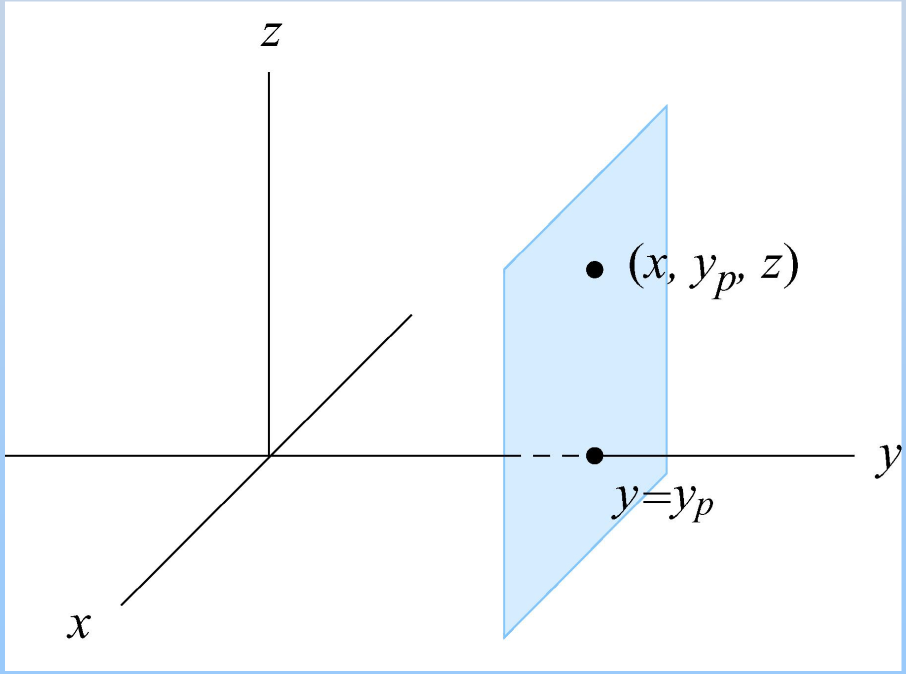
Cartesian Coordinate System and Vectors

Coordinate System

Coordinate system: used to describe the position of a point in space and consists of

1. An origin as the reference point
2. A set of coordinate axes with scales and labels
3. Choice of positive direction for each axis
4. Choice of unit vectors at each point in space



