

Module 3: Cartesian Coordinates and Vectors

“Philosophy is written in this grand book, the universe which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.”

Galileo Galilee in *Assayer*

3.1 Introduction

Mathematics and physics have been historically interwoven since the time of the ancient Greeks.

“The calculus had its origins in the logical difficulties encountered by the ancient Greek mathematicians in their attempt to express their intuitive ideas on the ratios and proportionalities of lines, which they vaguely recognized as continuous, in terms of numbers, which they regarded as discrete.”¹

The modern science of kinematics began in the 16th century, culminating with Galileo’s description of the motion of bodies, published in at the start of the 17th century. Galileo used geometric techniques derived from Euclid’s *Elements* to introduce the concepts of velocity and acceleration. The reliance on mathematics to describe nature is the foundation on which science is built.

With the introduction of analytic geometry, the systematic use of variables, and a use of the infinitesimal, Leibnitz and Newton developed algorithms for introducing and calculating the derivative and the integral, the core concepts of calculus. Although the calculus was not used by Newton in his development of the Principles of the Mechanics, its use was found to be indispensable in solving countless problems.

In physics, there are well-defined problem solving methodologies that provide step-by-step instructions on applying physics concepts but the key question is: How do these methodologies fit into the larger context of expert problem solving?

Although algorithmic steps can be articulated (and memorized), it is essential for students to understand the meaning of these methodologies. In particular how the methodologies are related to the core physics concepts. The process of understanding and mastering these methodologies is slow because it is built on practice. It is also difficult because it requires a simultaneous understanding of physical and mathematical concepts. The methodologies can bring out the subtleties of the physical concepts while the concepts illustrate the need for the mathematics. This is a non-algorithmic learning

¹ Carl B. Boyer, *The History of the Calculus and its Conceptual Development*, Dover, New York, 1949, p. 4

process of synthesizing two difficult abstract knowledge systems simultaneously, mathematical physics.

The conceptual foundation of the methodologies provides the framework for thinking about the physics problem. Thinking about physics problems and designing problem solving strategies is the fusion that turns students into expert problem solvers. Learning to think about a physics problem using force diagrams is an example of this type of fusion.

An emphasis on modeling of physical systems is an effective way to mitigate against two of the main hazards of mathematics classes:

1. The students recognize certain formulas as relevant to other subjects, but view mathematical reasoning as an isolated game with fixed rules that they are forced to master, but with no connection to other subjects.
2. Because they can pass the course by memorizing some templated techniques (and because of point 1) the mathematics they “learn” is not truly available to them when solving problems in other domains.

Today, the first-year student at MIT is expected to understand single variable calculus (18.01), multiple variable calculus (18.02), along with Newtonian Mechanics (8.01), and Electricity and Magnetism (8.02). These General Institute Requirements are in fact a rite of passage for the modern MIT student into the world of science and engineering.

The understanding of physics will deepen because students can solve more mathematically challenging homework problems. This will reinforce the idea that calculus is an important part of physics. The understanding of mathematics will deepen because physics provides a rich source of problems

3.2 Cartesian Coordinate System

Physics involve the study of phenomena that we observe in the world. In order to connect the phenomena to mathematics we begin by introducing the concept of a coordinate system. A coordinate system consists of four basic elements:

- (1) Choice of origin
- (2) Choice of axes
- (3) Choice of positive direction for each axis
- (4) Choice of unit vectors for each axis

There are three commonly used coordinate systems: Cartesian, cylindrical and spherical. What makes these systems extremely useful is the associated set of infinitesimal line, area, and volume elements that are key to making many integration

calculations in classical mechanics, such as finding the center of mass and moment of inertia.

Cartesian Coordinates

Cartesian coordinates consist of a set of mutually perpendicular axes, which intersect at a common point, the origin O . We live in a three-dimensional environment; for that reason, the most common system we will use has three axes, for which we choose the directions of the axes and position of the origin are.

(1) Choice of Origin

Choose an origin O . If you are given an object, then your choice of origin may coincide with a special point in the body. For example, you may choose the mid-point of a straight piece of wire.

(2) Choice of Axis

Now we shall choose a set of axes. The simplest set of axes are known as the Cartesian axes, x -axis, y -axis, and the z -axis. Once again, we adapt our choices to the physical object. For example, we select the x -axis so that the wire lies on the x -axis, as shown in Figure 3.1

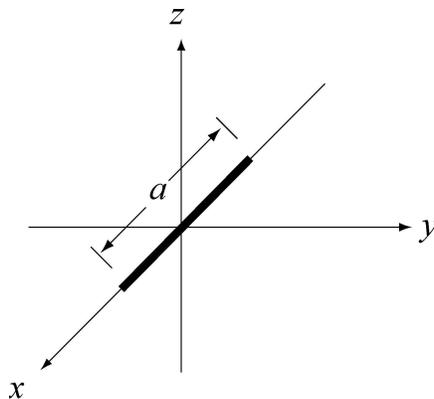


Figure 3.1 A segment of wire of length a lying along the x -axis of a Cartesian coordinate system.

Then each point P in space our S can be assigned a triplet of values (x_p, y_p, z_p) , the Cartesian coordinates of the point P . The ranges of these values are: $-\infty < x_p < +\infty$, $-\infty < y_p < +\infty$, $-\infty < z_p < +\infty$.

The collection of points that have the same the coordinate y_p is called a level surface. Suppose we ask what collection of points in our space S have the same value of $y = y_p$. This is the set of points $S_{y_p} = \{(x, y, z) \in S \text{ such that } y = y_p\}$. This set S_{y_p} is a

plane, the $x-z$ plane (Figure 3.2), called a *level set* for constant y_p . Thus, the y -coordinate of any point actually describes a plane of points perpendicular to the y -axis.

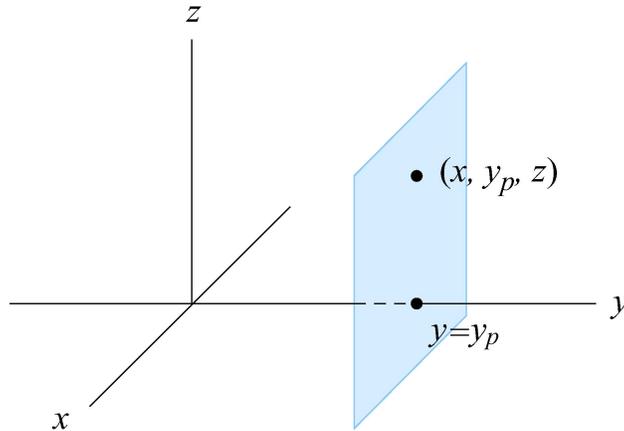


Figure 3.2 Level surface set for constant value y_p .

