf{k}}\,. \tag{27.7.52}$$

Therefore the torque equation

$$\vec{\boldsymbol{\tau}}_P = \frac{d\vec{\mathbf{L}}_P}{dt}\,, \tag{27.7.53}$$

becomes

$$RMg\sin\beta\hat{\mathbf{k}} = (I_{\text{cm}}\alpha_z + RMa_x)\hat{\mathbf{k}}\ . \tag{27.7.54}$$

Using the fact that $I_{\text{cm}} = (1/2)MR^2$ and $\alpha_x = a_x / R$, the z-component of Eq. (27.7.54) becomes

$$RMg\sin\beta = (1/2)MRa_x + Rma_x = (3/2)MRa_x\,. \tag{27.7.55}$$

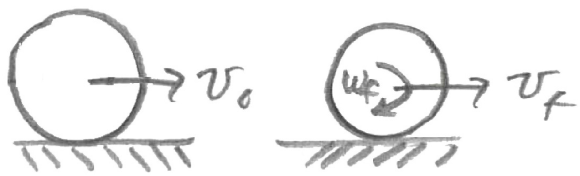
We can now solve Eq. (27.7.55) for the x-component of the acceleration

$$a_x = (2/3)g\sin\beta\,, \tag{27.7.56}$$

in agreement with Eq. (27.7.37).

27.7.5 Example *Bowling Ball*

A bowling ball of mass m and radius R is initially thrown down an alley with an initial speed v_0, and it slides without rolling but due to friction it begins to roll. The moment of inertia of the ball about its center of mass is $I_{\text{cm}} = (2/5)mR^2$. Using conservation of angular momentum about a point (you need to find that point) , find the speed v_f of the bowling ball when it just start to roll without slipping?



Solution: We begin by coordinates for our angular and linear motion. Choose an angular coordinate θ increasing in the clockwise direction. Choose positive $\hat{\mathbf{k}}$ unit vector pointing into the page. Then the angular velocity vector is defined to be a

$$\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}} = \frac{d\theta}{dt}\hat{\mathbf{k}}$$

and the angular acceleration vector is defined to be

$$\vec{\boldsymbol{\alpha}} = \alpha_z \hat{\mathbf{k}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}\,.$$

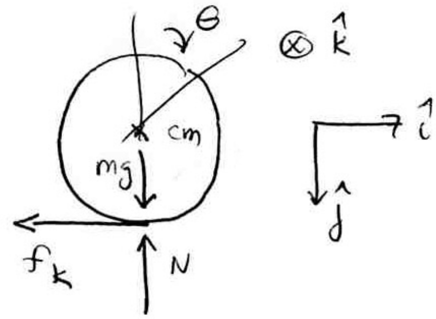
Choose the positive $\hat{\mathbf{i}}$ unit vector pointing to the right in the figure. Then the velocity of the center of mass is given by

$$\vec{\mathbf{v}}_{cm} = v_{cm,x}\hat{\mathbf{i}} = \frac{dx_{cm}}{dt}\hat{\mathbf{i}}$$

and the acceleration of the center of mass is given by

$$\vec{\mathbf{a}}_{cm} = a_{cm,x}\hat{\mathbf{i}} = \frac{d^2x_{cm}}{dt^2}\hat{\mathbf{i}}\,.$$

The free body force diagram is shown in the figure below.



At $t = 0$, when the ball is released, $\vec{\mathbf{v}}_{cm,0} = v_0\hat{\mathbf{i}}$ and $\vec{\boldsymbol{\omega}}_0 = \vec{\mathbf{0}}$, so the wheel is skidding and hence the friction force on the wheel due to the sliding of the wheel on the surface opposes the motion is kinetic friction and hence acts in the negative $\hat{\mathbf{i}}$-direction.

Our "rule to live by" for rotational motion is that

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt} \tag{57}$$

In order for angular momentum about some point to remain constant throughout the motion, the torque about that point must also be zero throughout the motion.