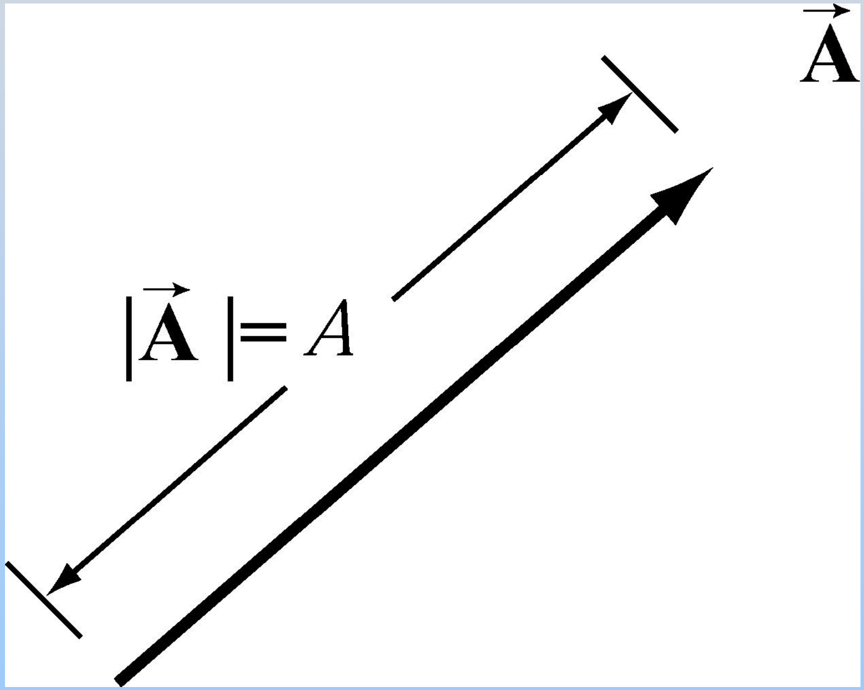
Cartesian Coordinate System

Vectors

Vector

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol $\vec{\mathbf{A}}$

The magnitude of $\vec{\mathbf{A}}$ is denoted by $|\vec{\mathbf{A}}| \equiv A$



Application of Vectors

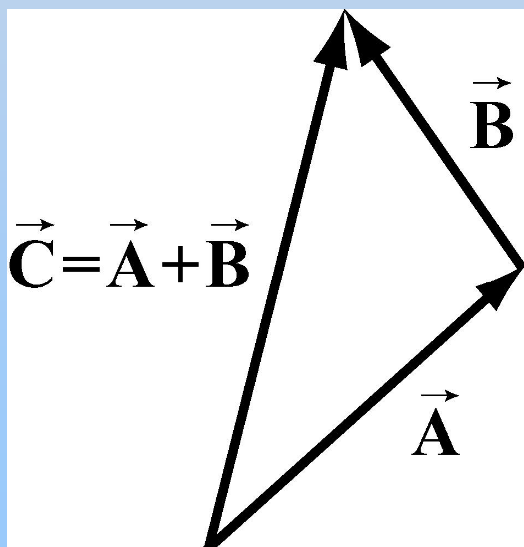
(1) Vectors can exist at any point *P* in space.

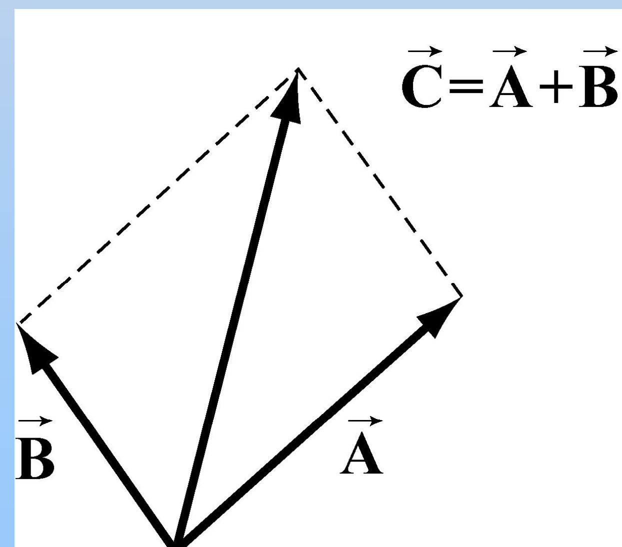
(2) Vectors have direction and magnitude.

(3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.

Vector Addition

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Define a new vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$,the “vector addition” of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ by the geometric construction shown in either figure





Summary: Vector Properties

Addition of Vectors

1. Commutativity $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$

2. Associativity $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$

3. Identity Element for Vector Addition $\vec{\mathbf{0}}$ such that $\vec{\mathbf{A}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{A}} = \vec{\mathbf{A}}$

4. Inverse Element for Vector Addition $-\vec{\mathbf{A}}$ such that $\vec{\mathbf{A}} + \left(-\vec{\mathbf{A}}\right) = \vec{\mathbf{0}}$

Scalar Multiplication of Vectors

1. Associative Law for Scalar Multiplication $b\,(c\vec{\mathbf{A}}) = (bc)\vec{\mathbf{A}} = (cb\,\vec{\mathbf{A}}) = c\,(b\vec{\mathbf{A}})$