(3) Choice of Positive Direction

Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol $+$ along the positive axis. Conventionally, Cartesian coordinates are drawn with the $y-z$ plane corresponding to the plane of the paper. The horizontal direction from left to right is taken as the positive y -axis, and the vertical direction from bottom to top is taken as the positive z -axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes. The endpoints of the wire now have coordinates $(a/2, 0, 0)$ and $(-a/2, 0, 0)$.

(4) Choice of Unit Vectors

We now associate to each point P in space, a set of three unit directions vectors $(\hat{\mathbf{i}}_p, \hat{\mathbf{j}}_p, \hat{\mathbf{k}}_p)$. A unit vector has magnitude one: $|\hat{\mathbf{i}}_p| = 1$, $|\hat{\mathbf{j}}_p| = 1$, and $|\hat{\mathbf{k}}_p| = 1$. We assign the direction of $\hat{\mathbf{i}}_p$ to point in the direction of the increasing x -coordinate at the point P . We define the directions for $\hat{\mathbf{j}}_p$ and $\hat{\mathbf{k}}_p$ in the direction of the increasing y -coordinate and z -coordinate respectively. (Figure 3.3).

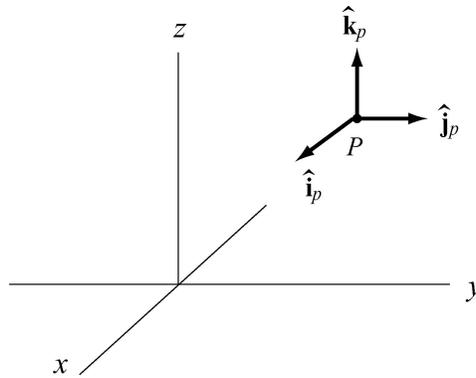


Figure 3.3 Choice of unit vectors.

3.3 Vector Analysis

Introduction to Vectors

Certain physical quantities such as mass or the absolute temperature at some point only have magnitude. These quantities can be represented by numbers alone, with the appropriate units, and they are called scalars. There are, however, other physical quantities which have both magnitude and direction; the magnitude can stretch or shrink, and the direction can reverse. These quantities can be added in such a way that takes into account both direction and magnitude. Force is an example of a quantity that acts in a certain direction with some magnitude that we measure in newtons. When two forces act on an object, the sum of the forces depends on both the direction and magnitude of the two forces. Position, displacement, velocity, acceleration, force, momentum and torque are all physical quantities that can be represented mathematically by vectors. We shall begin by defining precisely what we mean by a vector.

Properties of a Vector

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol \vec{A} . The magnitude of \vec{A} is $|\vec{A}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector (Figure 3.4).

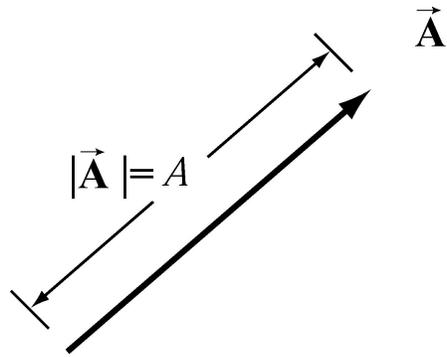


Figure 3.4 Vectors as arrows.

There are two defining operations for vectors:

(1) Vector Addition:

Vectors can be added. Let \vec{A} and \vec{B} be two vectors. We define a new vector, $\vec{C} = \vec{A} + \vec{B}$, the “vector addition” of \vec{A} and \vec{B} , by a geometric construction. Draw the arrow that represents \vec{A} . Place the tail of the arrow that represents \vec{B} at the tip of the arrow for \vec{A} as shown in Figure 3.5(a). The arrow that starts at the tail of \vec{A} and goes to the tip of \vec{B} is defined to be the “vector addition” $\vec{C} = \vec{A} + \vec{B}$. There is an equivalent construction for the law of vector addition. The vectors \vec{A} and \vec{B} can be drawn with their tails at the same point. The two vectors form the sides of a parallelogram. The diagonal of the parallelogram corresponds to the vector $\vec{C} = \vec{A} + \vec{B}$, as shown in Figure 3.5(b).

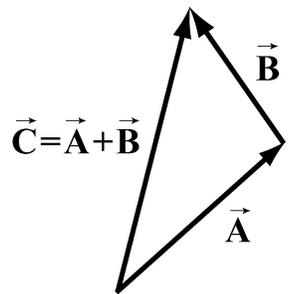


Figure 3.5(a) Geometric sum of vectors.

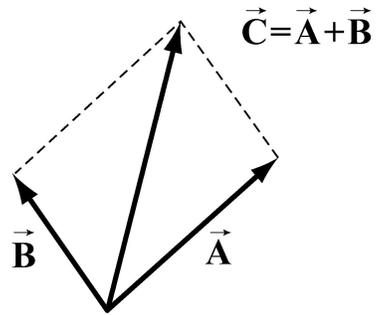


Figure 3.5 (b) Geometric sum of vectors.

Vector addition satisfies the following four properties:

(i) Commutivity:

The order of adding vectors does not matter;

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} . \tag{3.3.1}$$

Our geometric definition for vector addition satisfies the commutativity property (i) since in the parallelogram representation for the addition of vectors, it doesn't matter which side you start with, as seen in Figure 3.6.

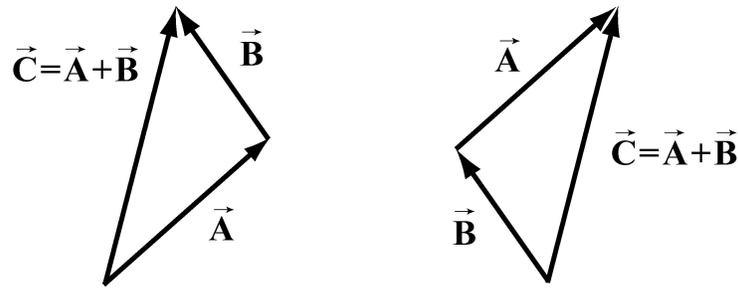


Figure 3.6 Commutative property of vector addition

(ii) Associativity:

When adding three vectors, it doesn't matter which two you start with

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) \quad (3.3.2)$$

In Figure 3.7, we add $(\vec{A} + \vec{B}) + \vec{C}$, and $\vec{A} + (\vec{B} + \vec{C})$ to arrive at the same vector sum in either case.

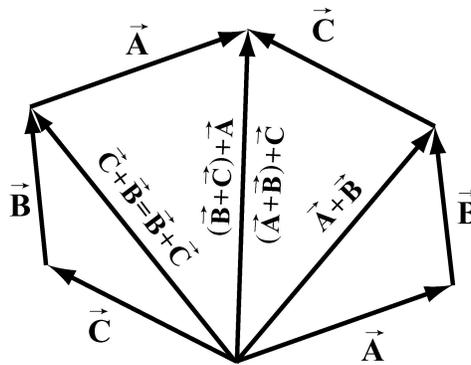


Figure 3.7 Associative law.

