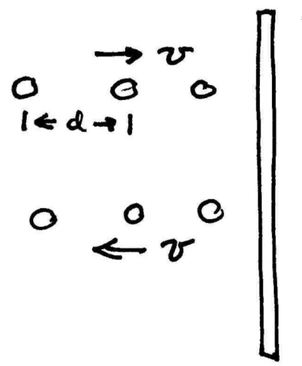
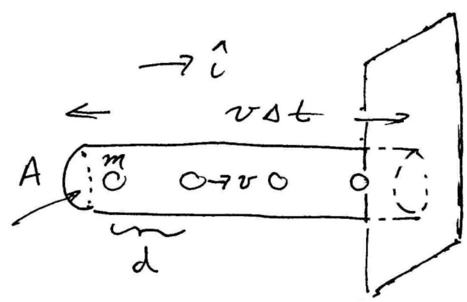
# Momentum and the Flow of Mass
## Challenge Problems Solutions

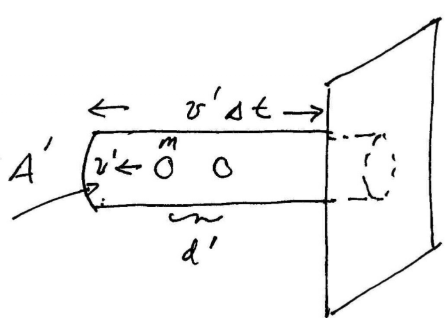
### Problem 1: Stream Bouncing off Wall

A stream of particles of mass $m$ and separation $d$ hits a perpendicular surface with speed $v$. The stream rebounds along the original line of motion with the same speed. The mass per unit length of the incident stream is $\lambda = m / d$. What is the magnitude of the force on the surface?



**Solution:** We begin by defining our system as a tube of water of cross sectional area $A$ and length $v\Delta t$.





**Mass Conservation:** The mass of the water in this tube is given by

$$\Delta m_i = \rho A v \Delta t = \lambda v \Delta t \qquad (1.1)$$

where $\lambda = \rho A = m / d$ is the mass per unit length . This means that all the water inside this tube will hit the wall during the interval $[t, t+\Delta t]$ and reflect. Note that if the collision with the wall is elastic, the reflected water will have the same speed as the incident water. Otherwise to allow for a slightly more general case, we shall assume that the reflected water travels backwards with speed $v'$. We assume that the water is incompressible so that the density has not changed. The mass of the water that has been reflected is given by

$$\Delta m_r = \rho A' v' \Delta t = \lambda' v' \Delta t \tag{1.2}$$

where $\lambda' = \rho A' = m / d'$ is the mass per unit length of the reflected water . We assume that the water is incompressible so that the density has not changed. We shall also assume for simplicity that all the water incident on the wall reflects with the speed $v'$. Thus

$$\Delta m_i = \Delta m_r \,. \tag{1.3}$$

Thus in the time interval

$$\frac{\Delta m_i}{\Delta t} = \frac{\Delta m_r}{\Delta t} \,. \tag{1.4}$$

This implies that

$$\lambda v = \lambda' v' \,. \tag{1.5}$$

Note that the instantaneous rate that mass is delivered to the surface and leaves is

$$\frac{dm}{dt} \equiv \frac{dm_i}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m_i}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta m_r}{\Delta t} = \frac{dm_r}{dt} = \lambda v = \lambda' v' \,. \tag{1.6}$$

Momentum Principle: The momentum principle states that the impulse from the surface changes the momentum of the water. The momentum of the incident water is given by

$$\vec{p}_i = \Delta m_i v \,\hat{i} = (\lambda v \Delta t) v \,\hat{i} = \lambda v^2 \Delta t \,\hat{i} \,. \tag{1.7}$$

The x-component of the momentum of the reflected water is given by

$$\vec{p}_r = -\Delta m_r v \,\hat{i} = -(\lambda' v' \Delta t) v' \,\hat{i} = -\lambda' v'^2 \Delta t \,\hat{i} \,. \tag{1.8}$$

The surface exerts an average force on the water, $\vec{F}_{sw}^{ave}$, hence an impulse $\vec{F}_{sw}^{ave} \Delta t$. So the momentum principle becomes

$$\vec{F}_{sw}^{ave} \Delta t = \Delta \vec{p} \,. \tag{1.9}$$

Based on our analysis of the momentum incident and reflected the momentum principle becomes

$$\vec{F}_{sw}^{ave}\Delta t = \vec{p}_r - \vec{p}_i = -\lambda' v'^2 \Delta t\,\hat{i} - \lambda v^2 \Delta t\,\hat{i} = -(\lambda' v' v' + \lambda v v)\Delta t\,\hat{i}\ . \qquad (1.10)$$

Therefore the force of the surface on the water is

$$\vec{F}_{sw}^{ave} = -(\lambda' v' v' + \lambda v v)\,\hat{i}\ . \qquad (1.11)$$

We can apply our condition for mass conservation, Eq. (1.5) and find that

$$\vec{F}_{sw}^{ave} = -\lambda v (v' + v)\,\hat{i}\ . \qquad (1.12)$$

By Newton's Third Law the average force of the water on the surface is

$$\vec{F}_{ws}^{ave} = -\vec{F}_{sw}^{ave} = \lambda v (v' + v)\,\hat{i}\ . \qquad (1.13)$$

In order to produce the maximum possible average force of water on the surface, the collision must be elastic since then the reflected speed is equal to the incident speed and

$$(\vec{F}_{ws}^{ave})_{max} = 2\lambda v^2\,\hat{i}\ . \qquad (1.14)$$

Note that the using the instantaneous rate that mass arrives or leaves the surface, we have that

$$(\vec{F}_{ws}^{ave})_{max} = 2\lambda v v\,\hat{i} = 2\frac{dm}{dt} v\,\hat{i}\ . \qquad (1.15)$$

So the magnitude of the force on the surface is

$$\left|(\vec{F}_{ws}^{ave})_{max}\right| = 2\lambda v^2\ . \qquad (1.16)$$

**Problem 2** A rocket has a dry mass (empty of fuel) $m_{r,0} = 2\times 10^7\,\mathrm{kg}$, and initially carries fuel with mass $m_{f,0} = 5\times 10^7\,\mathrm{kg}$. The fuel is ejected at a speed $u = 2.0\times 10^3\,\mathrm{m\cdot s^{-1}}$ relative to the rocket. What is the final speed of the rocket after all the fuel has burned?