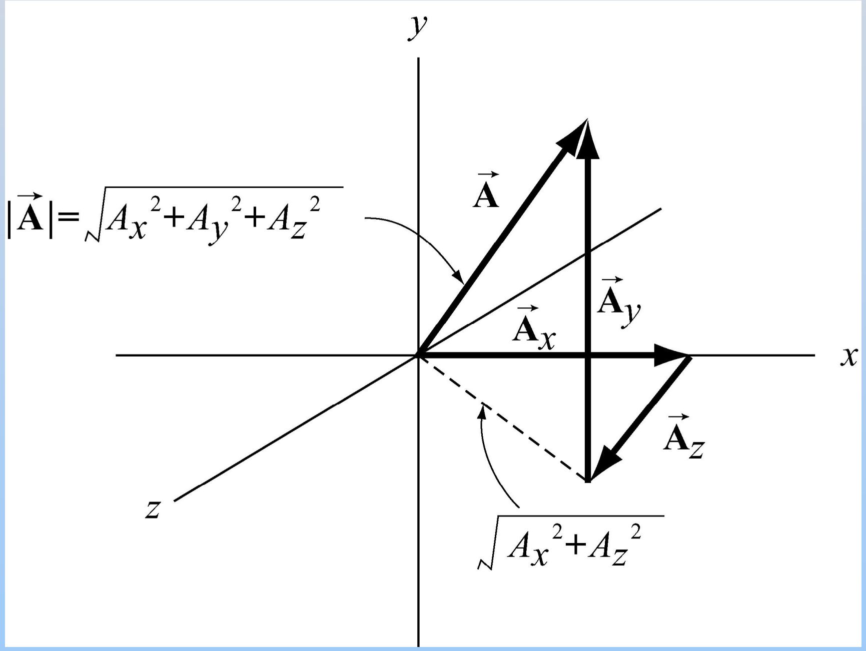
2. Distributive Law for Vector Addition $c\,(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = c\,\vec{\mathbf{A}} + c\,\vec{\mathbf{B}}$

3. Distributive Law for Scalar Addition $(b + c)\,\vec{\mathbf{A}} = b\,\vec{\mathbf{A}} + c\,\vec{\mathbf{A}}$

4. Identity Element for Scalar Multiplication: number 1 such that $1\,\vec{\mathbf{A}} = \vec{\mathbf{A}}$

Vector Decomposition

Choose a coordinate system with an origin and axes. We can decompose a vector into component vectors along each coordinate axis, for example along the x,y, and z-axes of a Cartesian coordinate system. A vector at *P* can be decomposed into the vector sum,

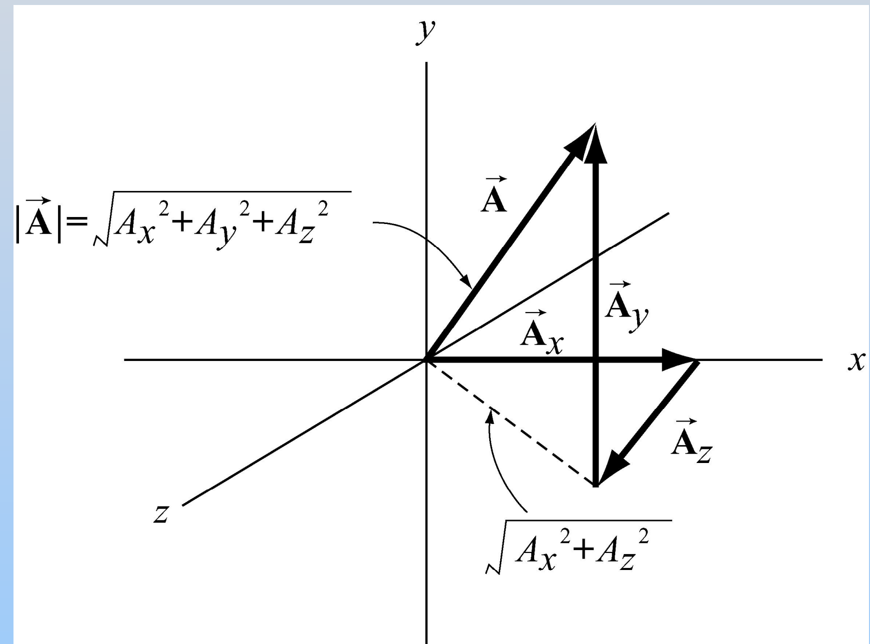


$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z$$

Unit Vectors and Components

The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$

with $|\hat{\mathbf{i}}|=1, |\hat{\mathbf{j}}|=1, |\hat{\mathbf{k}}|=1$



Components:

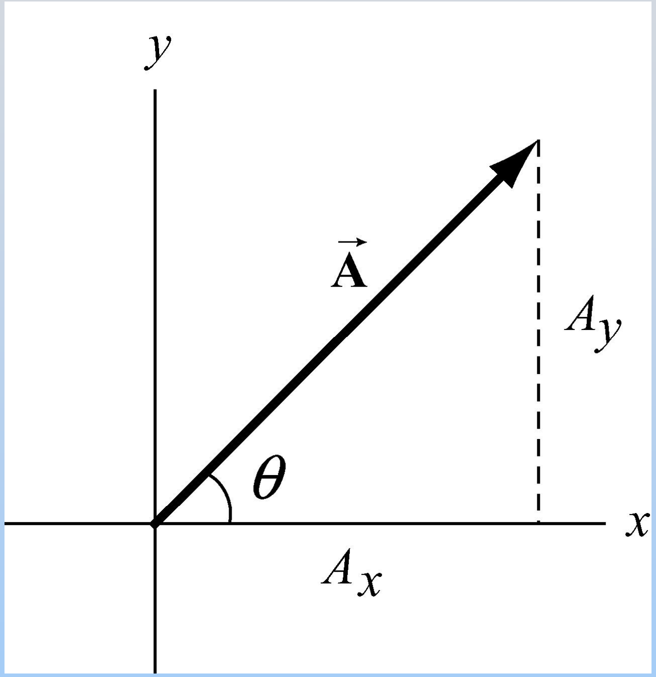
$$\vec{\mathbf{A}}=(A_x, A_y, A_z)$$

$$\vec{\mathbf{A}}_x = A_x\,\hat{\mathbf{i}},\quad \vec{\mathbf{A}}_y = A_y\,\hat{\mathbf{j}},\quad \vec{\mathbf{A}}_z = A_z\,\hat{\mathbf{k}} \qquad \vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$$

Vector Decomposition in Two Dimensions

Consider a vector

$$\vec{\mathbf{A}} = (A_x, A_y, 0)$$



x- and y components:

$$A_x = A\cos(\theta), \quad A_y = A\sin(\theta)$$

Magnitude: $A = \sqrt{A_x^2 + A_y^2}$

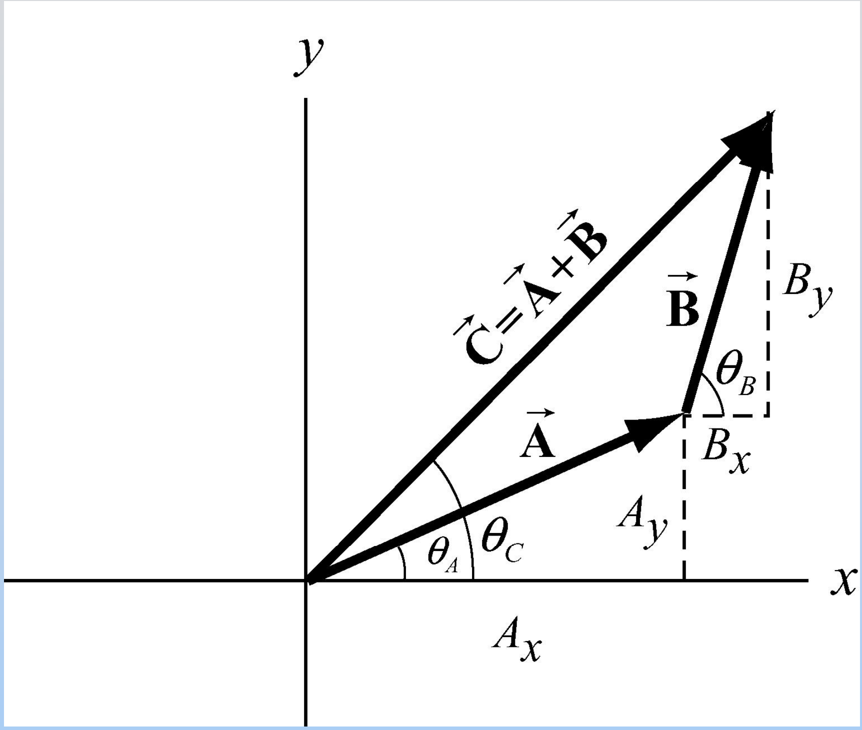
Direction: $\dfrac{A_y}{A_x} = \dfrac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta)$

$$\theta = \tan^{-1}(A_y / A_x)$$

Vector Addition

$$\vec{\mathbf{A}} = A\cos(\theta_A)\ \hat{\mathbf{i}} + A\sin(\theta_A)\ \hat{\mathbf{j}}$$

$$\vec{\mathbf{B}} = B\cos(\theta_B)\ \hat{\mathbf{i}} + B\sin(\theta_B)\ \hat{\mathbf{j}}$$



