Collision Theory
Challenge Problems Solutions

Problem 1 Estimate the energy loss in a completely inelastic collision between two identical cars that collide head-on traveling at highway speeds.

Solution: Consider small cars going fast, $m = 1000\,\text{kg}$ and $v = 30\,\text{m}\cdot\text{s}^{-1}$. The energy lost is the sum of the initial kinetic energies,

$$2\frac{1}{2}mv^2 = (1000\,\text{kg})(30\,\text{m}\cdot\text{s}^{-1})^2 \approx 10^6\,\text{J} \approx 1\text{MJ}\,. \tag{1.1}$$

Problem 2 You just witnessed a karate master breaking a brick with his hand. Estimate the impulse necessary to break the brick. Estimate an upper limit on the force per area that the hand can safely endure without breaking the hand.

Solution

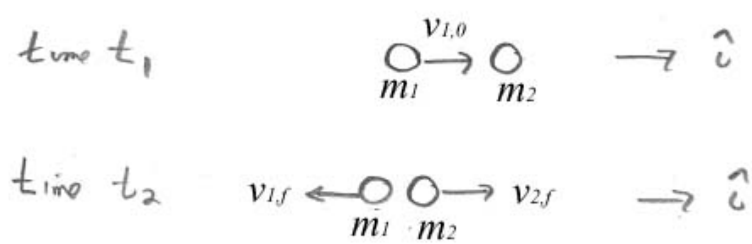
The motion is of course quite complicated. Taking the mass of the hand (should be the hand and some part of the forearm, at least) as $1\,\text{kg}$ and the initial speed of the hand to be $10\,\text{m}\cdot\text{s}^{-1}$, and assuming that all of the momentum of the hand is transferred to the brick (probably not the case), the impulse that breaks the brick is about $10\,\text{kg}\cdot\text{m}\cdot\text{s}^{-1}$. If this impulse, the change in the momentum of the hand, delivered over a time of roughly $1\,\text{ms} = 10^{-3}\,\text{s}$, represents the largest force that the hand can withstand, that force is roughly $10^{4}\,\text{kg}\cdot\text{m}\cdot\text{s}^{-2} = 10^{4}\,\text{N}$. Using an area of $10\,\text{cm}^{2} = 10^{-3}\,\text{m}^{2}$ gives a force per area of $10^{7}\,\text{N}\cdot\text{m}^{-2}$. (The SI unit for force per area is the pascal, $1\,\text{Pa} = 1\,\text{N}\cdot\text{m}^{-2}$.

Problem 3 Suppose a golf ball is hurled at a heavy bowling ball initially at rest and bounces elastically from the bowling ball. After the collision, which ball has the greater momentum? Which has the greater kinetic energy?

Solution:

We shall first begin with an argument that is based on assuming that the bowling ball acts like a wall. The golf ball bounces back at nearly its incident speed, whereas the bowling ball hardly budges. Thus the *change* in momentum of the golf ball is nearly $-2mv$, and the bowling ball must gain momentum $+2mv$ to conserve momentum. However, since the mass of the bowling ball is much larger than that of the golf ball, the bowling ball's speed and hence its kinetic energy are much smaller than those of the golf ball. Note that $K = p^2 / 2m$ so if the momentum are equal the object that has the larger mass has the smaller kinetic energy. Thus the kinetic energy of the bowling ball is smaller than the golf ball .

The golf ball has mass m_1 and the bowling ball has mass m_2. The initial x-component of the velocity of the golf ball is $v_{x1,0}$. The bowling ball is initially at rest ball, $v_{x2,0} = 0$. Draw a momentum diagram.



We can use our results from Course Notes Section 18.4, Eqs. 18.4.12 and 18.4.13:

$$v_{x1,f} = v_{x1,0} \frac{m_1 - m_2}{m_1 + m_2} \tag{3.1}$$

and a similar calculation yields

$$v_{x2,f} = v_{x1,0} \frac{2\,m_1}{m_2 + m_1}. \tag{3.2}$$

The final kinetic energy of the golf ball is

$$K_{1,f} = \frac{1}{2} m_1 v_{x1,f}{}^2 = \frac{1}{2} \frac{m_1}{(m_1 + m_2)^2} v_{x1,0}{}^2 (m_1 - m_2)^2 \quad (3.3)$$