**Solution** The initial mass of the rocket included the fuel is

$$m_{r,i} = m_{r,0} + m_{f,0} = 2\times 10^7\,\mathrm{kg} + 5\times 10^7\,\mathrm{kg} = 7\times 10^7\,\mathrm{kg} \tag{2.1}$$

The ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel) is

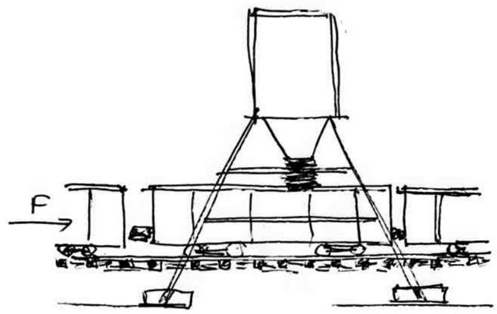
$$R = \frac{m_{r,i}}{m_{r,d}} = \frac{7\times 10^7\,\mathrm{kg}}{2\times 10^7\,\mathrm{kg}} = 3.5 \tag{2.2}$$

The final speed of the rocket is then

$$v_{r,f} = u \ln R = (2.0\times 10^3\,\mathrm{m\cdot s^{-1}}) \ln 3.5 = 2.5\times 10^3\,\mathrm{m\cdot s^{-1}}. \tag{2.3}$$

**Problem 3: Coal Car**

An empty coal car of mass $m_0$ starts from rest under an applied force of magnitude $F$. At the same time coal begins to run into the car at a steady rate $b$ from a coal hopper at rest along the track. Find the speed when a mass $m_c$ of coal has been transferred.




**Solution:** We shall analyze the momentum changes in the horizontal direction which we call the x-direction.. Since the coal does not have any horizontal velocity, the falling coal is not transferring any momentum to the coal car. So we shall take as our system the empty coal car and a mass $m_c$ of coal that has been transferred. Our initial state at $t = 0$ is when the coal car is empty and at rest before any coal has been transferred. The x-component of the momentum of this initial state is zero,

$$p_x(0) = 0 . \tag{3.1}$$

Our final state at $t = t_f$ is when all the coal of mass $m_c = bt_f$ has been transferred into the car which is now moving at speed $v_f$. The x-component of the momentum of this final state is

$$p_x(t_f) = (m_0 + m_c)v_f = (m_0 + bt_f)v_f . \tag{3.2}$$

There is an external constant force $F_x = F$ applied through the transfer. The momentum principle applied to the x-direction is

$$\int_0^{t_f} F_x dt = \Delta p_x = p_x(t_f) - p_x(0) . \tag{3.3}$$

Since the force is constant, the integral is simple and the momentum principle becomes

$$Ft_f = (m_0 + bt_f)v_f . \tag{3.4}$$

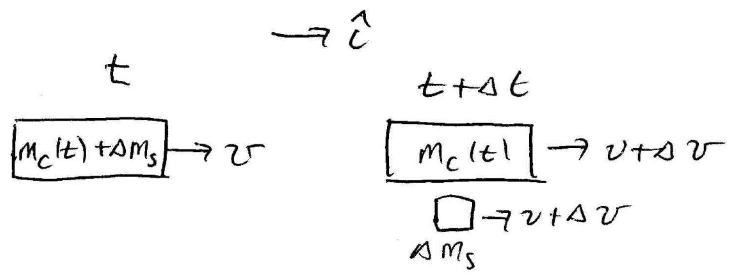
So the final speed is

$$v_f = \frac{Ft_f}{(m_0 + bt_f)}\,. \tag{3.5}$$

**Problem 4: Emptying a Freight Car**

A freight car of mass $m_c$ contains a mass of sand $m_s$. At $t = 0$ a constant horizontal force of magnitude $F$ is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s / dt$. Find the speed of the freight car when all the sand is gone. Assume that the freight car is at rest at $t = 0$.

**Solution:** Choose the positive-direction to point in the direction that the car is moving. Let's take as our system the amount of sand of mass $\Delta m_s$ that leaves the freight car during the time interval $[t, t + \Delta t]$, and the freight car and whatever sand is in it at time $t$.




At the beginning of the interval the car and sand is moving with speed $v$ so the x-component of the momentum at time $t$ is given by

$$p_x(t) = (\Delta m_s + m_c(t))v), \tag{4.1}$$

where $m_c(t)$ is the mass of the car and sand in it at time $t$. Denote by $m_{c,0} = m_c + m_s$ where the $m_c$ is the mass of the car and $m_s$ is the mass of the sand in the car at $t = 0$, and $m_s(t) = bt$ is the mass of the sand that has left the car at time $t$ since

$$m_s(t) = \int_0^t \frac{dm_s}{dt} dt = \int_0^t b dt = bt . \tag{4.2}$$

Thus

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt . \tag{4.3}$$

During the interval $[t, t + \Delta t]$, the small amount of sand of mass $\Delta m_s$ leaves the car with the speed of the car at the end of the interval $v + \Delta v$. So the x-component of the momentum at time $t + \Delta t$ is given by

$$p_x(t+\Delta t) = (\Delta m_s + m_c(t))(v+\Delta v)\,. \tag{4.4}$$

Throughout the interval a constant force $F$ is applied to the car so

$$F = \lim_{\Delta t \to 0} \frac{p_x(t+\Delta t) - p_x(t)}{\Delta t}. \tag{4.5}$$

From our analysis above Eq. (4.5) becomes

$$F = \lim_{\Delta t \to 0} \frac{(m_c(t)+\Delta m_s)(v+\Delta v) - (m_c(t)+\Delta m_s)v}{\Delta t}. \tag{4.6}$$

Eq. (4.6) simplifies to

$$F = \lim_{\Delta t \to 0} m_c(t)\frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_s \Delta v}{\Delta t}. \tag{4.7}$$

The second term vanishes when we take the $\Delta t \to 0$ because it is of second order in the infinitesimal quantities (in this case $\Delta m_s \Delta v$) and so when dividing by $\Delta t$ the quantity is of first order and hence vanishes since both $\Delta m_s \to 0$ and $\Delta v \to 0$. So Eq. (4.7) becomes

$$F = \lim_{\Delta t \to 0} m_c(t)\frac{\Delta v}{\Delta t}. \tag{4.8}$$

We now use the definition of the derivative:

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{4.9}$$

in Eq. (4.8) to fund the differential equation

$$F = m_c(t)\frac{dv}{dt}. \tag{4.10}$$

Using Eq. (4.3) we have

$$F = (m_c + m_s - bt)\frac{dv}{dt}. \tag{4.11}$$

(b) We can integrate this equation through the separation of variable technique. Rewrite Eq. (4.11) as

$$dv = \frac{Fdt}{(m_c + m_s - bt)}. \tag{4.12}$$

We can then integrate both sides of Eq. (4.12) with the limits as shown

$$\int_{v=0}^{v(t)} dv = \int_0^t \frac{Fdt}{m_c + m_s - bt}. \tag{4.13}$$

Integration yields the velocity of the car as a function of time

$$v(t) = -\frac{F}{b}\ln(m_c + m_s - bt)\Big|_0^t = -\frac{F}{b}\ln\left(\frac{m_c + m_s - bt}{m_c + m_s}\right). \tag{4.14}$$

**Problem 5: Falling Chain**

A chain of mass $m$ and length $l$ is suspended vertically with its lowest end touching a scale. The chain is released and fall onto the scale. What is the reading of the scale when a length of chain, $y$, has fallen? (Neglect the size of the individual links.) Let $g$ denote the gravitational constant.