(iii) Identity Element for Vector Addition:

There is a unique vector, $\vec{0}$, that acts as an identity element for vector addition. For all vectors \vec{A} ,

$$\vec{A} + \vec{0} = \vec{0} + \vec{A} = \vec{A} \quad (3.3.3)$$

(iv) Inverse Element for Vector Addition:

For every vector \vec{A} , there is a unique inverse vector

$$(-1)\vec{A} \equiv -\vec{A} \tag{3.3.4}$$

such that

$$\vec{A} + (-\vec{A}) = \vec{0}$$

The vector $-\vec{A}$ has the same magnitude as \vec{A} , $|\vec{A}| = |-\vec{A}| = A$, but they point in opposite directions (Figure 3.8).

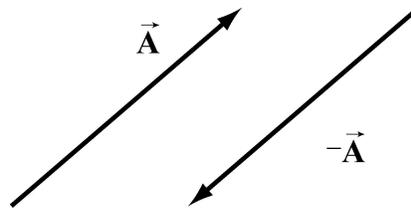


Figure 3.8 additive inverse.

(2) Scalar Multiplication of Vectors:

Vectors can be multiplied by real numbers. Let \vec{A} be a vector. Let c be a real positive number. Then the multiplication of \vec{A} by c is a new vector, which we denote by the symbol $c\vec{A}$. The magnitude of $c\vec{A}$ is c times the magnitude of \vec{A} (Figure 3.9a),

$$cA = Ac \tag{3.3.5}$$

Since $c > 0$, the direction of $c\vec{A}$ is the same as the direction of \vec{A} . However, the direction of $-c\vec{A}$ is opposite of \vec{A} (Figure 3.9b).

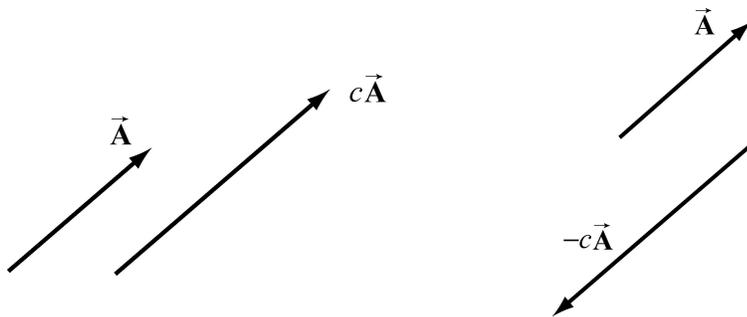


Figure 3.9 Multiplication of vector \vec{A} by (a) $c > 0$, and (b) $-c < 0$.

Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication:

The order of multiplying numbers is doesn't matter. Let b and c be real numbers. Then

$$b(c\vec{A}) = (bc)\vec{A} = (cb\vec{A}) = c(b\vec{A}) \quad (3.3.6)$$

(ii) Distributive Law for Vector Addition:

Vector addition satisfies a distributive law for multiplication by a number. Let c be a real number. Then

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B} \quad (3.3.7)$$

Figure 3.10 illustrates this property.

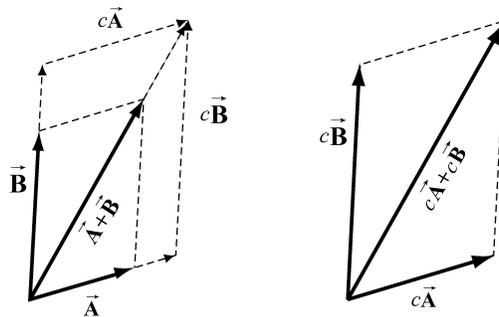


Figure 3.10 Distributive Law for vector addition.

(iii) Distributive Law for Scalar Addition:

The multiplication operation also satisfies a distributive law for the addition of numbers. Let b and c be real numbers. Then

$$(b + c)\vec{A} = b\vec{A} + c\vec{A} \quad (3.3.8)$$

Our geometric definition of vector addition satisfies this condition as seen in Figure 3.11.

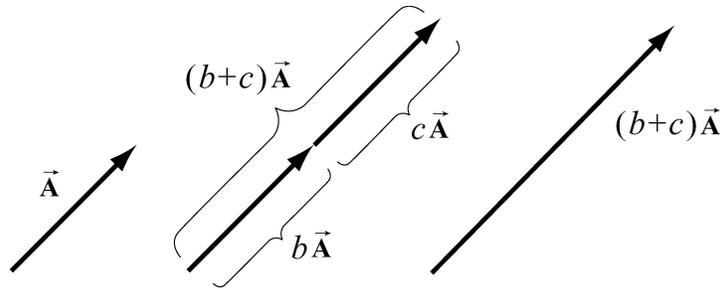


Figure 3.11 Distributive law for scalar multiplication

(iv) Identity Element for Scalar Multiplication:

The number 1 acts as an identity element for multiplication,

$$1 \vec{A} = \vec{A} \tag{3.3.9}$$

3.4 Application of Vectors

When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, momentum, torque, and angular momentum as vectors. We can't add force to velocity or subtract momentum from torque. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

- (1) Vectors can exist at any point P in space.
- (2) Vectors have direction and magnitude.
- (3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.
- (4) Vector Decomposition: Choose a coordinate system with an origin and axes. We can decompose a vector into component vectors along each coordinate axis. In Figure 3.12 we choose Cartesian coordinates for the x - y plane (we ignore the z -direction for simplicity but we can extend our results when we need to). A vector \vec{A} at P can be decomposed into the vector sum,

$$\vec{A} = \vec{A}_x + \vec{A}_y, \tag{3.4.1}$$

where \vec{A}_x is the x -component vector pointing in the positive or negative x -direction, and \vec{A}_y is the y -component vector pointing in the positive or negative y -direction (Figure 3.12).

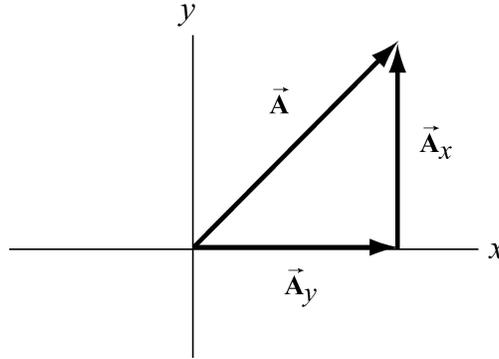


Figure 3.12 Vector decomposition

(5) Unit vectors: The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space. We associate to each point P in space, a set of three unit vectors $(\hat{i}, \hat{j}, \hat{k})$. A unit vector means that the magnitude is one: $|\hat{i}|=1$, $|\hat{j}|=1$, and $|\hat{k}|=1$. We assign the direction of \hat{i} to point in the direction of the increasing x -coordinate at the point P . We call \hat{i} the unit vector at P pointing in the $+x$ -direction. Unit vectors \hat{j} and \hat{k} can be defined in a similar manner (Figure 3.13).