The final kinetic energy of the bowling ball is

$$K_{2,f} = \frac{1}{2} m_2 v_{x2,f}{}^2 = \frac{1}{2} \frac{m_1}{(m_1 + m_2)^2} v_{x1,0}{}^2 (2 m_1 m_2) \quad (3.4)$$

Because $m_2 >> m_1$, $(m_1 - m_2)^2 \simeq m_2{}^2 > 2 m_1 m_2$. Therefore

$$K_{1,f} > K_{2,f} \,. \quad (3.5)$$

Problem 4 One Dimensional Collision

A proton makes a head-on collision with an unknown particle at rest. The proton rebounds straight back with $4/9$ of its initial kinetic energy. Find the ratio of the mass of the unknown particle to the mass of the proton, assuming that the collision is elastic.

Solution:

We choose as our system the proton and the unknown particle. Since we know that the collision is elastic the mechanical energy is constant. We also assume that there are no external forces acting so the momentum is constant. The collision is one dimensional so the constancy of momentum and energy provide two equations. In the statement of the problem the following quantities are not known: the mass of the proton m_1, the mass of the unknown particle m_2, the initial speed of the proton $v_{1,0}$, and the final speed of the unknown particle, $v_{2,f}$. Since we are told that the final kinetic energy of the proton is

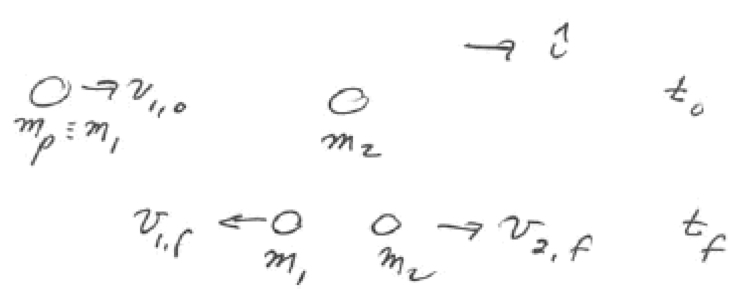
$$K_{1,f} = \frac{1}{2} m_1 v_{1,f}^2 = \frac{4}{9} K_{1,0} = \frac{4}{9}\left(\frac{1}{2} m_1 v_{1,i}^2\right), \tag{4.1}$$

we can determine the final speed of the proton

$$v_{1,f} = \frac{2}{3} v_{1,i} . \tag{4.2}$$

From the momentum condition we should be able to determine the final speed of the unknown particle, $v_{2,f}$ in terms of the initial speed of the proton $v_{1,0}$. Then the speeds of all the objects can be expressed in terms of the initial speed of the proton $v_{1,0}$. So the initial speed of the proton $v_{1,0}$ will cancel out in the energy equation which will then yield the desired ratio of the mass of the unknown particle to the mass of the proton, m_2 / m_1.

In the figure below we depict the states of the system before and after the collision.



The x-component of the momentum is constant so

$$m_1 v_{1,0} = -m_1 v_{1,f} + m_2 v_{2,f}\,. \tag{4.3}$$

Now substitute Eq. (4.2) into Eq. (4.3) yielding

$$m_1 v_{1,0} = -\frac{2}{3} m_1 v_{1,0} + m_2 v_{2,f}\,. \tag{4.4}$$

We can solve Eq. (4.4) for the final speed of the unknown particle in terms of the masses and the initial speed of the proton,

$$v_{1,f} = \frac{5m_1}{3m_2} v_{1,0}\,. \tag{4.5}$$

The kinetic energy is constant in the collision,

$$\frac{1}{2} m_1 v_{1,0}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2\,. \tag{4.6}$$

We first substitute Eq. (4.1) for the final kinetic energy of the proton into Eq. (4.6) yielding

$$\frac{1}{2} m_1 v_{1,0}^2 = \frac{4}{9}\frac{1}{2} m_1 v_{1,0}^2 + \frac{1}{2} m_2 v_{2,f}^2\,. \tag{4.7}$$

Eq. (4.7) simplifies to

$$\frac{5}{9} m_1 v_{1,0}^2 = m_2 v_{2,f}^2 \tag{4.8}$$

Now substitute our expression for the final speed of the unknown particle (Eq. (4.5)) into Eq. (4.8) yielding

$$\frac{5}{9} m_1 v_{1,0}^2 = m_2 \left(\frac{5m_1}{3m_2} v_{1,0} \right)^2 = \frac{25}{9} \frac{m_1^{\;2}}{m_2} v_{1,0}^2\,. \tag{4.9}$$