Recall that the torque about a point S is define as

$$\vec{\boldsymbol{\tau}}_S = \sum_i \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i$$

where $\vec{\mathbf{r}}_{S,i}$ is the vector form the point S to where the ith force $\vec{\mathbf{F}}_i$ acts on the object. As the ball moves down the alley, the contact point will move, but the friction force will always be parallel to the line of contact between the bowling bowl and the surface. So, if we pick any fixed point S along the line of contact between the bowling bowl and the surface then

$$\vec{\boldsymbol{\tau}}_{S,f_k} = \vec{\mathbf{r}}_{S,f_k} \times \vec{\mathbf{f}}_k = \vec{\mathbf{0}}$$

because $\vec{\mathbf{r}}_{S,f_k}$ and $\vec{\mathbf{f}}_k$ are anti-parallel. The gravitation force acts at the center of mass hence the torque due to gravity about the point S is

$$\vec{\boldsymbol{\tau}}_{S,mg} = \vec{\mathbf{r}}_{S,mg} \times m\vec{\mathbf{g}} = dmg\hat{\mathbf{k}}$$

where d is the distance from the point S tot eh contact point between the wheel and the ground.
The torque due to the normal force about the point S is

$$\vec{\boldsymbol{\tau}}_{S,N} = \vec{\mathbf{r}}_{S,N} \times m\vec{\mathbf{g}} = -dN\hat{\mathbf{k}}$$

with the same moment arm d. Because the wheel is not accelerating in the $\hat{\mathbf{j}}$-direction, from Newton's Second Law, we note that

$$mg - N = 0$$

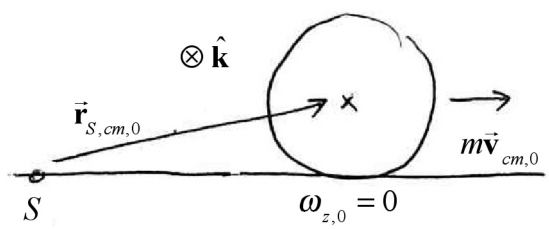
Therefore

$$\vec{\boldsymbol{\tau}}_{S,N} + \vec{\boldsymbol{\tau}}_{S,mg} = d(mg - N)\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Hence, there is no torque about the point S and the angular momentum about the point S is constant,

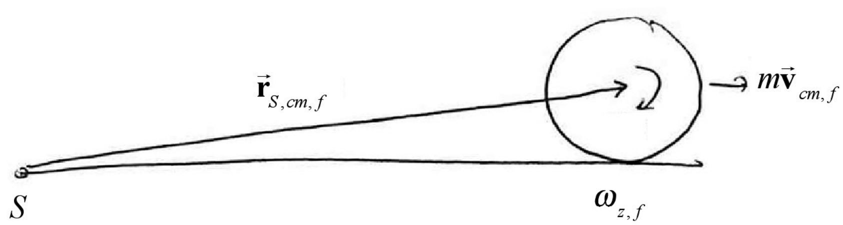
$$\vec{\mathbf{L}}_{S,0} = \vec{\mathbf{L}}_{S,f} \tag{58}$$

The initial angular momentum about the point S is only due to the translation of the center of mass,



$$\vec{\mathbf{L}}_{S,0} = \vec{\mathbf{r}}_{S,cm,0} \times m\vec{\mathbf{v}}_{cm,0} = mRv_{cm,0}\hat{\mathbf{k}}\,. \tag{59}$$

The final angular momentum about the point S has both a translational and rotational contribution



$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{r}}_{S,cm,f} \times m\vec{\mathbf{v}}_{cm,f} + I_{cm}\vec{\boldsymbol{\omega}}_f = mRv_{cm,f}\hat{\mathbf{k}} + I_{\text{cm}}\omega_{z,f}\hat{\mathbf{k}}\,, \tag{60}$$