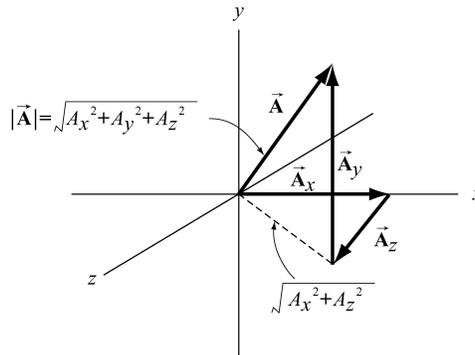


Figure 3.13 Choice of unit vectors in Cartesian coordinates.

(6) Vector Components: Once we have defined unit vectors, we can then define the x -component and y -component of a vector. Recall our vector decomposition, $\vec{A} = \vec{A}_x + \vec{A}_y$. We can write the x -component vector, \vec{A}_x , as

$$\vec{A}_x = A_x \hat{i} \quad (3.4.2)$$

In this expression the term A_x , (without the arrow above) is called the x -component of the vector $\vec{\mathbf{A}}$. The x -component A_x can be positive, zero, or negative. It is not the magnitude of $\vec{\mathbf{A}}_x$ which is given by $(A_x^2)^{1/2}$. Note the difference between the x -component, A_x , and the x -component vector, $\vec{\mathbf{A}}_x$.

In a similar fashion we define the y -component, A_y , and the z -component, A_z , of the vector $\vec{\mathbf{A}}$

$$\vec{\mathbf{A}}_y = A_y \hat{\mathbf{j}}, \quad \vec{\mathbf{A}}_z = A_z \hat{\mathbf{k}} \quad (3.4.3)$$

A vector $\vec{\mathbf{A}}$ can be represented by its three components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. We can also write the vector as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (3.4.4)$$

(7) Magnitude: In Figure 3.13, we also show the vector components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. Using the Pythagorean theorem, the magnitude of $\vec{\mathbf{A}}$ is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (3.4.5)$$

(8) Direction: Let's consider a vector $\vec{\mathbf{A}} = (A_x, A_y, 0)$. Since the z -component is zero, the vector $\vec{\mathbf{A}}$ lies in the x - y plane. Let θ denote the angle that the vector $\vec{\mathbf{A}}$ makes in the counterclockwise direction with the positive x -axis (Figure 3.14). Then the x -component and y -component are

$$A_x = A \cos(\theta), \quad A_y = A \sin(\theta) \quad (3.4.6)$$

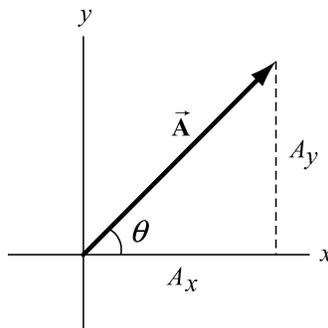


Figure 3.14 Components of a vector in the x - y plane.

We can now write a vector in the x - y plane as

$$\vec{\mathbf{A}} = A \cos(\theta) \hat{\mathbf{i}} + A \sin(\theta) \hat{\mathbf{j}} \quad (3.4.7)$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A \sin(\theta)}{A \cos(\theta)} = \tan(\theta), \quad (3.4.8)$$

that yields

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right). \quad (3.4.9)$$

Clearly, the direction of the vector depends on the sign of A_x and A_y . For example, if both $A_x > 0$ and $A_y > 0$, then $0 < \theta < \pi/2$, and the vector lies in the first quadrant. If, however, $A_x > 0$ and $A_y < 0$, then $-\pi/2 < \theta < 0$, and the vector lies in the fourth quadrant.

(9) Unit vector in the direction of $\vec{\mathbf{A}}$: Let $\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$. Let $\hat{\mathbf{A}}$ denote a unit vector in the direction of $\vec{\mathbf{A}}$. Then

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|} = \frac{A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}}{(A_x^2 + A_y^2 + A_z^2)^{1/2}} \quad (3.4.10)$$

(10) Vector Addition: Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors in the x - y plane. Let θ_A and θ_B denote the angles that the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ make (in the counterclockwise direction) with the positive x -axis. Then

$$\vec{\mathbf{A}} = A \cos(\theta_A) \hat{\mathbf{i}} + A \sin(\theta_A) \hat{\mathbf{j}} \quad (3.4.11)$$

$$\vec{\mathbf{B}} = B \cos(\theta_B) \hat{\mathbf{i}} + B \sin(\theta_B) \hat{\mathbf{j}} \quad (3.4.12)$$

In Figure 3.15, the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive x -axis.

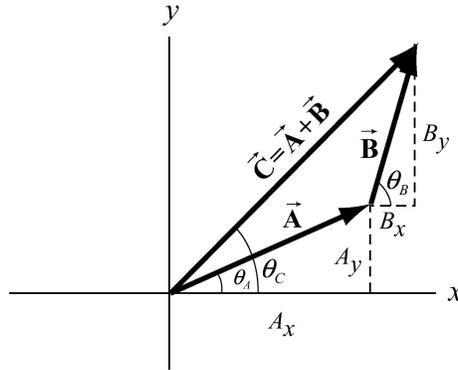


Figure 3.15 Vector addition with components

Then the components of \vec{C} are

$$C_x = A_x + B_x, \quad C_y = A_y + B_y \quad (3.4.13)$$

In terms of magnitudes and angles, we have

$$\begin{aligned} C_x &= C \cos(\theta_C) = A \cos(\theta_A) + B \cos(\theta_B) \\ C_y &= C \sin(\theta_C) = A \sin(\theta_A) + B \sin(\theta_B) \end{aligned} \quad (3.4.14)$$

We can write the vector \vec{C} as

$$\vec{C} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} = C \cos(\theta_C) \hat{i} + C \sin(\theta_C) \hat{j} \quad (3.4.15)$$

Example 1: Given two vectors, $\vec{A} = 2\hat{i} + (-3)\hat{j} + 7\hat{k}$ and $\vec{B} = 5\hat{i} + \hat{j} + 2\hat{k}$, find: (a) $|\vec{A}|$; (b) $|\vec{B}|$; (c) $\vec{A} + \vec{B}$; (d) $\vec{A} - \vec{B}$; (e) a unit vector \hat{A} pointing in the direction of \vec{A} ; (f) a unit vector \hat{B} pointing in the direction of \vec{B} .

