Module 1: Units and Significant Figures

1.1 The Speed of Light

When we observe and measure phenomena in the world, we try to assign numbers to the physical quantities with as much accuracy as we can possibly obtain from our measuring equipment. For example, we may want to determine the speed of light, which we can calculate by dividing the distance a known ray of light propagates over its travel time,

$$\text{speed of light} = \frac{\text{distance}}{\text{time}}. \tag{1.1.1}$$

In 1983 the General Conference on Weights and Measures defined the *speed of light* to be

$$c = 299{,}792{,}458 \text{ meters/second}. \tag{1.1.2}$$

This number was chosen to correspond to the most accurately measured value of the speed of light and is well within the experimental uncertainty.

1.2 International System of System of Units

The three quantities – time, length, and the speed of light – are directly intertwined. Which quantities should we consider as "base" and which ones as "derived" from the base quantities? For example, are length and time base quantities while speed is a derived quantity?

This question is answered by convention. The basic system of units used throughout science and technology today is the internationally accepted *Système International* (SI). It consists of seven base quantities and their corresponding base units:

Mechanics is based on just the first three of these quantities, the MKS or meter-kilogram-second system. An alternative metric system to this, still widely used, is the so-called CGS system (centimeter-gram-second). So far as distance and time measurements are concerned, there is also wide use of British Imperial units (especially in the USA) based on the foot (ft), the mile (mi), etc., as units of length, and also making use of the minute, hour, day and year as units of time.

Base Quantity	Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electric Current	ampere (A)
Temperature	Kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

We shall refer to the dimension of the base quantity by the quantity itself, for example

$$\text{dim length} \equiv \text{length} \equiv \text{L}, \text{dim mass} \equiv \text{mass} \equiv \text{M}, \text{dim time} \equiv \text{time} \equiv \text{T}. \qquad (1.2.1)$$

1.3 The Atomic Clock and the Definition of the Second

Isaac Newton, in the *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), distinguished between time as duration and an absolute concept of time,

> "Absolute true and mathematical time, of itself and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year. "[1].

The development of clocks based on atomic oscillations allowed measures of timing with accuracy on the order of 1 part in 10^{14}, corresponding to errors of less than one microsecond (one millionth of a second) per year. Given the incredible accuracy of this measurement, and clear evidence that the best available timekeepers were atomic in nature, the *second* (s) was redefined in 1967 by the International Committee on Weights and Measures as a certain number of cycles of electromagnetic radiation emitted by cesium atoms as they make transitions between two designated quantum states:

> *The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.*

1.4 The meter

The meter was originally defined as 1/10,000,000 of the arc from the Equator to the North Pole along the meridian passing through Paris. To aid in calibration and ease of comparison, the meter was redefined in terms of a length scale etched into a platinum bar

[1] Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 6.

preserved near Paris. Once laser light was engineered, the meter was redefined by the 17th Conférence Générale des Poids et Mèsures (CGPM) in 1983 to be a certain number of wavelengths of a particular monochromatic laser beam.

The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

Example 1: Light Year Astronomical distances are sometimes described in terms of *light-years* (ly). A light-year is the distance that light will travel in one year (yr). How far in meters does light travel in one year?

Solution: Using the relationship $\text{distance} = (\text{speed of light}) \cdot (\text{time})$, one light year corresponds to a distance. Since the speed of light is given in terms of meters per second, we need to know how many seconds are in a year. We can accomplish this by converting units. We know that

$$1\text{ year} = 365.25\text{ days},\ 1\text{ day} = 24\text{ hours},\ 1\text{ hour} = 60\text{ minutes},\ 1\text{ minute} = 60\text{ seconds}$$

Putting this together we find that the number of seconds in a year is

$$1\text{ year} = (365.25\text{ day})\left(\frac{24\text{ hours}}{1\text{ day}}\right)\left(\frac{60\text{ min}}{1\text{ hour}}\right)\left(\frac{60\text{ s}}{1\text{ min}}\right) = 31{,}557{,}600\text{ s}. \qquad (1.4.1)$$

So the distance that light travels in a one year is

$$1\ \text{ly} = \left(\frac{299{,}792{,}458\text{ m}}{1\text{ s}}\right)\left(\frac{31{,}557{,}600\text{ s}}{1\text{ yr}}\right)(1\text{ yr}) = 9.461\times 10^{15}\text{ m}. \qquad (1.4.2)$$

The distance to the nearest star, Alpha Centauri, is three light years.

A standard astronomical unit is the parsec. One parsec is the distance at which there is one arcsecond = 1/3600 degree angular separation between two objects that are separated by the distance of one astronomical unit, $1\text{AU} = 1.50\times 10^{11}\text{ m}$, which is the mean distance between the earth and sun. One astronomical unit is roughly equivalent to eight light minutes, $1\text{AU} = 8.31\text{-min}$ One parsec is equal to 3.26 light years, where one light year is the distance that light travels in one earth year, $1\text{pc} = 3.26\text{ly} = 2.06\times 10^{5}\text{ AU}$ where $1\text{ly} = 9.46\times 10^{15}\text{ m}$.

1.5 Mass

The unit of mass, the kilogram (kg), remains the only base unit in the International System of Units (SI) that is still defined in terms of a physical artifact, known as the "International Prototype of the Standard Kilogram." The prototype was made in 1879 by George Matthey (of Johnson Matthey) in the form of a cylinder, 39 mm high and 39 mm in diameter, consisting of an alloy of 90 % platinum and 10 % iridium. The international prototype is kept at the Bureau International des Poids et Mesures (BIPM) at Sevres, France under conditions specified by the 1st Conférence Générale des Poids et Mèsures (CGPM) in 1889 when it sanctioned the prototype and declared "This prototype shall henceforth be considered to be the unit of mass." It is stored at atmospheric pressure in a specially designed triple bell-jar. The prototype is kept in a vault with six official copies.