When the wheel is rolling without slipping,

$$v_{cm,f} = R\omega_{z,f} \tag{61}$$

and also $I_{\rm cm} = (2/5)mR^2$. Therefore the final angular momentum about the point S is

$$\vec{\mathbf{L}}_{S,f} = (mR + (2/5)mR)v_{cm,f}\hat{\mathbf{k}} = (7/5)mRv_{cm,f}\hat{\mathbf{k}}. \qquad (62)$$

Equating the z-components in Equations **(59)** and (62) yields

$$mRv_{cm,0} = (7/5)mRv_{cm,f}, \qquad \mathbf{(63)}$$

which we can solve for

$$v_{cm,f} = (5/7)v_{cm,0}. \qquad (64)$$

We could also solve this problem by calculating the analyzing the translational motion and the rotational motion about the center of mass.

Gravity exerts no torque about the center of mass, and the normal component of the contact force has a zero moment arm; the only force that exerts a torque is the frictional force, with a moment arm of R (the force vector and the radius vector are perpendicular).

The frictional force should be in the negative direction, to the left in the figure above. From the right-hand rule, the direction of the torque is into the page, and hence in the positive z-direction. Equating the z-component of the torque to the rate of change of angular momentum,

$$\tau_{cm} = Rf_k = I_{cm}\alpha_z, \qquad \mathbf{(65)}$$

where f_k is the *magnitude* of the *kinetic* friction force and α is the z-component of the angular acceleration of the bowling ball. Note that Equation (65) results in a positive angular acceleration, which is consistent with the ball tending to rotate as indicated in the figure.

The friction force is also the only force in the horizontal direction, and will cause an acceleration of the center of mass,

$$a_{cm,x} = -f_k / m; \qquad \mathbf{(66)}$$

Note that the acceleration will be negative, as expected.

Now we need to consider the kinematics. The bowling ball will increase its angular speed as given in Equation (65) and decrease its linear speed as given in Equation (66);

$$\omega_z(t) = \alpha_z\, t = \frac{Rf_k}{I_{\text{cm}}} t$$
$$v_{cm,x}(t) = v_{cm,0} - \frac{f_k}{m} t. \qquad \textbf{(67)}$$

As soon as the ball stops slipping, the kinetic friction no longer acts, static friction is zero, and the ball moves with constant angular and linear velocity. Denote the time when this happens as t_f; at this time, Eq. (61) holds and the relations in Equation (67) become

$$R^2 \frac{f_k}{I_{cm}} t_f = v_{cm,f}$$
$$v_{cm,0} - \frac{f_k}{m} t_f = v_{cm,f} \qquad \textbf{(68)}$$

We can now solve the first equation in Eq. (68) for t_f and find that

$$t_f = \frac{I_{cm}}{f_k R^2} v_{cm,f}. \qquad \textbf{(69)}$$

We now substitute Eq. (69) into the second equation in Eq. (68) and find that

$$v_{cm,f} = v_{cm,0} - \frac{f_k}{m} \frac{I_{cm}}{f_k R^2} v_{cm,f}$$
$$v_{cm,f} = v_{cm,0} - \frac{I_{cm}}{m R^2} v_{cm,f} \qquad \textbf{(70)}$$

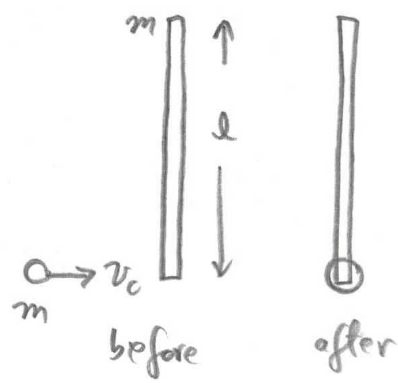
The second equation in (70) is easily solved for

$$v_{cm,f} = \frac{v_0}{1 + I_{cm} / mR^2} = \frac{5}{7} v_{cm,0}, \qquad \textbf{(71)}$$

agreeing with Eq. (64) where we have used $I_{\text{cm}} = (2/5) m R^2$ for a uniform sphere.