Vector Sum: $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$

Components

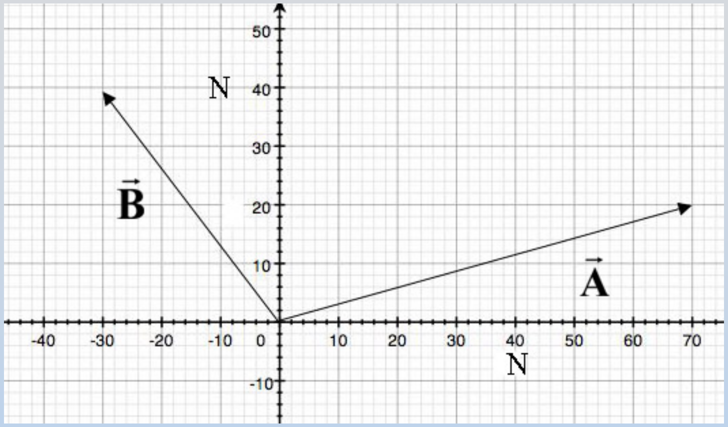
$$C_x = A_x + B_x, \qquad C_y = A_y + B_y$$

$$C_x = C\cos(\theta_C) = A\cos(\theta_A) + B\cos(\theta_B)$$

$$C_y = C\sin(\theta_C) = A\sin(\theta_A) + B\sin(\theta_B)$$

$$\vec{\mathbf{C}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} = C\cos(\theta_C)\ \hat{\mathbf{i}} + C\sin(\theta_C)\ \hat{\mathbf{j}}$$

Checkpoint Problem: Vector Decomposition



Two horizontal ropes are attached to a post that is stuck in the ground. The ropes pull the post producing the vector forces $\vec{\mathbf{A}} = 70\text{ N}\,\hat{\mathbf{i}} + 20\text{ N}\,\hat{\mathbf{j}}$ and $\vec{\mathbf{B}} = -30\text{ N}\,\hat{\mathbf{i}} + 40\text{ N}\,\hat{\mathbf{j}}$ as shown in the figure. Find the direction and magnitude of the horizontal component of a third force on the post that will make the vector sum of forces on the post equal to zero.

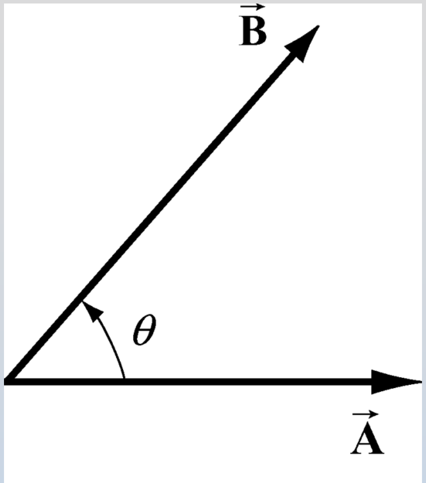
Checkpoint Problem: Sinking Sailboat

A Coast Guard ship is located 35 km away from a checkpoint in a direction 42^0 north of west. A distressed sailboat located in still water 20 km from the same checkpoint in a direction 36^0 south of east is about to sink. Draw a diagram indicating the position of both ships. In what direction and how far must the Coast Guard ship travel to reach the sailboat?