$$(a) \quad |\vec{A}| = \left(2^2 + (-3)^2 + 7^2\right)^{1/2} = \sqrt{62} = 7.87$$

$$(b) \quad |\vec{B}| = \left(5^2 + 1^2 + 2^2\right)^{1/2} = \sqrt{30} = 5.48$$

$$\begin{aligned} \vec{A} + \vec{B} &= (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} + (A_z + B_z) \hat{k} \\ (c) \quad &= (2 + 5) \hat{i} + (-3 + 1) \hat{j} + (7 + 2) \hat{k} \\ &= 7 \hat{i} - 2 \hat{j} + 9 \hat{k} \end{aligned}$$

$$\begin{aligned}
\vec{\mathbf{A}} - \vec{\mathbf{B}} &= (A_x - B_x) \hat{\mathbf{i}} + (A_y - B_y) \hat{\mathbf{j}} + (A_z - B_z) \hat{\mathbf{k}} \\
\text{(d)} \quad &= (2 - 5) \hat{\mathbf{i}} + (-3 - 1) \hat{\mathbf{j}} + (7 - 2) \hat{\mathbf{k}} \\
&= -3 \hat{\mathbf{i}} - 4 \hat{\mathbf{j}} + 5 \hat{\mathbf{k}}
\end{aligned}$$

(e) A unit vector $\hat{\mathbf{A}}$ in the direction of $\vec{\mathbf{A}}$ can be found by dividing the vector $\vec{\mathbf{A}}$ by the magnitude of $\vec{\mathbf{A}}$. Therefore

$$\hat{\mathbf{A}} = \vec{\mathbf{A}} / |\vec{\mathbf{A}}| = (2 \hat{\mathbf{i}} + -3 \hat{\mathbf{j}} + 7 \hat{\mathbf{k}}) / \sqrt{62}$$

(f) In a similar fashion, $\hat{\mathbf{B}} = \vec{\mathbf{B}} / |\vec{\mathbf{B}}| = (5 \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2 \hat{\mathbf{k}}) / \sqrt{30}$.

Example 2: Consider two points located at $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, separated by distance $r_{12} = |\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|$. Find a vector $\vec{\mathbf{A}}$ from the origin to the point on the line between $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ at a distance xr_{12} from the point at $\vec{\mathbf{r}}_1$, where x is some number.

Solution: Consider the unit vector pointing from $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ given by $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 / |\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2| = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 / r_{12}$. The vector $\vec{\mathbf{a}}$ in the figure connects $\vec{\mathbf{A}}$ to the point at $\vec{\mathbf{r}}_1$, therefore we can write $\vec{\mathbf{a}} = xr_{12}\hat{\mathbf{r}}_{12} = xr_{12}(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 / r_{12}) = x(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)$. The vector $\vec{\mathbf{r}}_1 = \vec{\mathbf{A}} + \vec{\mathbf{a}}$. Therefore $\vec{\mathbf{A}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{a}} = \vec{\mathbf{r}}_1 - x(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) = \vec{\mathbf{r}}_1(1 - x) + x\vec{\mathbf{r}}_2$.

3.5 Dot Product

We shall now introduce a new vector operation, called the “dot product” or “scalar product” that takes any two vectors and generates a scalar quantity (a number). We shall see that the physical concept of work can be mathematically described by the dot product between the force and the displacement vectors.

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Since any two non-collinear vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure 3.16. Note that θ can vary from 0 to π .

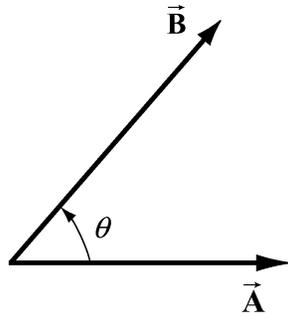


Figure 3.16 Dot product geometry.

Definition: Dot Product

The dot product $\vec{A} \cdot \vec{B}$ of the vectors \vec{A} and \vec{B} is defined to be product of the magnitude of the vectors \vec{A} and \vec{B} with the cosine of the angle θ between the two vectors:

$$\vec{A} \cdot \vec{B} = AB \cos(\theta) \tag{3.5.1}$$

Where $A = |\vec{A}|$ and $B = |\vec{B}|$ represent the magnitude of \vec{A} and \vec{B} respectively. The dot product can be positive, zero, or negative, depending on the value of $\cos \theta$. The dot product is always a scalar quantity.

The angle formed by two vectors is therefore

$$\theta = \cos^{-1} \left(\frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} \right) \tag{3.5.2}$$

The magnitude of a vector \vec{A} is given by the square root of the dot product of the vector \vec{A} with itself.

$$|\vec{A}| = (\vec{A} \cdot \vec{A})^{1/2} \tag{3.5.3}$$

We can give a geometric interpretation to the dot product by writing the definition as

$$\vec{A} \cdot \vec{B} = (A \cos(\theta)) B \tag{3.5.4}$$

In this formulation, the term $A \cos \theta$ is the projection of the vector \vec{B} in the direction of the vector \vec{A} . This projection is shown in Figure 3.17(a). So the dot product is the product of the projection of the length of \vec{A} in the direction of \vec{B} with the length of \vec{B} . Note that we could also write the dot product as

$$\vec{A} \cdot \vec{B} = A(B \cos(\theta)) \quad (3.5.5)$$

Now the term $B \cos(\theta)$ is the projection of the vector \vec{B} in the direction of the vector \vec{A} as shown in Figure 3.17(b). From this perspective, the dot product is the product of the projection of the length of \vec{B} in the direction of \vec{A} with the length of \vec{A} .

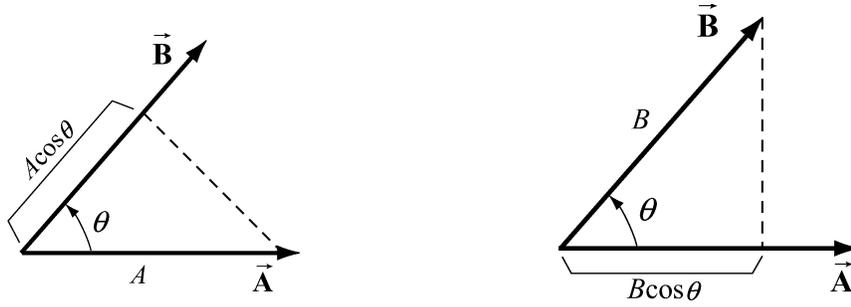


Figure 3.17(a) and 2.17(b) Projection of vectors and the dot product.

From our definition of the dot product we see that the dot product of two vectors that are perpendicular to each other is zero since the angle between the vectors is $\pi/2$ and $\cos(\pi/2) = 0$.