27.7.6 Example Rotation and Translation: *Object and Stick Collision*

A long narrow uniform stick of length l and mass m lies motionless on ice (assume the ice provides a frictionless surface). The center of mass of the stick is the same as the geometric center (at the midpoint of the stick). The moment of inertia of the stick about its center of mass is I_{cm}. A puck (with putty on one side) has the same mass m as the stick. The puck slides without spinning on the ice with a speed of v_0 toward the stick, hits one end of the stick, and attaches to it. You may assume that the radius of the puck is much less than the length of the stick so that the moment of inertia of the puck about its center of mass is negligible compared to I_{cm}.



a) How far from the midpoint of the stick is the center of mass of the stick-puck combination after the collision?

b) What is the linear velocity of the stick plus puck after the collision?

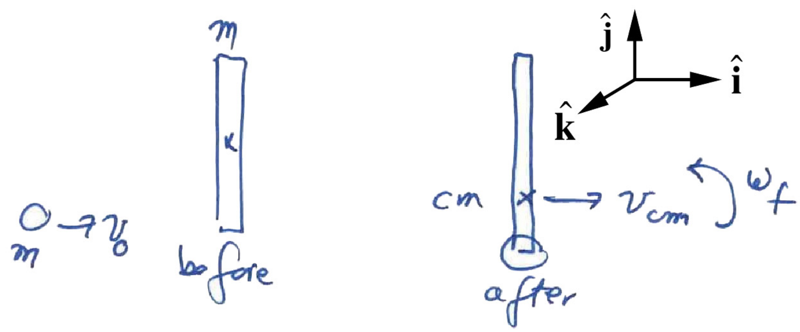
c) Is mechanical energy conserved during the collision? Explain your reasoning.

d) What is the angular velocity of the stick plus puck after the collision?

e) How far does the stick's center of mass move during one rotation of the stick?

Solution:

In this problem we will calculate the center of mass of the puck-stick system after the collision. There are no external forces or torques acting on this system so the momentum of the center of mass is constant before and after the collision and the angular momentum about the center of mass of the puck-stick system is constant before and after the collision. We shall use these relations to compute the final angular velocity of the puck-stick about the center of mass. We note that the mechanical energy is not constant because the puck collides completely inelastically with the stick.



a) With respect to the center of the stick, the center of mass of the stick-puck combination is (neglecting the radius of the puck)

$$d_{\text{cm}} = \frac{m_{\text{stick}} d_{\text{stick}} + m_{\text{puck}} d_{\text{puck}}}{m_{\text{stick}} + m_{\text{puck}}} = \frac{m(l/2)}{m+m} = \frac{l}{4}. \qquad (27.7.72)$$

b) During the collision, the only net forces on the system (the stick-puck combination) are the internal forces between the stick and the puck (transmitted through the putty).

Hence, linear momentum is conserved. Initially only the puck had linear momentum $p_0 = m v_0$. After the collision, the center of mass of the system is moving with speed v_f. Equating initial and final linear momenta,

$$m v_0 = (2m) v_f \quad \Rightarrow \quad v_f = \frac{v_0}{2}\ . \qquad (27.7.73)$$

The direction of the velocity is the same as the initial direction of the puck's velocity.

Note that the result of part a) was not needed for part b); if the masses are the same, Equation (27.7.73) would hold for any mass distribution of the stick.

c) The forces that deform the putty do negative work (the putty is compressed somewhat), and so mechanical energy is not conserved; the collision is totally inelastic.