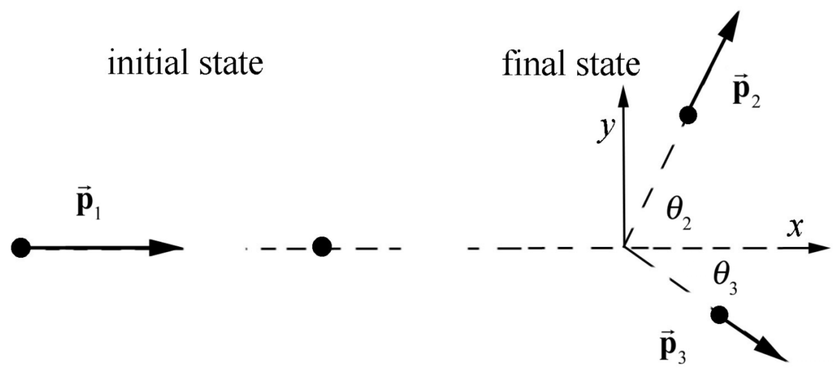
We can solve Eq. (4.9) for the ratio of the mass of the unknown particle and the proton

$$\frac{m_2}{m_1} = 5\,. \tag{4.10}$$

(Note: There are no known stable nuclei with mass equal to five time the proton mass.)

Problem 5 Two-Dimensional Collision

An object with momentum $\vec{\mathbf{p}}_1$ collides with a stationary particle. After the collision, two particles emerge, one with momentum $\vec{\mathbf{p}}_2$ and the other with momentum $\vec{\mathbf{p}}_3$. The direction of the vector $\vec{\mathbf{p}}_2$ makes an angle θ_2 with respect to the direction of the vector $\vec{\mathbf{p}}_1$ and the direction of the vector $\vec{\mathbf{p}}_3$ makes an angle θ_3 with respect to the direction of the vector $\vec{\mathbf{p}}_1$. In terms of $p_1 = |\vec{\mathbf{p}}_1|$, θ_2 and θ_3, what are the magnitudes $p_2 = |\vec{\mathbf{p}}_2|$ and $p_3 = |\vec{\mathbf{p}}_3|$?



Solutions:

In this problem, we are given two directions and are asked to find two magnitudes in terms of the given angles and the magnitude of the initial momentum; two unknowns, two relations from equating initial and final momentum vectors.

One "straightforward" method and one "tricky" method will be presented.

Take the x-direction to be the direction of the initial momentum $\vec{\mathbf{p}}_1$. Then, equating the components of momentum before and after the collision,

$$\begin{aligned} p_1 &= p_2 \cos\theta_2 + p_3 \cos\theta_3 \\ 0 &= p_2 \sin\theta_2 - p_3 \sin\theta_3. \end{aligned} \qquad (5.1)$$

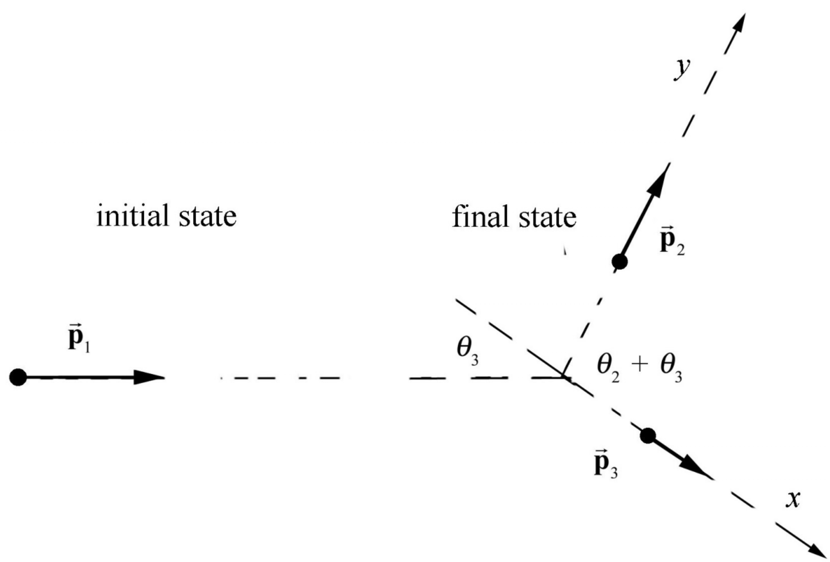
The relations in (5.1) are two equations in the two unknowns p_2 and p_3, and may be solved in a variety of ways. The method presented here is equivalent to Cramer's Rule. Multiply the first expression in (5.1) by $\sin\theta_3$ and the second by $\cos\theta_3$ and add, to cancel the terms in p_3, yielding