Algebraic Properties of Dot Product

The first property involves the dot product between a vector $c\vec{A}$ where c is a scalar and a vector \vec{B} ,

$$(1a) \quad c\vec{A} \cdot \vec{B} = c(\vec{A} \cdot \vec{B}) \quad (3.5.6)$$

The second involves the dot product between the sum of two vectors \vec{A} and \vec{B} with a vector \vec{C} ,

$$(2a) \quad (\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C} \quad (3.5.7)$$

Since the dot product is a commutative operation,

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}, \quad (3.5.8)$$

similar definitions hold;

$$(1b) \quad \vec{A} \cdot c\vec{B} = c(\vec{A} \cdot \vec{B}) \quad (3.5.9)$$

$$(2b) \quad \vec{C} \cdot (\vec{A} + \vec{B}) = \vec{C} \cdot \vec{A} + \vec{C} \cdot \vec{B}. \quad (3.5.10)$$

Vector Decomposition and the Dot Product

With these properties in mind we can now develop an algebraic expression for the dot product in terms of components. Let's choose a Cartesian coordinate system with the vector \vec{B} pointing along the positive x -axis with positive x -component B_x , *i.e.*, $\vec{B} = B_x \hat{i}$. The vector \vec{A} can be written as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad (3.5.11)$$

We first calculate that the dot product of the unit vector \hat{i} with itself is unity:

$$\hat{i} \cdot \hat{i} = |\hat{i}| |\hat{i}| \cos(0) = 1 \quad (3.5.12)$$

since the unit vector has magnitude $|\hat{i}| = 1$ and $\cos(0) = 1$. We note that the same rule applies for the unit vectors in the y and z directions:

$$\hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \quad (3.5.13)$$

The dot product of the unit vector \hat{i} with the unit vector \hat{j} is zero because the two unit vectors are perpendicular to each other:

$$\hat{i} \cdot \hat{j} = |\hat{i}| |\hat{j}| \cos(\pi / 2) = 0 \quad (3.5.14)$$

Similarly, the dot product of the unit vector \hat{i} with the unit vector \hat{k} , and the unit vector \hat{j} with the unit vector \hat{k} are also zero:

$$\hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0 \quad (3.5.15)$$

The dot product of the two vectors now becomes

$$\begin{aligned} \vec{A} \cdot \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot B_x \hat{i} \\ &= A_x \hat{i} \cdot B_x \hat{i} + A_y \hat{j} \cdot B_x \hat{i} + A_z \hat{k} \cdot B_x \hat{i} \quad \text{property (2a)} \\ &= A_x B_x (\hat{i} \cdot \hat{i}) + A_y B_x (\hat{j} \cdot \hat{i}) + A_z B_x (\hat{k} \cdot \hat{i}) \quad \text{property (1a) and (1b)} \\ &= A_x B_x \end{aligned} \quad (3.5.16)$$

This third step is the crucial one because it shows that it is only the unit vectors that undergo the dot product operation.

Since we assumed that the vector $\vec{\mathbf{B}}$ points along the positive x -axis with positive x -component B_x , our answer can be zero, positive, or negative depending on the x -component of the vector $\vec{\mathbf{A}}$. In Figure 3.18, we show the three different cases.

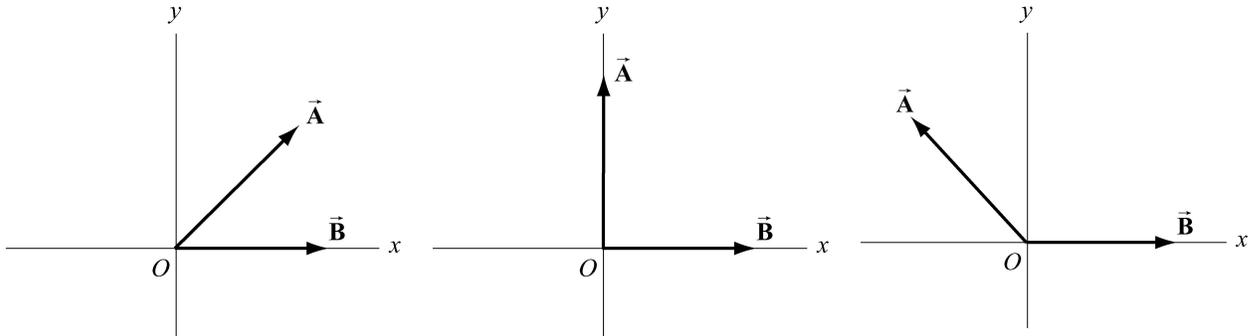


Figure 3.18 Dot product that is (a) positive, (b) zero or (c) negative.

The result for the dot product can be generalized easily for arbitrary vectors

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (3.5.17)$$

and

$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}} \quad (3.5.18)$$

to yield

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z \quad (3.5.19)$$

Example 3: Given two vectors, $\vec{\mathbf{A}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$.

$$\begin{aligned} \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} &= A_x B_x + A_y B_y + A_z B_z \\ &= (2)(5) + (3)(1) + (7)(2) = 21 \end{aligned}$$

Example 4: Find the cosine of the angle between the vectors $\vec{\mathbf{A}} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

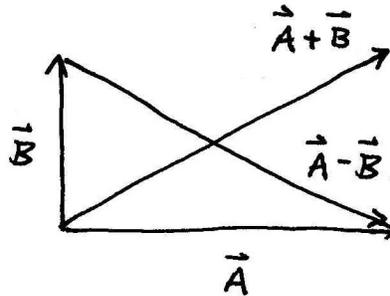
Solution: The dot product of two vectors is equal to $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\cos\theta$ where θ is the angle between the two vectors. Therefore

$$\begin{aligned}
\cos \theta &= \vec{A} \cdot \vec{B} / |\vec{A}| |\vec{B}| \\
&= (A_x B_x + A_y B_y + A_z B_z) / (A_x^2 + A_y^2 + A_z^2)^{1/2} (B_x^2 + B_y^2 + B_z^2)^{1/2} \\
&= ((3)(-2) + (1)(-3) + (1)(-1)) / ((3)^2 + (1)^2 + (1)^2)^{1/2} ((-2)^2 + (-3)^2 + (-1)^2)^{1/2} \\
&= (-10) / (11)^{1/2} (14)^{1/2} = -0.806
\end{aligned}$$

Example 5: Show that if $|\vec{A} - \vec{B}| = |\vec{A} + \vec{B}|$, then \vec{A} is perpendicular to \vec{B} .

Solution: Recall that $|\vec{A} - \vec{B}| = ((\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}))^{1/2} = ((\vec{A} \cdot \vec{A} - 2\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}))^{1/2}$.

Similarly $|\vec{A} + \vec{B}| = ((\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}))^{1/2} = ((\vec{A} \cdot \vec{A} + 2\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}))^{1/2}$. So if $|\vec{A} - \vec{B}| = |\vec{A} + \vec{B}|$, then $\vec{A} \cdot \vec{B} = 0$ hence \vec{A} is perpendicular to \vec{B} .