d) Choose the center of mass of the stick-puck combination, as found in part a), as the point about which to find angular momentum. This choice means that after the collision there is no angular momentum due to the translation of the center of mass. Before the collision, the angular momentum was entirely due to the motion of the puck,

$$\vec{\mathbf{L}}_0 = \vec{\mathbf{r}}_{\text{puck}} \times \vec{\mathbf{p}}_0 = (l/4)(m v_0)\hat{\mathbf{k}}, \qquad (27.7.74)$$

where $\hat{\mathbf{k}}$ is directed out of the page in the figure above. After the collision, the angular momentum is

$$\vec{\mathbf{L}}_f = I_{\text{cm}'}\omega_f\,\hat{\mathbf{k}}\,, \qquad (27.7.75)$$

where $I_{\text{cm}'}$ is the moment of inertia about the center of mass of the stick-puck combination. This moment of inertia of the stick about the new center of mass is found from the parallel axis theorem, and the moment of inertia of the puck is $m(l/4)^2$, and so

$$I_{\text{cm}'} = I_{\text{cm}',\,\text{stick}} + I_{\text{cm}',\,\text{puck}} = (I_{\text{cm}} + m(l/4)^2) + m(l/4)^2 = I_{\text{cm}} + \frac{ml^2}{8}. \qquad (27.7.76)$$

Inserting this expression into Equation (27.7.75), equating the expressions for $\vec{\mathbf{L}}_0$ and $\vec{\mathbf{L}}_f$ and solving for ω_f yields

$$\omega_f = \frac{m(l/4)}{I_{\text{cm}} + ml^2/8}v_0\,. \qquad (27.7.77)$$

If the stick is uniform, $I_{\text{cm}} = ml^2/12$ and Equation (27.7.77) reduces to

$$\omega_f = \frac{6}{5}\frac{v_0}{l}. \qquad (27.7.78)$$

It may be tempting to try to calculate angular momentum about the contact point, where the putty hits the stick. If this is done, there is no initial angular momentum, and after the collision the angular momentum will be the sum of two parts, the angular momentum of the center of mass of the stick and the angular moment about the center of the stick,

$$\vec{\mathbf{L}}_f = \vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{p}}_{\text{cm}} + I_{\text{cm}}\vec{\boldsymbol{\omega}}_f\,. \qquad (27.7.79)$$

There are two crucial things to note: First, the speed of the center of mass is not the speed found in part b); the rotation must be included, so that $v_{\text{cm}} = v_0/2 - \omega_f(l/4)$. Second, the direction of $\vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{p}}_{\text{cm}}$ with respect to the contact point is, from the right-hand rule, *into* the page, or the $-\hat{\mathbf{k}}$-direction, opposite the direction of $\vec{\boldsymbol{\omega}}_f$. This is to be expected, as the sum in Equation (27.7.79) must be zero. Adding the $\hat{\mathbf{k}}$-components (the only components) in Equation (27.7.79),

$$-(l/2)m(v_0/2 - \omega_f(l/4)) + I_{\text{cm}}\omega_f = 0\,. \qquad (27.7.80)$$

Solving Equation (27.7.80) for ω_f yields Equation (27.7.77).

This alternative derivation should serve two purposes. One is that it doesn't matter which point we use to find angular momentum. The second is that use of foresight, in this case choosing the center of mass of the system so that the final velocity does not contribute to the angular momentum, can prevent extra calculation. It's often a matter of trial and error ("learning by misadventure") to find the "best" way to solve a problem.

e) The time of one rotation will be the same for all observers, independent of choice of origin. This fact is crucial in solving problems, in that the angular velocity will be the same (this was used in the alternate derivation for part d) above). The time for one rotation is the period $T = 2\pi / \omega_f$ and the distance the center of mass moves is

$$\begin{aligned} x_{\text{cm}} &= v_{\text{cm}} T = 2\pi \frac{v_{\text{cm}}}{\omega_f} \\ &= 2\pi \frac{v_0 / 2}{\left(\dfrac{m(l/4)}{I_{\text{cm}} + ml^2/8} \right) v_0} \\ &= 2\pi \frac{I_{\text{cm}} + ml^2/8}{m(l/2)}. \end{aligned} \qquad (27.7.81)$$

Using $I_{\text{cm}} = ml^2 / 12$ for a uniform stick gives

$$x_{\text{cm}} = \frac{5}{6}\pi l. \qquad (27.7.82)$$