Newtonian Mechanics
8.01

Units and Dimensional Analysis

Dimensions in Mechanics

Physical quantities have dimensions.

These quantities are the basic dimensions:

- mass, length, time with dimension symbols M, L, T

Other quantities' dimensions are more complex:

- [velocity] = length/time = LT^{-1}
- [force] = (mass)(length) /(time)^2 = MLT^{-2}
- [any mechanical quantity] = $M^a L^b T^c$ where a b, and c can be negative and/or non-integer

Base Quantities

Name	Symbol for quantity	Symbol for dimension	SI base unit
Length	l	**L**	meter
Time	t	**T**	second
Mass	m	**M**	kilogram
Electric current	I	**I**	ampere
Thermodynamic Temperature	T	**Θ**	kelvin
Amount of substance	n	**N**	mole
Luminous intensity	I_V	**J**	candela

SI Base Units:

Second: The *second* (s) is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

Meter: The *meter* (m) is now defined as the distance traveled by a light wave in vacuum in 1/299,792,458 seconds.

Mass: The SI standard of mass is a platinum-iridium cylinder assigned a mass of 1 kg

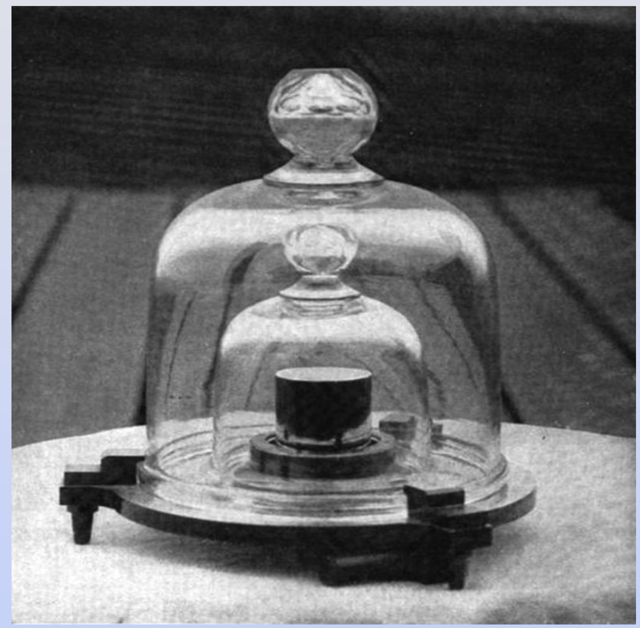

Image courtesy of the National Bureau of Standards

Speed of Light

In 1983 the General Conference on Weights and Measures defined the *speed of light* to be the best measured value at that time:

$$c = 299,792,458 \text{ meters/second}$$

This had the effect that length became a derived quantity, but the meter was kept around for practicality

Meter

The meter was originally defined as 1/10,000,000 of the arc from the Equator to the North Pole along the meridian passing through Paris.

To aid in calibration and ease of comparison, the meter was redefined in terms of a length scale etched into a platinum bar preserved near Paris.

Once laser light was engineered, the meter was redefined to be a certain number of wavelengths of a particular monochromatic laser beam.

The *meter* (m) is now defined as the distance traveled by a light wave in vacuum in 1/299,792,458 seconds.

Worked Example: Proportions of the Standard Kilogram

The standard kilogram is a cylindrical alloy of 90 % platinum and 10 % iridium. The density of alloy is

$$\rho = 21.56 \text{ g} \cdot \text{cm}^{-3}$$

Design a **strategy** for finding the optimal height and radius for the standard kilogram keeping in mind that the surface is occasionally cleaned of unwelcome atoms (dust). You don't have to solve this.

Proportions of Standard Kilogram

Strategy; Since atoms collect on the surface, chose the radius and height to minimize surface area.

Constant volume for cylinder:

$$V = \pi r^2 h$$

The surface area is

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2V / r$$

Minimize the area with respect to radius:

$$dA / dr = 4\pi r - 2V / r^2 = 0$$

Radius is one half height:

$$V = 2\pi r_0^3 \quad \Rightarrow \quad r_0 = h / 2$$

The volume determined from density

$$V = m / \rho \cong 1000 \text{ g} / 22 \text{ g} \cdot \text{cm}^{-3} \cong 46.38 \text{cm}^3$$

The radius is

$$r_0 = \left(V / 2\pi \right)^{1/3} \quad 1.95 \text{ cm}$$

The height is

$$h = 2r_0 \quad 3.90 \text{ cm}$$

Fundamental and Derived Quantities: Dimensions and Units

The dimensions of (new) physical quantities follow from the equations that involve them

$$F = ma$$

implies that

$$[\text{Force}] = \mathrm{M}^1\,\mathrm{L}^1\,\mathrm{T}^{-2} = \mathrm{M}\,\mathrm{L}\,\mathrm{T}^{-2}$$

Since we use force so often, we define new units to measure it: Newtons, Pounds, Dynes, Troy Oz.

Worked Example: Dimensions

Determine the dimensions of the following mechanical quantities:

1. momentum
2. pressure
3. kinetic energy

Worked Example Solution: Dimensions

Determine the dimensions of the following mechanical quantities:

$$[\text{momentum}] = (\text{mass})(\text{velocity}) = \text{M L T}^{-1}$$

$$[\text{pressure}] = [\text{force/area}] = \text{M L T}^{-2} / \text{L}^2 = \text{M L}^{-1}\,\text{T}^{-2}$$

$$[\text{kinetic energy}] = [(\text{mass})(\text{velocity})^2] = \text{M}(\text{L T}^{-1})^2 = \text{M L}^2\,\text{T}^{-2}$$

Checkpoint Problem: Dimensions

Determine the dimensions of the following mechanical quantities:

1. Work

2. power

Dimensional Analysis: Strategy

When trying to find a dimensionally correct formula for a quantity from a set of given quantities, an answer that is dimensionally correct will scale properly and is generally off by a constant of order unity

Since:

[desired quantity] = $M^{\alpha} L^{\beta} T^{\gamma}$ where α β, and γ are known

Combine the given quantities correctly so that:

[desired quantity] = $M^{\alpha} L^{\beta} T^{\gamma}$ = $(\text{given1})^{X} (\text{given2})^{Y} (\text{given3})^{Z}$

- solve for X, Y, Z to match correct dimensions of desired quantity

Checkpoint Problem Dimensional Analysis: Period of a Pendulum

The length *l* of a simple pendulum, the mass *m* of the pendulum bob, the gravitational acceleration *g* and the angular amplitude of the bob are all possible quantities that may enter into a relationship for the period of the pendulum swing. Using dimensional analysis, find (up to a dimensionless multiplicative function) a function

$$T_{\text{period}} = f\left(l, m, g, \theta_0\right)$$

for the time it takes the pendulum to complete one full swing (the *period* of the pendulum).

Checkpoint Problem: Dimensional Analysis

The speed of a sail-boat or other craft that does not plane is limited by the wave it makes – it can't climb uphill over the front of the wave. What is the maximum speed you'd expect?

Hint: relevant quantities might be the length *l* of the boat, the density ρ of the water, and the gravitational acceleration *g.*

$$v_{\text{boat}} = l^{X} \rho^{Y} g^{Z}$$

Hint: Dimensions of quantities that may describe the maximum speed for boat

Name of Quantity	Symbol	Dimension
Maximum speed	v	LT^{-1}
density	ρ	ML^{-3}
Gravitational acceleration	g	LT^{-2}
Length	l	L