$$
\begin{aligned}
p_1 \sin\theta_3 &= p_2(\cos\theta_2 \sin\theta_3 + \sin\theta_2 \cos\theta_3) \\
&= p_2 \sin(\theta_2 + \theta_3) \qquad (5.2) \\
p_2 &= p_1 \frac{\sin\theta_3}{\sin(\theta_2 + \theta_3)}
\end{aligned}
$$

with a similar result for p_3.

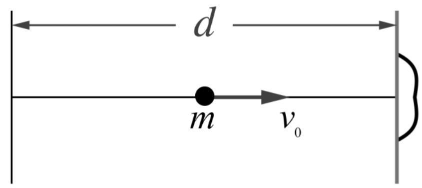
Alternate Solution:

The "trick" is to use coordinate systems with one axis parallel to the direction of one of the final momenta, one of which is shown. Setting the x-direction to be along the direction of $\vec{\mathbf{p}}_3$ and considering y-components yields the result of (5.2) immediately, with a similar result obtained by setting the x-direction to be along the direction of $\vec{\mathbf{p}}_2$.

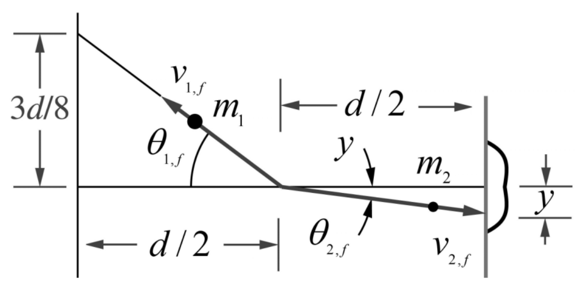


Problem 6 Exploding Hockey Puck

A hockey player shoots a "trick" hockey puck along the ice towards the center of the goal from a position d directly in front of the goal. The initial speed of the puck is v_0 and the puck has a mass m.



Half way to the goal the puck explodes into two fragments. One piece of mass $m_1 = (3/5)m$ comes back towards the player and passes $3d/8$ to the side of the spot it was initially shot from with a speed $v_{1,f} = (5/12)v_0$. The other piece of the puck with mass $m_2 = (2/5)m$ continues on towards the goal with a speed $v_{2,f}$.



Assume that there is no friction as the puck slides along the ice and that the mass of explosive in the puck is negligible. The goal of this problem is to find the distance, y, that the piece that continues towards the goal misses the center of the goal? Express your answer in terms of d.

a) What concepts will you apply to this problem and briefly explain why.

b) By what distance, y, does the piece that continues towards the goal miss the center of the goal? Express your answer in terms of d.

Solutions:

a) Since the puck explodes the mechanical energy of the system consisting ofn the puck is not constant (it increases due to the explosion, converting chemical energy into kinetic energy). However there are no external forces acting on the puck or the fragments of the puck since we assume the ice is frictionless. From the geometry fo the problem $\tan\theta_{2,f} = y/(d/2)$ and so we should be able to use the fact that the momentum is constant to determine $\tan\theta_{2,f}$ and hence find the distance y.

The equations for the constancy of the components of momentum are

$$mv_0 = -m_1(v_{1,f})\cos\theta_{1,f} + m_2(v_{2,f})\cos\theta_{2,f} \tag{6.1}$$

$$0 = m_1(v_{1,f})\sin\theta_{1,f} - m_2(v_{2,f})\sin\theta_{2,f} \tag{6.2}$$

Substitute the masses $m_1 = (3/5)m$, $m_2 = (2/5)m$, and $v_{1,f} = (5/12)v_0$ into Eqs. (6.1) and (6.2), yielding

$$v_0 = -\frac{1}{4}v_0\cos\theta_{1,f} + \frac{2}{5}v_{2,f}\cos\theta_{2,f} \tag{6.3}$$

$$0 = \frac{1}{4}v_0\sin\theta_{1,f} - \frac{2}{5}v_{2,f}\sin\theta_{2,f} \tag{6.4}$$