**Solution:**

Suppose the chain is released at time $t=0$. At time $t$, a length $y$ of the chain has fallen onto the scale and the rest of the chain has fallen a distance $y$. (Note that we have chosen downward as the positive y-direction.)

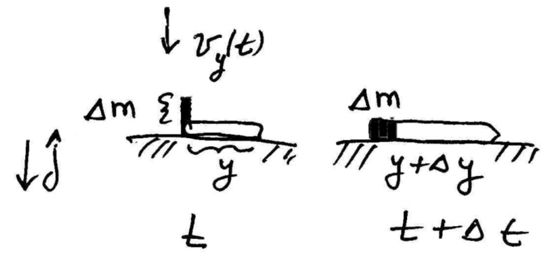
We can use our energy condition, applied to the portion of the chain that is freely falling (mass $m(t)$), to find that

$$(1/2)m(t)v_y^{\,2} = -m(t)g\Delta h = m(t)gy\,. \qquad (5.1)$$

Hence at time $t$, the entire chain is falling with a y-component of the velocity given by

$$v_y = \sqrt{2gy}\,. \qquad (5.2)$$

Note at the bottom of the chain, a small increment of chain, (length $\Delta y$ and mass $\Delta m = \lambda \Delta y = (m/l)\Delta y$), is moving with speed $v_y = \sqrt{2gy}$. After a small interval of time $\Delta t$ has elapsed (we shall shortly consider the limit as $\Delta t \to 0$), this small piece of chain has come to rest on the scale. Note that we have assumed that the mass per unit length $\lambda = m/l$ is uniform.




Let's take as our system this small piece of chain. There is a contact force between the chain and the scale, $\vec{\mathbf{F}}_{c,s}(t)$, that acts on this piece. This force $\vec{\mathbf{F}}_{c,s}(t)$ varies in time since the speed of the chain increases in time as the chain falls. It also varies due to the uneven collisions between the chain links and the scale. We will average this force over the uneven collisions in order to smooth out these small variations, and so the average external force on our system is $\vec{\mathbf{F}}_{c,s}^{ave}(t)$.

The y-component of the momentum of the system at time $t+\Delta t$ is at time is

$$p_y(t+\Delta t)=0\,. \qquad (5.3)$$

The y-component of the momentum of the system at time $t$ is at time is

$$p_y(t)=\Delta m v_y=(m/l)\Delta y\, v_y\,, \qquad (5.4)$$

The change in the y-component of the momentum is therefore

$$\Delta p_y=p_y(t+\Delta t)-p_y(t)=-(m/l)\Delta y\, v_y\,, \qquad (5.5)$$

Since the external force is equal to the change in momentum we have that

$$\vec{\mathbf{F}}_{c,s}^{ave}(t)=\lim_{\Delta t\to 0}\frac{\Delta p_y}{\Delta t}\hat{\mathbf{j}}=-(m/l)v_y\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\hat{\mathbf{j}}=-(m/l)v_y\frac{dy}{dt}\hat{\mathbf{j}}=-(m/l){v_y}^2\,\hat{\mathbf{j}}\,, \qquad (5.6)$$

.

where we have used the fact that $dy/dt=v_y$. Note that we have chosen downward as the positive y-direction so the force on the chain is upwards as we expect in order to stop the chain. Recall from the energy condition that ${v_y}^2=2gy$. So the magnitude of the external force is given by

$$\left|\vec{\mathbf{F}}_{c,s}^{ave}(t)\right|=2(m/l)gy\,. \qquad (5.7)$$

The scale reading is equal to the magnitude of this force plus the weight of the chain, $(m/l)yg$, that is already on the scale

$$\text{Scale reading} = \left|\vec{\mathbf{F}}_{c,s}^{ave}(t)\right| + (m/l)yg = 2(m/l)yg + (m/l)yg = 3(m/l)gy\,. \qquad (5.8)$$

Note that at time t, the amount of chain that is rested on the scale is

$$y(t) = \frac{1}{2}gt^2\,. \qquad (5.9)$$

So the scale reading as a function of time is given by

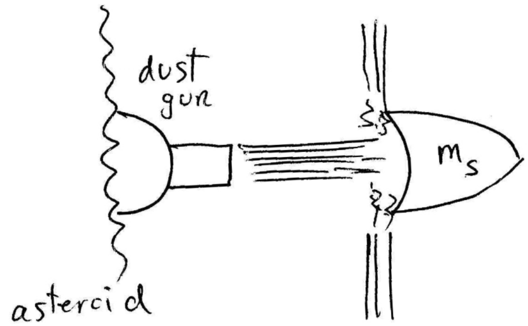
$$\text{Scale reading} = \frac{3}{2}(m/l)g^2t^2\,. \qquad (5.10)$$

We note that when the entire chain has just come to rest on the scale, $y = l$, the scale reads

$$\text{Scale reading} = 3mg\,. \qquad (5.11)$$

After the collision has ended the scale reading drops to the weight of the chain. $\text{Scale reading} = mg$.

**Problem 6** A spacecraft is launched from an asteroid by being bombarded by a stream of rock dust. The stream of dust is ejected from the dust gun at a constant rate $dm_e / dt = b$ at a speed $u$ with respect to the asteroid, which we take to be stationary. Assume that the dust comes momentarily to rest at the spacecraft and then slips away sideways; the effect is to keep the spacecraft's mass $m_s$ constant.