Preview: Vector Description of Motion

- Position $\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$
- Displacement $\Delta\vec{\mathbf{r}}(t) = \Delta x(t)\,\hat{\mathbf{i}} + \Delta y(t)\,\hat{\mathbf{j}}$
- Velocity $\vec{\mathbf{v}}(t) = \dfrac{dx(t)}{dt}\,\hat{\mathbf{i}} + \dfrac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}$
- Acceleration $\vec{\mathbf{a}}(t) = \dfrac{dv_x(t)}{dt}\,\hat{\mathbf{i}} + \dfrac{dv_y(t)}{dt}\,\hat{\mathbf{j}} \equiv a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}$

Dot Product

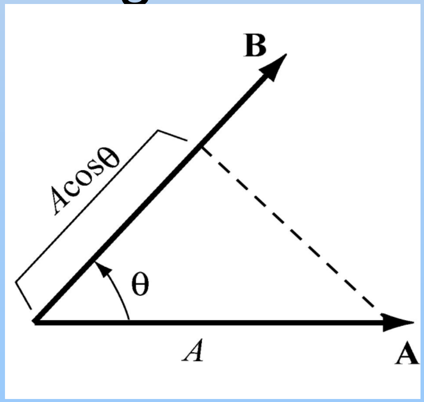


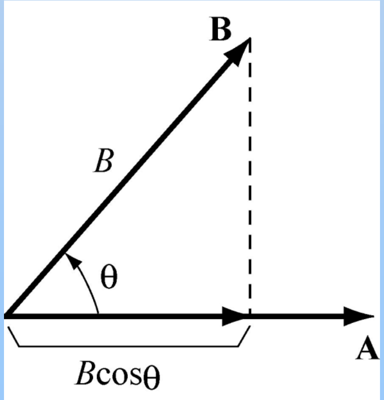
A scalar quantity

Magnitude: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \cos\theta$

The dot product can be positive, zero, or negative

Two types of projections: the dot product is the parallel component of one vector with respect to the second vector times the magnitude of the second vector





$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = A\ \left|\vec{\mathbf{B}}\right|$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right| B$$

Dot Product Properties

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$$

$$c\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = c(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})$$

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \cdot \vec{\mathrm{C}} = \vec{\mathbf{A}} \cdot \vec{\mathrm{C}} + \vec{\mathbf{B}} \cdot \vec{\mathrm{C}}$$

Dot Product in Cartesian Coordinates

With unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}| \cos(0) = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \cos(\pi/2) = 0$$

Example:

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}, \qquad \vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

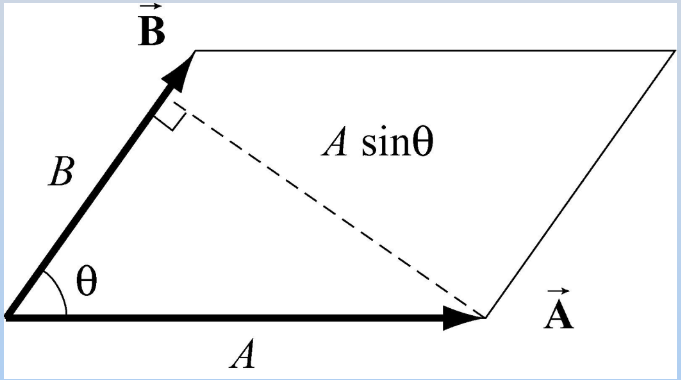
Checkpoint Problem: Scalar Product

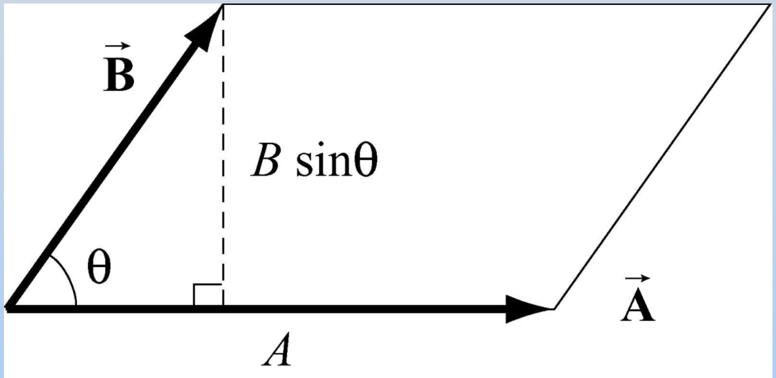
In the methane molecule, CH_4 , each hydrogen atom is at the corner of a tetrahedron with the carbon atom at the center. In a coordinate system centered on the carbon atom, if the direction of one of the C--H bonds is described by the vector $\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and the direction of an adjacent C--H is described by the vector $\vec{\mathbf{B}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$, what is the angle between these two bonds.