From the geometry of the collision,

$$\cos\theta_{1,f} = \frac{4d/8}{5d/8} = \frac{4}{5} \tag{6.5}$$

$$\sin\theta_{1,f} = \frac{3d/8}{5d/8} = \frac{3}{5} \tag{6.6}$$

The Eqs. (6.3) and (6.4) become

$$v_0 = -\frac{1}{5}v_0 + \frac{2}{5}v_{2,f}\cos\theta_{2,f} \tag{6.7}$$

$$0 = \frac{3}{20}v_0 - \frac{2}{5}v_{2,f}\sin\theta_{2,f} \tag{6.8}$$

Eq. (6.7) becomes

$$3v_0 = v_{2,f}\cos\theta_{2,f} \tag{6.9}$$

Eq. (6.8)

$$\frac{3}{8}v_0 = v_{2,f}\sin\theta_{2,f} \tag{6.10}$$

Now divide Eq. (6.10) by Eq. (6.9) yielding

$$\tan\theta_{2,f} = \frac{1}{8} \tag{6.11}$$

From the geometry of the problem

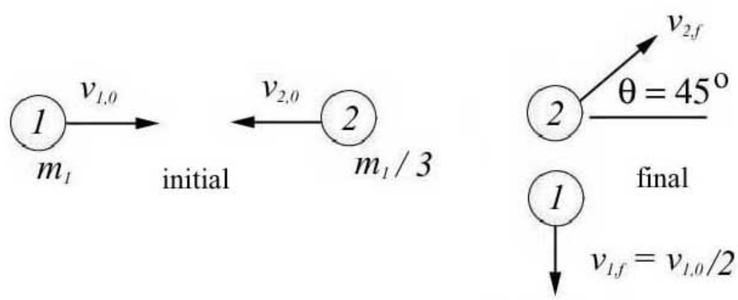
$$\tan\theta_{2,f} = \frac{y}{d/2} \tag{6.12}$$

So comparing Eqs. (6.11) and (6.12), we can solve for the distance y,

$$y = d/16 \tag{6.13}$$

Problem 7 Two- Dimensional Particle Collision

Particle 1 of mass m_1 is initially moving in the positive x-direction (to the right in the figure) with a speed $v_{1,0}$ and collides with a second particle 2 of mass $m_2 = m_1/3$, which is initially moving in the opposite direction (to the left in the figure) with an unknown speed $v_{2,0}$. Assume that the total external force acting on the particles is zero. **You do not know whether or not the collision is elastic.** After the collision particle 1 moves with a speed $v_{1,f} = v_{1,0}/2$, at an angle $\theta_{1,f} = 90^\circ$ with respect to the positive x-direction (downward in the figure). After the collision, particle 2 moves with an unknown speed $v_{2,f}$, at an angle $\theta_{2,f} = 45^\circ$ with respect to the positive x-direction (upward and to the right in the figure). In this problem you will find $v_{2,0}$ and $v_{2,f}$ in terms of $v_{1,0}$. You also will determine whether or not the collision is elastic. Note: $\sin 45^\circ = \cos 45^\circ = \sqrt{2}/2$.



a) Find the speed $v_{2,0}$ in terms of $v_{1,0}$.

b) Find the speed $v_{2,f}$ in terms of $v_{1,0}$.

c) Is the mechanical energy conserved in this collision? Justify your answer.

Solution:

a) The problem statement specifies that there are no net external forces acting on the system. Therefore, momentum is conserved.
In the above diagram, take the positive $\hat{\mathbf{i}}$-direction to be to the right and the positive $\hat{\mathbf{j}}$-direction to be upward.

To conserve momentum in the $\hat{\mathbf{i}}$-direction,

$$m_1 v_{1,0} - m_2 v_{2,0} = m_2 v_{2,f} \frac{\sqrt{2}}{2}$$
$$v_{1,0} - \frac{v_{2,0}}{3} = \frac{v_{2,f}}{3\sqrt{2}}, \tag{7.1}$$

where $m_2 = m_1 / 3$ and the known trigonometric relations have been used.

To conserve momentum in the $\hat{\mathbf{j}}$-direction,

$$0 = m_2 v_{2,f} \frac{\sqrt{2}}{2} - m_1 v_{1,f}$$
$$0 = \frac{v_{2,f}}{3\sqrt{2}} - \frac{v_{1,0}}{2}, \tag{7.2}$$

where the same relations have been used.