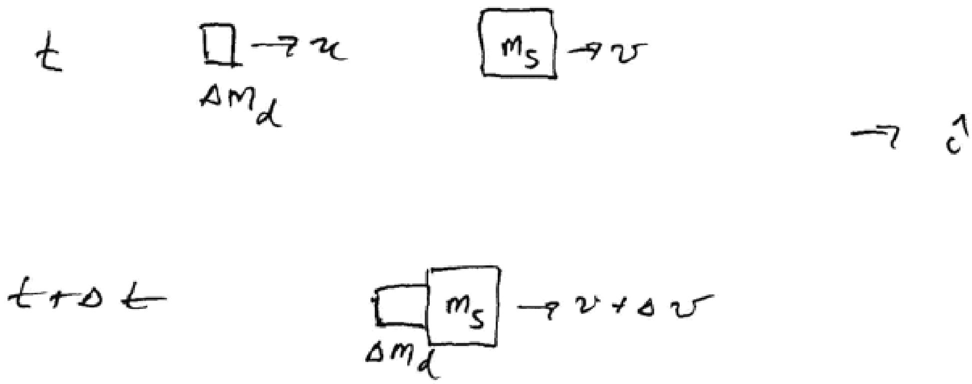



a) Derive an equation for the acceleration $dv_s / dt$ of the spacecraft at time $t$, in terms of the rate that the dust mass hits the surface of the spacecraft $dm_d / dt$, the speed of the dust relative to the asteroid $u$, the mass of the spacecraft $m_s$, and the velocity of the spacecraft $v_s$. Show your momentum flow diagrams at time $t$ and time $t + \Delta t$. Clearly identify your system and label all the objects in your system. What is the terminal velocity of the spacecraft? Hint: $dm_d / dt \neq b$.

b) Using conservation of mass, at time $t$, find an expression for the rate that the dust mass hits the spacecraft, $dm_d / dt$, as a function of the speed of the spacecraft $v_s$, the rate that the dust mass is shot from the asteroid $dm_e / dt = b$, and the speed $u$ of the dust relative to the asteroid. Hint: $dm_d / dt \neq b$.

c) Use your results from part b) in part a) to find the speed $v_s(t)$ of the spacecraft as a function of time, assuming $v_s(t = 0) = 0$. (If you get an integral that you are not sure how to integrate, you can leave your answer in integral form.)

**Solution:** We choose as our system, the small amount of dust $\Delta m_d$ that hits the spacecraft during the interval $[t, t + \Delta t]$ and the spacecraft of mass $m_s$. Because we assumed that the dust comes momentarily to rest at the spacecraft and then slips away sideways; at the end of the interval the dust has the same speed as the spacecraft and the mass $m_s$ of the spacecraft remains constant. We show the momentum diagrams for the system at times $t$ and $t + \Delta t$ below.




Because we are ignoring the gravitational force (it is very small), the x-component of the momentum of the system is constant during the interval $[t, t+\Delta t]$ and so

$$0 = \lim_{\Delta t \to 0} \frac{p_x(t+\Delta t) - p_x(t)}{\Delta t}\,. \tag{6.1}$$

Using the information from the figure above, Eq. (4.5) becomes

$$0 = \lim_{\Delta t \to 0} \frac{(m_s + \Delta m_d)(v_s + \Delta v_s) - (\Delta m_d u + m_s v_s)}{\Delta t}\,. \tag{6.2}$$

Eq. (4.6) simplifies to

$$0 = \lim_{\Delta t \to 0} m_s \frac{\Delta v_s}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_d}{\Delta t} v_s + \lim_{\Delta t \to 0} \frac{\Delta m_d \Delta v_s}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta m_d}{\Delta t} u\,. \tag{6.3}$$

The third term vanishes when we take the $\Delta t \to 0$ because it is of second order in the infinitesimal quantities (in this case $\Delta m_d \Delta v_s$) and so when dividing by $\Delta t$ the quantity is of first order and hence vanishes since both $\Delta m_d \to 0$ and $\Delta v_s \to 0$. So Eq. (4.7) becomes

$$0 = \lim_{\Delta t \to 0} m_s \frac{\Delta v_s}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_d}{\Delta t} v_s - \lim_{\Delta t \to 0} \frac{\Delta m_d}{\Delta t} u\,. \tag{6.4}$$

We now use the definition of the derivatives:

$$\lim_{\Delta t \to 0} \frac{\Delta v_s}{\Delta t} = \frac{dv_s}{dt}; \;\; \lim_{\Delta t \to 0} \frac{\Delta m_d}{\Delta t} = \frac{dm_d}{dt}\,. \tag{6.5}$$

in Eq. (4.8) to fund the differential equation describing the relation between the acceleration of the spacecraft and the time rate of change of the mass of dust striking the spacecraft

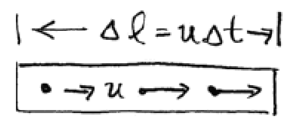
$$0 = m_s \frac{dv_s}{dt} + \frac{dm_d}{dt}(v_s - u) . \quad (6.6)$$

In the limit as $t \to \infty$, $dv_s / dt \to 0$, hence $v_s(\infty) = u$ the spacecraft can go no faster than the speed of the dust.

In order to solve the above differential equation, we begin by observing that

$$\frac{dm_d}{dt} \neq \frac{dm_e}{dt} = b \quad (6.7)$$

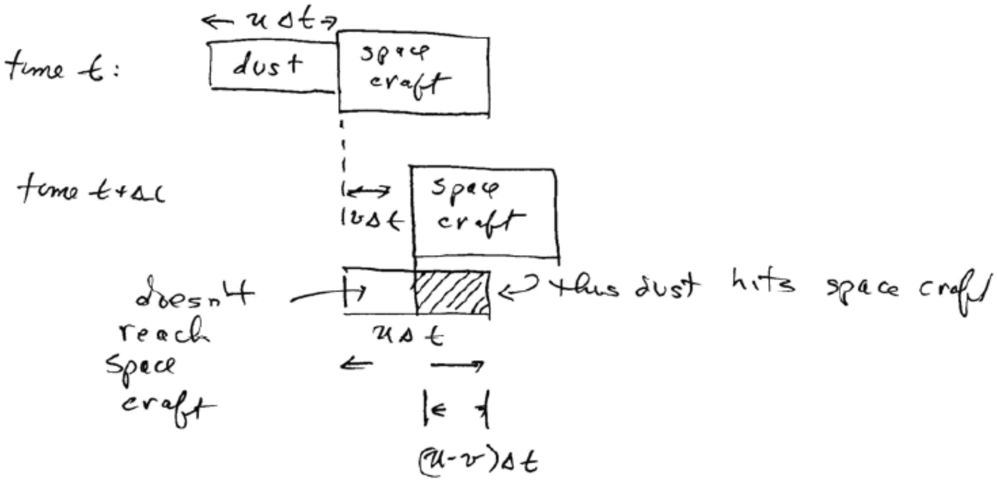
The reason is that $b$ is the rate that the dust is ejected from the asteroid. In the figure below a column of ejected dust of length $\Delta l = u\Delta t$ has mass $\Delta m_{ejected} = \lambda u \Delta t$ where $\lambda$ is the mass per unit length and is assumed to be constant.




The rate that the mass is ejected from the asteroid is then

$$b = \frac{dm_e}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m_e}{\Delta t} = \lambda u . \quad (6.8)$$

In the figure below we consider a column of dust of length $u\Delta t$ that is just behind a perpendicular surface of the spacecraft at time $t$.