Work:

A typical physics application of the dot product between two vectors involves the calculation of the work done by a force \vec{F} on an object that undergoes a displacement $\Delta \vec{r}$. If the force is uniform in space and constant in time during the entire displacement, then the work done by the force on the object is given by

$$W_F = \vec{F} \cdot \Delta \vec{r} \quad (3.5.20)$$

3.6 Cross Product

We shall now introduce our second vector operation, called the “cross product” that takes any two vectors and generates a new vector. The cross product is a type of “multiplication” law that turns our vector space (law for addition of vectors) into a vector algebra (a vector algebra is a vector space with an additional rule for multiplication of vectors). The first application of the cross product will be the physical concept of torque about a point P that can be described mathematically by the cross product between two vectors: a vector from P to where the force acts, and the force vector.

Definition: Cross Product

Let \vec{A} and \vec{B} be two vectors. Since any two non-parallel vectors form a plane, we define the angle θ to be the angle between the vectors \vec{A} and \vec{B} as shown in Figure 3.19. The magnitude of the cross product $\vec{A} \times \vec{B}$ of the vectors \vec{A} and \vec{B} is defined to be product of the magnitude of the vectors \vec{A} and \vec{B} with the sine of the angle θ between the two vectors,

$$|\vec{A} \times \vec{B}| = AB \sin(\theta), \quad (3.6.1)$$

where A and B denote the magnitudes of \vec{A} and \vec{B} , respectively. The angle θ between the vectors is limited to the values $0 \leq \theta \leq \pi$ insuring that $\sin(\theta) \geq 0$.

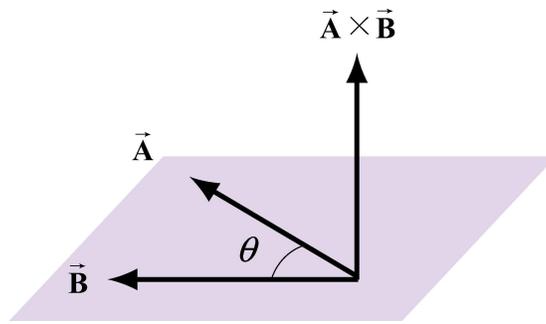


Figure 3.19 Cross product geometry.

The direction of the cross product is defined as follows. The vectors \vec{A} and \vec{B} form a plane. Consider the direction perpendicular to this plane. There are two possibilities: we shall choose one of these two (shown in Figure 3.19) for the direction of the cross product $\vec{A} \times \vec{B}$ using a convention that is commonly called the “right-hand rule”.

Right-hand Rule for the Direction of Cross Product

The first step is to redraw the vectors \vec{A} and \vec{B} so that their tails are touching. Then draw an arc starting from the vector \vec{A} and finishing on the vector \vec{B} . Curl your right fingers the same way as the arc. Your right thumb points in the direction of the cross product $\vec{A} \times \vec{B}$ (Figure 3.20).

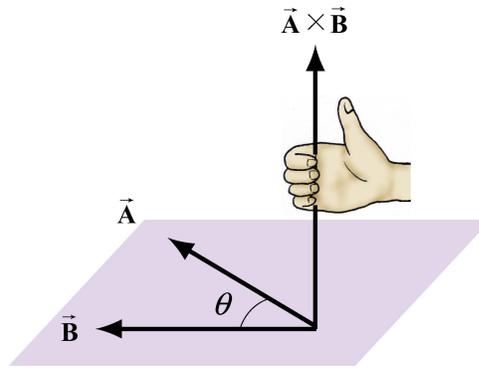


Figure 3.20 Right-Hand Rule.

You should remember that the direction of the cross product $\vec{A} \times \vec{B}$ is perpendicular to the plane formed by \vec{A} and \vec{B} .

We can give a geometric interpretation to the magnitude of the cross product by writing the magnitude as

$$|\vec{A} \times \vec{B}| = A(B \sin \theta) \quad (3.6.2)$$

The vectors \vec{A} and \vec{B} form a parallelogram. The area of the parallelogram is equal to the height times the base, which is the magnitude of the cross product. In Figure 3.22, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure 3.21(a), the term $B \sin(\theta)$ is the projection of the vector \vec{B} in the direction perpendicular to the vector \vec{A} . We could also write the magnitude of the cross product as

$$|\vec{A} \times \vec{B}| = (A \sin(\theta))B \quad (3.6.3)$$

Now the term $A \sin(\theta)$ is the projection of the vector \vec{A} in the direction perpendicular to the vector \vec{B} as shown in Figure 3.21(b).

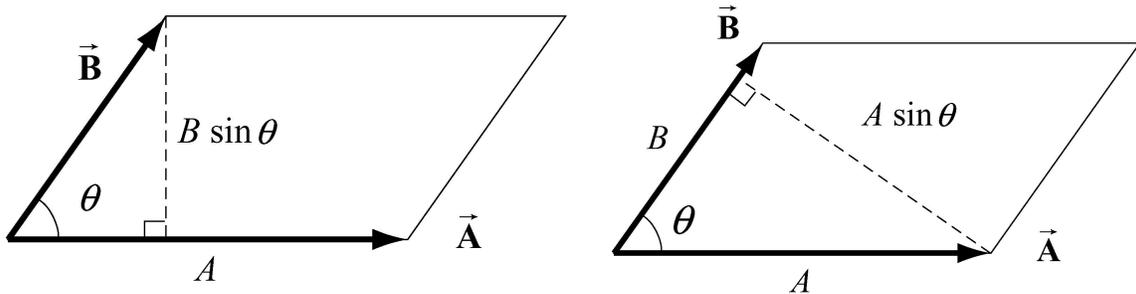


Figure 3.21(a) and 3.21(b) Projection of vectors and the cross product

The cross product of two vectors that are parallel (or anti-parallel) to each other is zero since the angle between the vectors is 0 (or π) and $\sin(0) = 0$ (or $\sin(\pi) = 0$). Geometrically, two parallel vectors do not have a unique component perpendicular to their common direction.

Properties of the Cross Product

- (1) The cross product is anti-commutative since changing the order of the vectors cross product changes the direction of the cross product vector by the right hand rule:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}} \quad (3.6.4)$$

- (2) The cross product between a vector $c\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\vec{\mathbf{A}} \times \vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \quad (3.6.5)$$

Similarly,

$$\vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \quad (3.6.6)$$

- (3) The cross product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}} \quad (3.6.7)$$

Similarly,

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}} \quad (3.6.8)$$

Vector Decomposition and the Cross Product

We first calculate that the magnitude of cross product of the unit vector $\hat{\mathbf{i}}$ with $\hat{\mathbf{j}}$:

$$|\hat{\mathbf{i}} \times \hat{\mathbf{j}}| = |\hat{\mathbf{i}}| |\hat{\mathbf{j}}| \sin(\pi / 2) = 1 \quad (3.6.9)$$

since the unit vector has magnitude $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = 1$ and $\sin(\pi / 2) = 1$. By the right hand rule, the direction of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is in the $+\hat{\mathbf{k}}$ as shown in Figure 3.22. Thus $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$.