A simple substitution of the second expression in (7.2) into the second expression in (7.1) gives $v_{1,0} = -v_{2,0} / 3 = v_{1,0} / 2$, readily solved for

$$v_{2,0} = \frac{3}{2} v_{1,0}. \tag{7.3}$$

b) The second expression in (7.2) immediately gives $v_{2,f} = \left(3/\sqrt{2}\right) v_{1,0}$.

c) The initial and final kinetic energies are

$$E_0 = \frac{1}{2} m_1 v_{1,0}^2 + \frac{1}{2} m_2 v_{2,0}^2 = \frac{1}{2} m_1 v_{1,0}^2 + \frac{1}{2} \frac{m_1}{3} \left(\frac{3}{2} v_{1,0}^2 \right) = \frac{7}{8} m_1 v_{1,0}^2$$
$$E_f = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 = \frac{1}{2} m_1 \left(\frac{v_{1,0}}{2} \right)^2 + \frac{1}{2} \frac{m_1}{3} \left(\frac{3}{\sqrt{2}} v_{1,0} \right)^2 = \frac{7}{8} m_1 v_{1,0}^2. \tag{7.4}$$

The energies before and after the collision are the same; mechanical energy is conserved.

Problem 8 Pendulums and Collisions

A simple pendulum consists of a bob of mass m_1 that is suspended from a pivot by a string of negligible mass. The bob is pulled out and released from a height h_0 as measured from the bob's lowest point directly under the pivot point and then swings downward in a circular orbit (Figure 18.a). At the bottom of the swing, the bob collides with a block of mass m_2 that is initially at rest on a frictionless table. Assume that there is no friction at the pivot point.

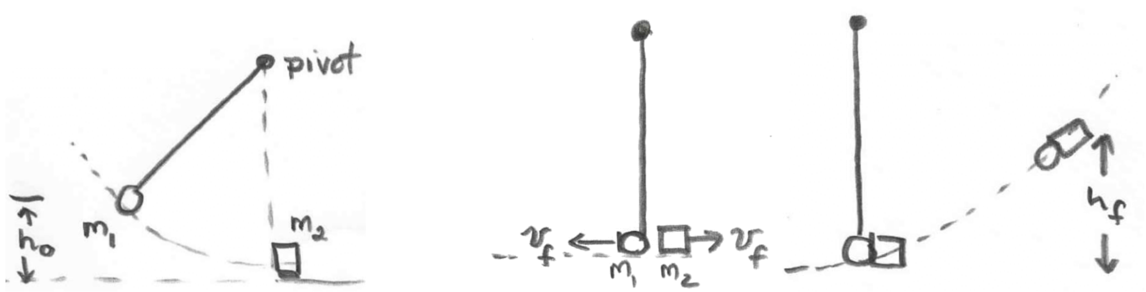
a) What is the speed of the bob immediately before the collision at the bottom of the swing?

b) Assume that the kinetic energy of the bob before the collision is equal to the kinetic energy of the bob and the block after the collision (the collision is elastic). Also assume that the bob and the block move in opposite directions but with the same speed after the collision (Figure b). What is the mass m_2 of the block?

c) Suppose the bob and block stick together after the collision due to some putty that is placed on the block. What is the speed of the combined system immediately after the collision? (Assume now that m_2 is the combined mass of the block and putty.)

d) What is the change in kinetic energy of the block and bob due to the collision in part c)? What is the ratio of the change in kinetic energy to the kinetic energy before the collision?

e) After the collision in part d), the bob and block move together in circular motion. What is the height h_f above the low point of the bob's swing when they both first come to rest after the collision (Figure c)? Ignore any air resistance.