During the time interval $[t, t+\Delta t]$, the spacecraft is moving a distance $v\Delta t$ so the entire column of dust does not hit the spacecraft. Only a fraction of the column hits the spacecraft with the mass of the dust that strikes the spacecraft given by

$$\Delta m_d = \lambda(u - v_s)\Delta t = \frac{b}{u}(u - v_s)\Delta t \tag{6.9}$$

where we used $\lambda = b/u$. Dividing the equation through by $\Delta t$ and taking limits we have that

$$\frac{dm_d}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m_d}{\Delta t} = \frac{b}{u}(u - v_s)\,. \tag{6.10}$$

Substituting into Eq. (4.10) and after some rearranging yields

$$m_s \frac{dv_s}{dt} = \frac{b}{u}(v_s - u)^2\,. \tag{6.11}$$

We can integrate this equation by separating variables to find an integral expression for the mass of the spacecraft as a function of time

$$\int_{v_s'=0}^{v_s'=v_s(t)} \frac{dv_s'}{(v_s' - u)^2} = \frac{b}{um_s} \int_{t'=0}^{t'=t} dt'\,. \tag{6.12}$$

We can easily integrate both sides of the equation, yielding

$$-\frac{1}{(v_s' - u)}\Bigg|_{v_s'=0}^{v_s'=v_s(t)} = \frac{b}{um_s} t\,. \tag{6.13}$$

Evaluating the endpoints of the integral yields

$$-\frac{1}{u}-\frac{1}{(v_s(t)-u)}=\frac{b}{um_s}t\,. \qquad (6.14)$$

A little algebraic rearrangement yields

$$-\frac{u}{(v_s(t)-u)}=\frac{m_s+bt}{m_s}\,. \qquad (6.15)$$

Inverting yields

$$v_s(t)-u=-u\frac{m_s}{m_s+bt}\,. \qquad (6.16)$$

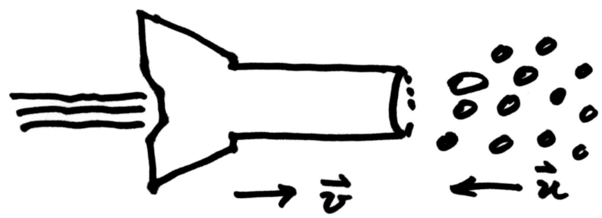
So the speed of the spacecraft as a function of time is then

$$v_s(t)=\frac{ubt}{m_s+bt}\,. \qquad (6.17)$$

Note that in the limit as $t\to\infty$, $v_s(\infty)=u$ in agreement with what expect, that the spacecraft can go no faster than the speed of the dust

**Problem 7 Space Junk**

A spacecraft of cross-sectional area $A$, proceeding along the positive $x$-direction, enters an asteroid storm at time $t = 0$, in which the mean mass density of the asteroid storm is $\rho$ and the average asteroid velocity is $\vec{\mathbf{u}} = -u\,\hat{\mathbf{i}}$ in the negative $x$-direction. As the spacecraft proceeds through the storm, all of the asteroids that hit the spacecraft stick to it.



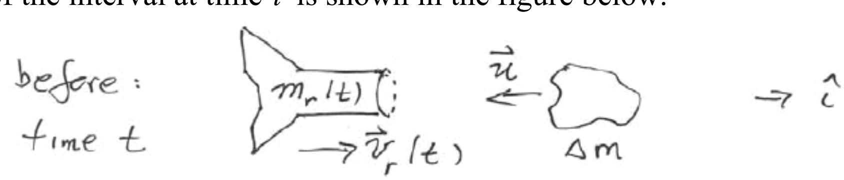
a) Suppose that at time $t$ the velocity of the spacecraft is $\vec{\mathbf{v}} = v\,\hat{\mathbf{i}}$ in the positive x-direction, and its mass is $m$. Further, suppose that in an interval $\Delta t$, the mass of the spacecraft increases by an amount $\Delta m$. Given that there are no external forces, using conservation of momentum find an equation for the change of the spacecraft velocity $\Delta v$, in terms of $\Delta m$, $u$, and $v$?

b) When the spacecraft enters the asteroid storm, the magnitude of its velocity and mass are $v_0$ and $m_0$, respectively. Integrate your differential equation in part a) to find the velocity $v$ of the spacecraft when the mass is $m$.

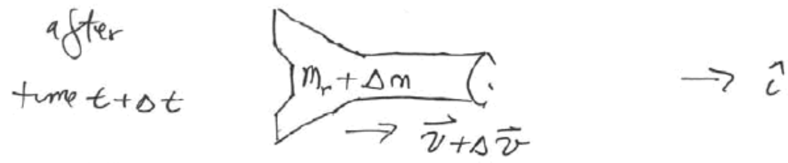
c) Find an expression for the mass of the asteroids $\Delta m$ that sticks to the spacecraft within the time interval $\Delta t$? (Hint: consider the volume of asteroids swept up by the spacecraft in time $\Delta t$).

d) When the spacecraft enters the asteroid storm, the magnitude of its velocity and mass are $v_0$ and $m_0$, respectively. What is the mass of the spacecraft at time $t$? (Use your results from parts c) and b).)

**Solution:** Let's choose as our system a small element of the asteroid cloud $\Delta m$ that is absorbed by the rocket during the interval $[t, t+\Delta t]$ and the rocket itself. Choose positive x-direction as the direction the rocket is moving. The momentum diagram at the beginning of the interval at time $t$ is shown in the figure below.




The momentum diagram at time $t + \Delta t$ is also shown below.




Because we are assuming that there are no external forces acting on our system,

$$0 = \lim_{\Delta t \to 0} \frac{p_x(t+\Delta t) - p_x(t)}{\Delta t}. \tag{7.1}$$

Using the information from the figure above, Eq. (7.1) becomes

$$0 = \lim_{\Delta t \to 0} \frac{(m+\Delta m)(v+\Delta v) - (-\Delta m u + m v)}{\Delta t}. \tag{7.2}$$

Eq. (7.2) simplifies to

$$0 = \lim_{\Delta t \to 0} m\frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} v + \lim_{\Delta t \to 0} \frac{\Delta m \Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} u . \tag{7.3}$$

The third term vanishes when we take the $\Delta t \to 0$ because it is of second order in the infinitesimal quantities (in this case $\Delta m \Delta v$) and so when dividing by $\Delta t$ the quantity is of first order and hence vanishes since both $\Delta m \to 0$ and $\Delta v \to 0$. So Eq. (7.3) becomes

$$0 = \lim_{\Delta t \to 0} m\frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} v + \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} u . \tag{7.4}$$

We now use the definition of the derivatives:

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}; \ \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \frac{dm}{dt}. \tag{7.5}$$

in Eq. (7.5) to find the differential equation describing the relation between the acceleration of the rocket and the time rate of change of the mass of the rocket

$$0 = m\frac{dv}{dt} + \frac{dm}{dt}(v+u). \tag{7.6}$$

b) We can integrate this equation through the separation of variable technique. Rewrite Eq. (7.6) as (cancel the common factor $dt$)

$$-\frac{dv}{u+v} = \frac{dm}{m}. \tag{7.7}$$

We can then integrate both sides of Eq. (7.7) with the limits as shown

$$-\int_{v'=v_0}^{v'=v(m)} \frac{dv'}{u+v'} = \int_{m'=m_0}^{m'=m} \frac{dm'}{m'} \tag{7.8}$$

Integration yields

$$-\ln\left(\frac{u+v(m)}{u+v_0}\right) = \ln\left(\frac{m}{m_0}\right) \tag{7.9}$$

Recall that $\ln(a/b) = -\ln(b/a)$ so Eq. (7.9) becomes

$$\ln\left(\frac{u+v_0}{u+v}\right) = \ln\left(\frac{m}{m_0}\right) \tag{7.10}$$

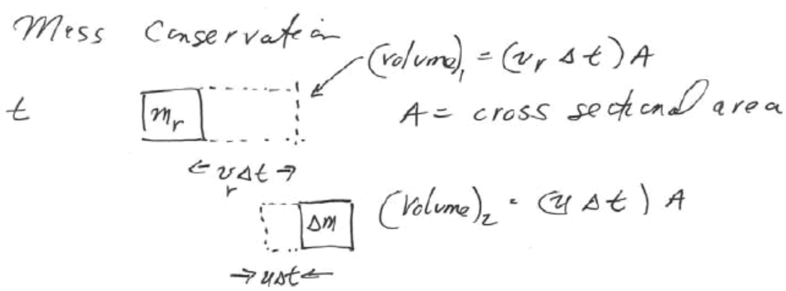
Also recall that $\exp(\ln(a/b)) = a/b$ and so exponentiating both sides of Eq. (7.10) yields

$$\frac{u+v_0}{u+v(m)} = \frac{m}{m_0} \tag{7.11}$$

After some rearranging, the speed $v(m)$ of the rocket as a function of $m$ can be expressed as

$$\begin{aligned} m(u+v(m)) &= m_0(u+v_0) \\ u+v(m) &= \frac{m_0}{m}(u+v_0) \\ v(m) &= \frac{m_0}{m}(u+v_0) - u \end{aligned} \tag{7.12}$$

In the above expression for the speed $v(m)$ of the rocket is a function of the mass of the rocket. We can determine how the mass of the rocket behaves as a function of time by considering mass conservation. In the interval $[t, t+\Delta t]$ the rocket sweeps out a tube of length $\Delta l = v\Delta t$. All the asteroids inside this tube are collected by the rocket adding an amount of mass to the rocket $\Delta m_1 = \rho A v \Delta t$. In addition, because the asteroid are moving towards the rocket an additional amount of mass $\Delta m_2 = \rho A u \Delta t$ also is collected by the rocket (see figure below).



Therefore the total amount of mass collected by the rocket during the interval is

$$\Delta m = \Delta m_1 + \Delta m_2 = \rho A v \Delta t + \rho A u \Delta t = \rho A (v + u)\Delta t \, . \qquad (7.13)$$

Dividing Eq. through by $\Delta t$ and taking limits we have that

$$\frac{dm}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \rho A (v + u) \, . \qquad (7.14)$$

We now substitute the second line in Eq. (7.12) into Eq. (7.14) and find that

$$\frac{dm}{dt} = \rho A \frac{m_0}{m}(u + v_0) \, . \qquad (7.15)$$

We can integrate this equation by separating variables to find an integral expression for the mass of the rocket as a function of time

$$\int_{m'=m_0}^{m'=m(t)} m' dm'_b = \int_{t'=0}^{t'=t} \rho A m_0 (u + v_0) dt' \, . \qquad (7.16)$$

We can easily integrate both sides of the equation yielding

$$\frac{1}{2}(m(t)^2 - m_0{}^2) = \rho A m_0 (u + v_0) t \, . \qquad (7.17)$$

So the mass of the rocket as a function of time is then

$$m(t) = m_0 \sqrt{1 + \frac{2\rho A (u + v_0) t}{m_0}} \, . \qquad (7.18)$$

We now substitute Eq. (7.18) into the third line of Eq. (7.12) yielding the speed of the rocket as a function of time

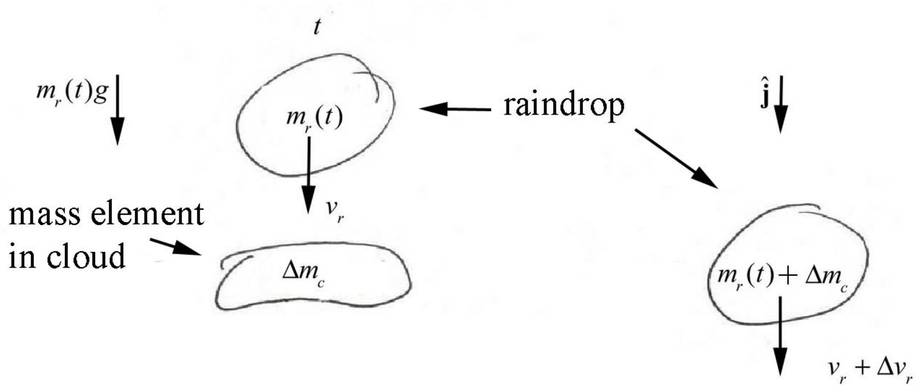
$$v(t) = \frac{u + v_0}{\sqrt{1 + \frac{2\rho A (u + v_0) t}{m_0}}} - u \, . \qquad (7.19)$$

**Problem 8 Continuous Mass Transport: falling raindrop** A raindrop of initial mass $m_0$ starts falling from rest under the influence of gravity. Assume that the raindrop gains mass from the cloud at a rate proportional to the momentum of the raindrop, $dm_r / dt = km_r v_r$, where $m_r$ is the instantaneous mass of the raindrop, $v_r$ is the instantaneous velocity of the raindrop, and $k$ is a constant with units $[\text{m}^{-1}]$. You may neglect air resistance.

a) Derive a differential equation for the velocity of the raindrop.

b) Show that the speed of the drop eventually becomes effectively constant and give an expression for the terminal speed.

**Solution:** At time $t$ choose the raindrop with mass $m_r(t)$ and a small mass element $\Delta m_c$ in the cloud that will be added to the raindrop during the time interval $\Delta t$. Choose the positive $y$ direction downward. Then the momentum flow diagram is shown in the figure below.