Summary: Cross Product

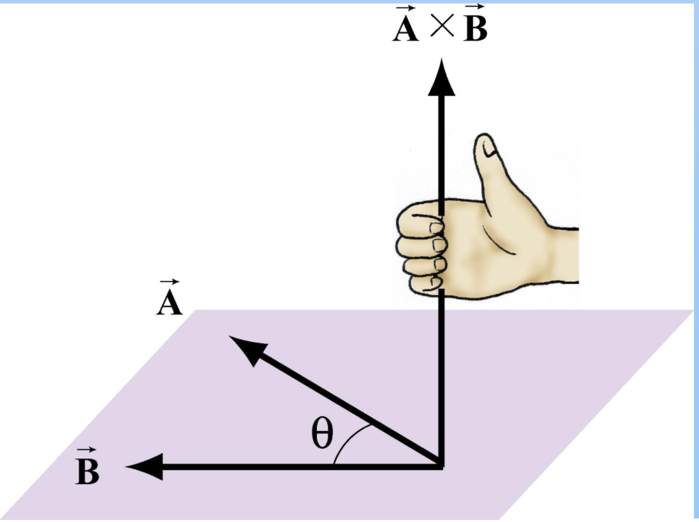
Magnitude: equal to the area of the parallelogram defined by the two vectors

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\theta = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right) = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right| \quad (0 \le \theta \le \pi)$$





Direction: determined by the Right-Hand-Rule



Properties of Cross Products

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

$$c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c\vec{\mathbf{A}} \times \vec{\mathbf{B}}$$

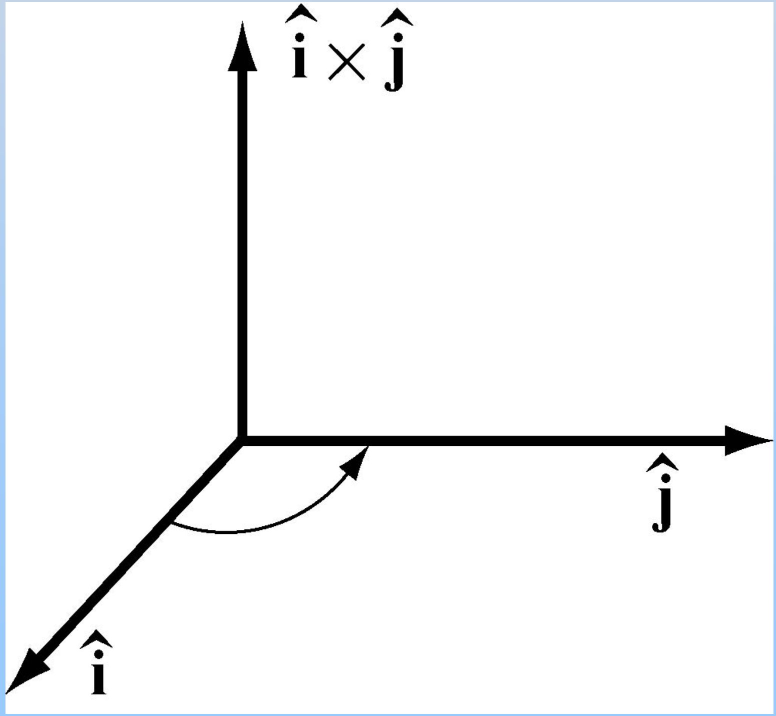
$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}}$$

Cross Product of Unit Vectors

- Unit vectors in Cartesian coordinates

$$\left|\hat{\mathbf{i}}\times\hat{\mathbf{j}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(\pi/2) = 1$$

$$\left|\hat{\mathbf{i}}\times\hat{\mathbf{i}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(0) = 0$$



$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{i}}\times\hat{\mathbf{i}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \hat{\mathbf{k}}\times\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Components of Cross Product

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \quad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Checkpoint Problem: Vector Product

Find a unit vector perpendicular to

$$\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

and

$$\vec{\mathbf{B}} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}} \qquad .$$