(a) (b) (c)

Solution:

a) The mechanical energy of the bob is constant between when it is released and the bottom of the swing. We can use

$$\frac{1}{2}m_1 v_{1,0}^2 = m_1 g h_0 \tag{8.1}$$

to calculate the speed of the bob at the low point of the swing just before the collision,

$$v_{1,0} = \sqrt{2 g h_0}\,. \tag{8.2}$$

b) Consider the bob and the block as the system. Although tension in the string and the gravitation force are now acting as external forces, both are particular to the motion of the bob and block during the collision. If we additionally assume that the collision is nearly instantaneous, then the momentum is constant in the direction of the bob's motion,

$$m_1 v_{1,0} = m_2 v_{2,f} - m_1 v_{1,f}\,, \tag{8.3}$$

where $v_{2,f}$ is the speed of the block immediately after the collision.
Since the bob and block are given to have the same speeds after the collision, define $v_f \equiv v_{2,f} = v_{1,f}$ and rewrite Eq. (8.3) as

$$m_1 v_{1,0} = (m_2 - m_1) v_f\,. \tag{8.4}$$

Solve Eq. (8.4) for the speed of the bob and block after the collision,

$$v_f = v_{1,0}\frac{m_1}{m_2 - m_1} \tag{8.5}$$

(at this point we see explicitly what we might have guessed, that $m_2 > m_1$).

The collision is given to be elastic,

$$\frac{1}{2}m_1 v_{1,0}^2 = \frac{1}{2}(m_1 + m_2) v_f^2\,. \tag{8.6}$$

Substituting Eq. (8.5) into Eq. (8.6) yields

$$m_1 v_{1,0}^2 = (m_1 + m_2)\left(\frac{m_1}{m_2 - m_1}\right)^2 v_{1,0}^2 . \tag{8.7}$$

Canceling the common factor of $m_1 v_{1,0}^2$ from both sides of Eq. (8.7) and rearranging gives

$$(m_2 - m_1)^2 = (m_1 + m_2) m_1 . \tag{8.8}$$

Expanding the square and canceling m_1^2 yields

$$m_2(m_2 - 3\, m_1) = 0 , \tag{8.9}$$

and so the block has mass

$$m_2 = 3 m_1 \tag{8.10}$$

and the final speed is

$$v_f = \frac{v_{1,0}}{2} = \sqrt{\frac{g\, h_0}{2}} . \tag{8.11}$$

c) The bob and block stick together and move with a speed v_f' after the collision. The external forces are still perpendicular to the motion, and if we assume that the collision time is negligible, then the momentum in the direction of the motion is constant,

$$m_1 v_{1,0} = (m_1 + m_2) v_f' \tag{8.12}$$

The speed immediately after the collision is (recalling that $m_2 = 3 m_1$)

$$v_f' = \frac{m_1 v_{1,0}}{m_1 + m_2} = \frac{1}{4} v_{1,0} . \tag{8.13}$$

Using Eq. (8.2) in Eq. (8.13) yields

$$v_f' = \frac{1}{4} v_{1,0} = \frac{1}{4}\sqrt{2 g h_0} = \sqrt{\frac{g h_0}{8}} . \tag{8.14}$$

d) The change in kinetic energy of the bob and block due to the collision in part c) is given by

$$\Delta K = K_{\text{after}} - K_{\text{before}} = \frac{1}{2}(m_1 + m_2)v_f'^2 - \frac{1}{2}m_1 v_{1,0}^2 . \tag{8.15}$$

Using Eq. (8.13), (8.2) and (8.13) in Eq. (8.15),

$$\begin{aligned} \Delta K &= \frac{1}{2}(4m_1)\frac{gh_0}{8} - \frac{1}{2}m_1 2gh_0 \\ &= -\frac{3}{4}m_1 gh_0 . \end{aligned} \tag{8.16}$$

The kinetic energy before the collision was $m_1 gh_0$, and so the ratio of the change in kinetic energy to the kinetic energy before the collision is

$$\frac{\Delta K}{K_{\text{before}}} = -3/4 . \tag{8.17}$$

Note that this agrees with our result in worked example 18.5.2, with $m_2 = 3m_1$

$$\frac{\Delta K}{K_{\text{before}}} = -\frac{m_2}{m_1 + m_2} = -\frac{3}{4} . \tag{8.18}$$

for completely inelastic collisions when the target object (the block of mass m_2 in this example) is initially stationary.

e) After the collision, the tension is acting on the bob-block system but the tension force is perpendicular to the motion so does no work on the bob-block system and the mechanical energy after the collision is the same as when the bob-block combination reaches its highest point,

$$\begin{aligned} K_{\text{after}} &= (m_1 + m_2)g\, h_f' \\ \frac{m_1 gh_0}{4} &= 4m_1 g\, h_f' \end{aligned} \tag{8.19}$$

$$h_f' = \frac{h_0}{16} . \tag{8.20}$$