At time $t$, the small mass element $\Delta m_c$ is at rest in the cloud at time $t$. The raindrop is moving downward with velocity

$$\vec{\mathbf{v}}_r(t) = v_r\ \hat{\mathbf{j}}. \tag{8.1}$$

So the momentum of the drop is equal to the momentum of the system

$$\vec{\mathbf{p}}_{\text{sys}}(t) = m_r v_r\ \hat{\mathbf{j}} \tag{8.2}$$

At time $t + \Delta t$, the raindrop has added some mass $\Delta m_c$ from the cloud and is moving downward with velocity

$$\vec{\mathbf{v}}_r(t + \Delta t) = (v_r + \Delta v_r)\ \hat{\mathbf{j}}. \tag{8.3}$$

where $\Delta v_r$ is the infinitesimal change in the *y*-component of the velocity of the raindrop. So the momentum of the drop is

$$\vec{\mathbf{p}}_{\text{sys}}(t+\Delta t) = (m_r + \Delta m_c)\,(v_r + \Delta v_r)\,\hat{\mathbf{j}}\,. \tag{8.4}$$

During this interval there is an external gravitational force acting on the system and a buoyant force in the cloud that is keeping the mass element $\Delta m_c$ from falling. Since the total force on the mass element $\Delta m_c$ is zero, the total external force acting on the system is just the gravitational force acting on the raindrop,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = m_r g\,\hat{\mathbf{j}}\,. \tag{8.5}$$

Newton's Second Law and Third Law for the system state that he total external force is equal to the derivative of the momentum

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt}\,. \tag{8.6}$$

Recall the definition of the derivative of the system momentum,

$$\frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{\text{sys}}(t+\Delta t) - \vec{\mathbf{p}}_{\text{sys}}(t)}{\Delta t}\,. \tag{8.7}$$

Substitute Eq.(8.4), Eq.(8.2) , and Eq.(8.5) into Eq.(8.6) yields

$$m_r g\,\hat{\mathbf{j}} = \lim_{\Delta t \to 0} \frac{(m_r + \Delta m_c)\,(v_r + \Delta v_r)\,\hat{\mathbf{j}} - m_r v_r\,\hat{\mathbf{j}}}{\Delta t}\,. \tag{8.8}$$

Expanding the numerator in Eq.(8.8),

$$m_r g\,\hat{\mathbf{j}} = \lim_{\Delta t \to 0} \frac{(m_r \Delta v_r + \Delta m_c v_r + \Delta m_c \Delta v_r)\,\hat{\mathbf{j}}}{\Delta t}\,. \tag{8.9}$$

The third term in the numerator on the right hand side of Eq.(8.9) is a second oreder term in infinitesimals and hence in the limit as $\Delta t \to 0$ vanishes,

$$\lim_{\Delta t \to 0} \frac{\Delta m_c \Delta v_r\,\hat{\mathbf{j}}}{\Delta t} \to 0\,. \tag{8.10}$$

Thus Eq.(8.9) becomes

$$m_r g\,\hat{\mathbf{j}} = m_r \lim_{\Delta t \to 0} \frac{\Delta v_r}{\Delta t}\hat{\mathbf{j}} + v_r \lim_{\Delta t \to 0} \frac{\Delta m_c}{\Delta t}\hat{\mathbf{j}} = m_r \frac{dv_r}{dt}\hat{\mathbf{j}} + v_r \frac{dm_r}{dt}\hat{\mathbf{j}} \qquad (8.11)$$

The $y$-components of Eq.(8.11) are equal,

$$m_r g = m_r \frac{dv_r}{dt} + v_r \frac{dm_r}{dt} \qquad (8.12)$$

From the statement of the problem, we have modeled the rate that the rate that raindrop adds mass is proportional to the momentum of the raindrop

$$\frac{dm_r}{dt} = k m_r v_r . \qquad (8.13)$$

Thus Eq.(8.12) becomes

$$m_r g = m_r \frac{dv_r}{dt} + k m_r {v_r}^2 . \qquad (8.14)$$

Divide out the mass of the raindrop from Eq.(8.14) giving

$$g = \frac{dv_r}{dt} + k {v_r}^2 \qquad (8.15)$$

Thus the acceleration of the raindrop is thus non-uniform and given by

$$\frac{dv_r}{dt} = g - k {v_r}^2 . \qquad (8.16)$$

The raindrop will reach a terminal velocity when the acceleration is zero,

$$0 = g - k {v_{r,\mathrm{ter}}}^2 . \qquad (8.17)$$

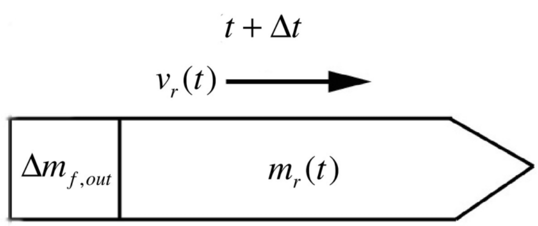
So the terminal velocity is

$$v_{r,\mathrm{ter}} = \sqrt{\frac{g}{k}} . \qquad (8.18)$$

**Problem 9: Rocket Problem** A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed $u$ relative to the rocket. Assume that the rate at which mass is expelled is given by $dm_f / dt = \gamma m_r$, where $m_r$ is the instantaneous mass of the rocket and $\gamma$ is a constant. The rocket is retarded by air resistance with a force $F = bm_r v_r$ proportional to the instantaneous momentum of the rocket where $b$ is a constant and $v_r$ is the speed of the rocket. Find the speed of the rocket as a function of time.

**Solution:**

To find the rocket's velocity as a function of time, we first need to find how the velocity changes with respect to time, the differential equation mentioned in the problem statement. Take the positive y-direction to be upward and assume that the rocket velocity is upward as well. For purposes of space and clarity, the positive upward direction is shown as being to the right in the figures. The gravitational force and air resistance force are then in the negative direction, to the left in the figures.




As suggested by the figure above, take the system to be the rocket and fuel combination. The smaller square represents the small amount of fuel of mass $\Delta m_{f,out}$ that is ejected during the interval $[t, t+\Delta t]$. In the above figure

$$m_r(t) = m_{r,0} + m_{f,in}(t) \tag{9.1}$$

is the combined mass of the rocket where $m_{r,0}$ is the dry mass of the rocket and $m_{f,in}(t)$ is the mass of fuel inside the rocket at time $t$ that does not leave the rocket during the interval. Note that differentiating the above equation yields

$$\frac{dm_r(t)}{dt} = \frac{dm_{f,in}(t)}{dt}. \tag{9.2}$$

Denote by $m_{f,0}$ the fuel in the rocket at $t = 0$. Then

$$m_{f,0} = m_{f,in}(t) + m_{f,out}(t) \tag{9.3}$$

where $m_{f,out}(t)$ is the mass of the fuel that has been ejected during the interval $[0,t]$. Since $m_{f,0}$ is constant, we can differentiate the above equation yielding

$$0 = \frac{dm_{f,0}}{dt} = \frac{dm_{f,in}(t)}{dt} + \frac{dm_{f,out}(t)}{dt}. \tag{9.4}$$