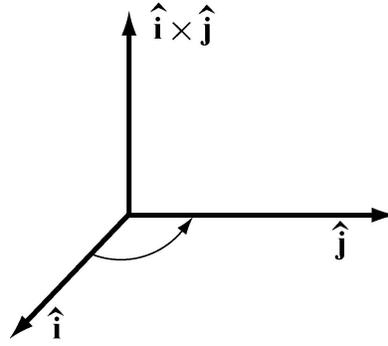


Figure 3.22 Cross product of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the y and z directions,

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \quad (3.6.10)$$

Note that by the anti-commutativity property (1) of the cross product,

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \quad (3.6.11)$$

The cross product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, ($\sin(0) = 0$),

$$|\hat{\mathbf{i}} \times \hat{\mathbf{i}}| = |\hat{\mathbf{i}}| |\hat{\mathbf{i}}| \sin(0) = 0 \quad (3.6.12)$$

The cross product of the unit vector $\hat{\mathbf{j}}$ with itself and the unit vector $\hat{\mathbf{k}}$ with itself are also zero for the same reason.

$$|\hat{\mathbf{j}} \times \hat{\mathbf{j}}| = 0, \quad |\hat{\mathbf{k}} \times \hat{\mathbf{k}}| = 0 \quad (3.6.13)$$

With these properties in mind we can now develop an algebraic expression for the cross product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive x -axis with positive x -component B_x . Then the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (3.6.14)$$

and

$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} \quad (3.6.15)$$

respectively. The cross product in vector components is

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}) \times B_x \hat{\mathbf{i}} \quad (3.6.16)$$

This becomes, using properties (3) and (2),

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_x \hat{\mathbf{i}} \times B_x \hat{\mathbf{i}}) + (A_y \hat{\mathbf{j}} \times B_x \hat{\mathbf{i}}) + (A_z \hat{\mathbf{k}} \times B_x \hat{\mathbf{i}}) \\ &= A_x B_x (\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_y B_x (\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + A_z B_x (\hat{\mathbf{k}} \times \hat{\mathbf{i}}) \\ &= -A_y B_x \hat{\mathbf{k}} + A_z B_x \hat{\mathbf{j}}\end{aligned}\quad (3.6.17)$$

The vector component expression for the cross product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (3.6.18)$$

and

$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}} \quad (3.6.19)$$

to yield

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y) \hat{\mathbf{i}} + (A_z B_x - A_x B_z) \hat{\mathbf{j}} + (A_x B_y - A_y B_x) \hat{\mathbf{k}}. \quad (3.6.20)$$

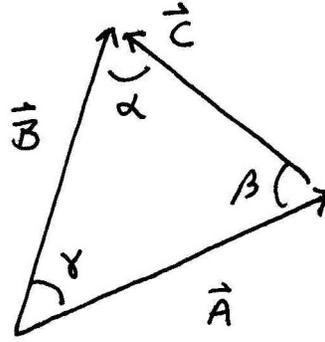
Example 6: Given two vectors, $\vec{\mathbf{A}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$.

Solution:

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_y B_z - A_z B_y) \hat{\mathbf{i}} + (A_z B_x - A_x B_z) \hat{\mathbf{j}} + (A_x B_y - A_y B_x) \hat{\mathbf{k}} \\ &= ((-3)(2) - (7)(1)) \hat{\mathbf{i}} + ((7)(5) - (2)(2)) \hat{\mathbf{j}} + ((2)(1) - (-3)(5)) \hat{\mathbf{k}} \\ &= -13 \hat{\mathbf{i}} + 31 \hat{\mathbf{j}} + 17 \hat{\mathbf{k}}\end{aligned}$$

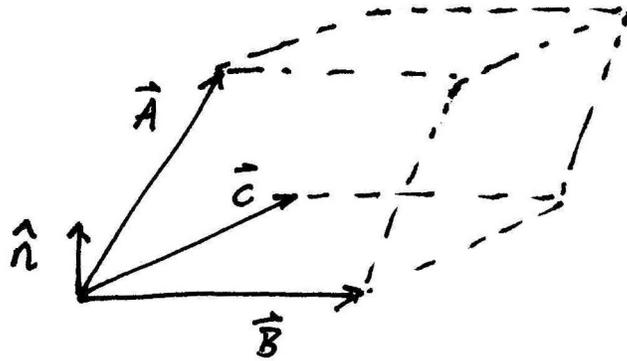
Example 6: Law of Sines: Prove that $|\vec{\mathbf{B}}|/\sin \beta = |\vec{\mathbf{A}}|/\sin \alpha$ and $|\vec{\mathbf{B}}|/\sin \beta = |\vec{\mathbf{C}}|/\sin \gamma$ using the cross product. (Hint: Consider the area of a triangle formed by three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$, where $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = 0$.)

Solution: Since $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = 0$, we have that $0 = \vec{\mathbf{A}} \times (\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$. Thus $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{A}} \times \vec{\mathbf{C}}$ or $|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = |\vec{\mathbf{A}} \times \vec{\mathbf{C}}|$.



From the figure we see that $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin \gamma$ and $|\vec{A} \times \vec{C}| = |\vec{A}||\vec{C}|\sin \beta$. Therefore $|\vec{A}||\vec{B}|\sin \gamma = |\vec{A}||\vec{C}|\sin \beta$, and hence $|\vec{B}|/\sin \beta = |\vec{C}|/\sin \gamma$. A similar argument shows that $|\vec{B}|/\sin \beta = |\vec{A}|/\sin \alpha$ proving the law of sines.

Example 8: Show that the volume of a parallelepiped with edges formed by the vectors \vec{A} , \vec{B} , and \vec{C} is given by $\vec{A} \cdot (\vec{B} \times \vec{C})$.



Solution: The volume of a parallelepiped is given by area of the base times height. If the base is formed by the vectors \vec{B} and \vec{C} , then the area of the base is given by the magnitude of $\vec{B} \times \vec{C}$. The vector $\vec{B} \times \vec{C} = |\vec{B} \times \vec{C}|\hat{n}$ where \hat{n} is a unit vector perpendicular to the base. The projection of the vector \vec{A} along the direction \hat{n} gives the height of the parallelepiped. This projection is given by taking the dot product of \vec{A} with a unit vector and is equal to $\vec{A} \cdot \hat{n} = \text{height}$. Therefore $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{A} \cdot (|\vec{B} \times \vec{C}|\hat{n}) = (|\vec{B} \times \vec{C}|)\vec{A} \cdot \hat{n} = (\text{area})(\text{height}) = (\text{volume})$.

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