Units and Dimensional Analysis
Challenge Problem Solutions

Problem 1: An ideal (non-viscous) liquid with a density of ρ is poured into a cylindrical vessel with a cross-sectional area of A_1 to a level at a height h from the bottom. The bottom has an opening with a cross-sectional area A_2. Find the time it takes the k=liquid to flow out.

Problem 1 Solution:

The time should be a function of

$$t = b\rho^V h^W g^X A_1^{\,Y} A_2^{\,Z}\,. \tag{1.1}$$

Therefore the dimension must satisfy the relation

$$[T] = [\rho]^V [h]^W [g]^X [A_1]^Y [A_2]^Z\,. \tag{1.2}$$

Using the fact density has dimensions $[\rho] = [M \cdot L^{-3}]$ Eq. (1.2) becomes

$$[T] = [M \cdot L^{-3}]^V [L]^W [L \cdot T^{-2}]^X [L^2]^Y [L^2]^Z\,. \tag{1.3}$$

Therefore we have the following set of algebraic relations

$$0 = -3V + W + X + 2(Y + Z)\,, \tag{1.4}$$

$$0 = V\,, \tag{1.5}$$

$$1 = -2X\,. \tag{1.6}$$

We have five unknowns and three equations but two of the unknowns can be determined immediately

$$X = -1/2,\ \ V = 0, \tag{1.7}$$

So the time does not depend on the density of the liquid. To proceed further we must make some additional assumptions. We can assume that the speed of the liquid leaving the container does not depend on the cross-sectional area of the hole. This implies that the flow rate Φ through the hole (volume per second) is equal to the speed v times the cross-sectional area A_2,

$$\Phi = vA_2\,. \tag{1.8}$$

In addition we shall make the approximation that the flow rate remains essentially constant so that the total volume hA_1 must be equal to the flow rate times the time it takes for the volume to leave the container,

$$hA_1 \approx \Phi t = vA_2 t \,. \tag{1.9}$$

Since we do not know the relationship between the speed and the height, we can only conclude that

$$t \approx A_1 / A_2 \,. \tag{1.10}$$

So therefore the coefficients $Y = 1$, $Z = -1$. So we can solve Eq. (1.4) for the remaining coefficient $W = -X = 1/2$. Therefore the time that it takes the liquid to leave the container is

$$t = b\left(\frac{h}{g}\right)^{1/2} \frac{A_1}{A_2}$$

Problem 2: Non-Uniform Acceleration: *Terminal Velocity of Raindrop*

A raindrop of initial mass m_0 starts falling from rest under the influence of gravity. If we assume the air resistance is proportional to the square of the velocity, the resulting acceleration is given by the equation

$$\frac{dv}{dt} = g - k\,v^2 \tag{2.1}$$

where $g = 9.8\ \mathrm{m \cdot s^{-2}}$ and k is a constant. Note that the positive direction is downward.

a) What are the dimensions of k?

b) What is the terminal velocity of the raindrop? Make an order of magnitude estimate for the constant k based on your experience. Do all raindrops fall at the same terminal velocity? On which quantities do you think the terminal velocity may depend? A nice simulation of falling raindrops can be found at raindrop terminal velocity.

Problem 2 Solution:

a) The product of k and the square of the speed must have dimensions of acceleration;

$$\begin{aligned} \dim[k] \cdot \frac{\mathrm{L}^2}{\mathrm{T}^2} &= \frac{\mathrm{L}}{\mathrm{T}^2} \\ \dim[k] &= \frac{1}{\mathrm{L}} = \mathrm{L}^{-1}, \end{aligned} \tag{2.2}$$

inverse length.

b) The terminal velocity is given by the condition that the acceleration is zero,

$$0 = g - k\,{v_{\mathrm{term}}}^2 . \tag{2.3}$$

Solving Equation (2.3), we find that

$$v_{\mathrm{term}} = \sqrt{\frac{g}{k}} . \tag{2.4}$$

The terminal velocity of a raindrop will vary depending on the size of the drop. A nice simulation can be found at raindrop terminal velocity. For a typical raindrop (diameter $0.2\ \mathrm{cm}$), the terminal velocity is approximately $5\ \mathrm{m \cdot s^{-1}}$. Then

$$k = \frac{g}{{v_{\mathrm{term}}}^2} \simeq \frac{10\ \mathrm{m \cdot s^{-2}}}{(5\ \mathrm{m \cdot s^{-1}})^2} = 0.4\ \mathrm{m^{-1}} . \tag{2.5}$$

Problem 3: Dimensional Analysis Solution

The speed of a sail-boat or any other craft that does not plane is limited by the wave it makes – it can't climb uphill over the front of the wave. What is the maximum speed you'd expect?

Hint: relevant quantities might be the length l of the boat, the density ρ of the water, and the gravitational acceleration g:

$$v_{\text{boat}} \sim l^X \rho^Y g^Z \tag{3.1}$$

Problem 3 Solution: The dimensions of the three quantities: the length of the boat, the density of water, and the gravitational acceleration are respectively $[l]=\text{L}$, $[\rho]=\text{ML}^{-3}, [g]=\text{LT}^{-2}$. The dimensions of the speed of the boat is $[v_{\text{boat}}]=\text{LT}^{-1}$. Since this is independent of mass, the speed of the boat cannot depend on the density of water, so the coefficient $Y = 0$. Therefore the dimensions of

$$v_{\text{boat}} = l^X g^Z \tag{3.2}$$

become

$$\text{LT}^{-1} = (\text{L}^X)(\text{LT}^{-2})^Z. \tag{3.3}$$

Comparing powers we see that

$$-2\text{Z} = -1 \tag{3.4}$$

and

$$\text{X} + \text{Z} = 1. \tag{3.5}$$

Equations (3.4) and (3.5) can be solved simultaneously yielding

$$\text{X} = \text{Z} = 1/2. \tag{3.6}$$

Thus by dimensional analysis, the maximum speed in SI units is proportional to

$$v_{\text{boat}} \sim \sqrt{lg}\,. \tag{3.7}$$

In in SI units Eq. (3.7) becomes

$$v_{\text{boat}} \sim (3.1\,\text{m}^{1/2}\text{s}^{-1})\sqrt{l} \tag{3.8}$$

The maximum speed of a single-hull displacement boat in SI units is given by[1]

$$v_{\text{boat}} = (1.25\,\text{m}^{1/2}\text{s}^{-1})\sqrt{l} \tag{3.9}$$

where l length of the hull at the waterline is measured in meters. So our estimation based on dimensional analysis requires a coefficient

$$v_{\text{boat}} = 0.40\sqrt{lg} \tag{3.10}$$

for the single-hull displacement boat**.**

[1] Rousmaniere, John The Annapolis Book of Seamanship Boat Selection. Chapter 1 p35 Simon & Schuster, New York, New York.