Thus combining equations Eq. (9.2) and Eq. (9.4) yields

$$\frac{dm_{f,out}(t)}{dt} = -\frac{dm_r(t)}{dt}. \tag{9.5}$$

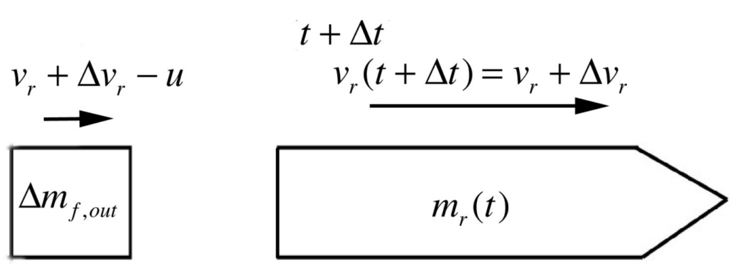
From the statement of the problem, the burn rate of fuel is then equal to the negative of the rate that the mass of the rocket is decreasing and from the statement of the problem is given by

$$\gamma m_r(t) = \frac{dm_{f,out}(t)}{dt} = -\frac{dm_r(t)}{dt}. \tag{9.6}$$

Returning to our momentum diagram we see that at time $t$, the y-component of the momentum of the system $p_y^{sys}(t)$ is given by

$$p_y^{sys}(t) = (m_r(t) + \Delta m_{f,out})v_r \tag{9.7}$$

During the interval the fuel is ejected backwards relative to the rocket with speed $u$. After the interval has ended the momentum diagram of the system is shown below.




At time $t + \Delta t$, the y-component of the momentum of the system $p_y^{sys}(t + \Delta t)$ is given by

$$p_y^{sys}(t + \Delta t) = m_r(t)(v_r + \Delta v_r) + \Delta m_{f,out}(v_r + \Delta v_r - u) \tag{9.8}$$

and so the change in y-component of the momentum of the system during the interval $[t, t+\Delta t]$ is

$$\begin{aligned}\Delta p_y^{sys} &= p_y^{sys}(t+\Delta t) - p_y^{sys}(t) \\ &= m_r(t)(v_r + \Delta v_r) + \Delta m_{f,out}(v_r + \Delta v_r - u) - (m_r(t) + \Delta m_{f,out})v_r \; . \\ &= m_r(t)\Delta v_r + \Delta m_{f,out}\Delta v_r - \Delta m_{f,out} u\end{aligned} \tag{9.9}$$

The external force acting on the system is given by

$$F_y^{ext} = -m_r(t)g - bm_r(t)v_r \; . \tag{9.10}$$

The momentum principle

$$F_y^{ext} = \lim_{\Delta t \to 0} \frac{\Delta p_y^{sys}}{\Delta t} \tag{9.11}$$

then becomes

$$-m_r(t)g - bm_r(t)v_r = \lim_{\Delta t \to 0} \frac{m_r(t)\Delta v_r + \Delta m_{f,out}\Delta v_r - \Delta m_{f,out} u}{\Delta t} \; . \tag{9.12}$$

We note that

$$\lim_{\Delta t \to 0} \frac{\Delta m_{f,out}\Delta v_r}{\Delta t} = 0 \; . \tag{9.13}$$

Then using the definition of the derivative, we find that the differential equation describing the motion of the system is given by

$$-m_r(t)g - bm_r(t)v_r = m_r(t)\frac{dv_r}{dt} - \frac{d\,m_{f,out}}{dt}u \tag{9.14}$$

We can now substitute Eq. (9.6) into the above equation and find that

$$-m_r(t)g - bm_r(t)v_r = m_r(t)\frac{dv_r}{dt} - \gamma m_r(t)u \; . \tag{9.15}$$

We can divide through by the mass of the rocket in the above equation yielding

$$\frac{dv_r}{dt} = \gamma u - g - bv_r \tag{9.16}$$

**On to the Differential Equation:**

We've done the physics, but there are several interesting aspects to the calculation of the velocity as a function of time. Our differential equation is separable and so

Rewrite Eq. (9.16) as

$$\frac{dv_r}{\gamma u - g - bv_r dt} = dt\ . \tag{9.17}$$

We can integrate both sides

$$\int_{v_r'=0}^{v_r'=v_r(t)} \frac{dv_r'}{\gamma u - g - bv_r'} = \int_{t'=0}^{t'=t} dt'\ . \tag{9.18}$$

Integration yields

$$-\frac{1}{b}\ln\left(\frac{\gamma u - g - bv_r(t)}{\gamma u - g}\right) = t\ . \tag{9.19}$$

After a slight rearrangement, this expression can be exponentiated yielding

$$\frac{\gamma u - g - bv_r(t)}{\gamma u - g} = e^{-bt}\ . \tag{9.20}$$

After some rearrangement we find that the speed of the rocket is given by

$$v_r(t) = \frac{1}{b}(\gamma u - g)(1 - e^{-bt})\ . \tag{9.21}$$

There are several aspects of the result in Eq. (9.21) that are worth considering. First, unless $\gamma u > g$, the rocket doesn't get off the ground. Next, the rocket continues to accelerate, only reaching terminal speed $v_{\text{terminal}} = (\gamma u - g)/b$ in the limit $t \to \infty$; the air resistance force seems to have wimped out.

The key point in considering the result of Eq. (9.21) is in the model for the rate at which the fuel is exhausted

$$\frac{dm_{f,out}}{dt} = -\frac{dm_r}{dt} = \gamma\, m_r(t) \ , \tag{9.22}$$

Solving Equation (9.22) would yield that the mass of the rocket and remaining fuel is

$$m_r(t) = m_0\, e^{-\gamma t} \tag{9.23}$$

where $m_0 = m_{r,0} + m_{f,0}$ The mass being accelerated decreases exponentially, the gravitation force and the air resistance force, being proportional to the mass, decrease as well, and so the rocket can continue to accelerate indefinitely.

The product $u(dm_{f,out}\,/\,dt)$ is sometimes called the "thrust" (ask your Course XVI friends); check to see that the thrust has dimensions of force. In the model for this problem, we would have that the "thrust force" is

$$F_{\text{thrust}} = u\frac{dm_{f,out}}{dt} = \gamma\, u\, m_r\,, \tag{9.24}$$

an unlikely feature of realistic rocket design. At best, even if Equation (9.24) were an approximate model, the thrust would have to vanish when the fuel runs out.

The bottom line is that the model for the fuel burn rate was given in the form it was in order to make solving for the velocity as a function of time possible. Maybe a neat math problem, but the physics was done once we found the rate of change of momentum.