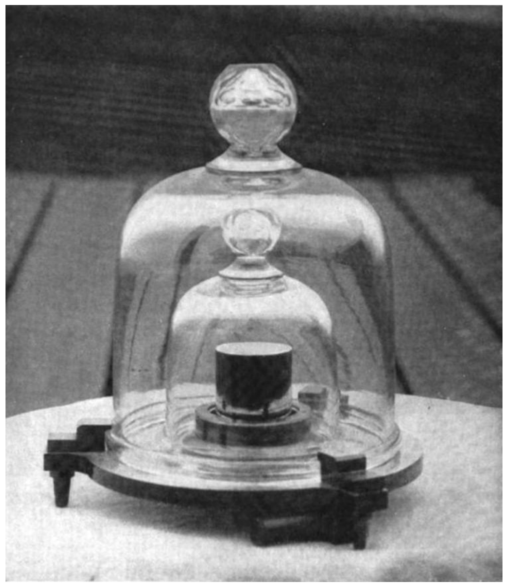

Image courtesy of the National Bureau of Standards.

Figure 1.1 International Prototype of the Standard Kilogram

The 3rd CGPM (1901), in a declaration intended to end the ambiguity in popular usage concerning the word "weight" confirmed that:

> *The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.*

There is a stainless steel one-kilogram standard that can travel for comparisons. In practice it is more common to quote a conventional mass value (or weight-in-air, as measured with the effect of buoyancy), than the standard mass. Standard mass is normally only used in specialized measurements wherever suitable copies of the prototype are stored.

Example 2: The International Prototype Kilogram Determine the type of shape and dimensions of the platinum-iridium prototype kilogram such that it has the smallest surface area for a given volume. The standard kilogram is an alloy of 90 % platinum and 10 % iridium. The density of the alloy is $\rho = 21.56 \text{ g} \cdot \text{cm}^{-3}$. You may want to consider the following questions:

1) Is there any reason that the surface area of the standard could be important?

2) What is the appropriate density to use?

3) What shape (that is, sphere, cube, right cylinder, parallelepiped, etc.) has the smallest surface area for a given volume?

4) Why was a right-circular cylinder chosen?

Solution: The standard kilogram is an alloy of 90 % platinum and 10 % iridium. The density of platinum is $21.45\ \text{g} \cdot \text{cm}^{-3}$ and the density of iridium is $22.55\ \text{g} \cdot \text{cm}^{-3}$. Thus the density of the standard kilogram, $\rho = 21.56\ \text{g} \cdot \text{cm}^{-3}$, and its volume is

$$V = m/\rho \cong 1000\ \text{g} / 22\ \text{g} \cdot \text{cm}^{-3} \cong 46.38\ \text{cm}^3. \tag{1.5.1}$$

Corrosion would affect the mass through chemical reaction; platinum and iridium were chosen for the standard's composition as they resist corrosion.

To further minimize corrosion, the shape should be chosen to have the least surface area. Ideally, this would be a sphere, but as spheres roll easily they become impractical, whereas cylinders have flat surfaces which prevent this. The volume for a cylinder or radius r and height h is a constant and given by

$$V = \pi r^2 h. \tag{1.5.2}$$

The surface area can be expressed in terms of the radius r as

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2V}{r}. \tag{1.5.3}$$

To find the smallest surface area, minimize the area with respect to the radius

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2} = 0. \tag{1.5.4}$$

Solve for the radius

$$r^3 = \frac{V}{2\pi} = \frac{\pi r^2 h}{2\pi}. \tag{1.5.5}$$

Thus the radius is one half the height,

$$r = \frac{h}{2}. \tag{1.5.6}$$

For the standard mass, the radius is

$$r = \left(\frac{V}{2\pi}\right)^{1/3} = \left(\frac{46.38 \text{ cm}^3}{2\pi}\right)^{1/3} \cong 1.95 \text{ cm}. \tag{1.5.7}$$

Twice this radius is the diameter of the standard kilogram.

Alternative Definition of Mass

Since the prototype kilogram is an artifact, there are some intrinsic problems associated with its use as a standard. It may be damaged, or destroyed. The prototype gains atoms due to environment wear and cleaning, at a rate of change of mass corresponding to approximately $1\ \mu\text{g} / \text{year}$ ($1\ \mu\text{g} \equiv 1 \text{microgram} \equiv 1 \times 10^{-6}$ g).

Several new approaches to defining the SI unit of mass (kg) are currently being explored. One possibility is to define the kilogram as a fixed number of atoms of a particular substance, thus relating the kilogram to an atomic mass. Silicon is a good candidate for this approach because it can be grown as a large single crystal, in a very pure form.

Example 3: Mass of a Silicon Crystal

A given standard unit cell of silicon has a volume V_0 and contains N_0 atoms. The number of molecules in a given mole of substance is given by Avogadro's constant $N_A = 6.0221415 \times 10^{23} \text{ mole}^{-1}$. The molar mass of silicon is given by M_{molar}. Find the mass m of a volume V in terms of V_0, N_0, V, M_{molar}, and N_A.

Solution: The mass m_0 of the unit cell is the density ρ of silicon cell multiplied by the volume of the cell V_0,

$$m_0 = \rho V_0. \tag{1.5.8}$$

The number of moles in the unit cell is the total mass, m_0, of the cell, divided by the molar mass M_{molar},

$$n_0 = m_0 / M_{\text{molar}} = \rho V_0 / M_{\text{molar}}. \tag{1.5.9}$$

The number of atoms in the unit cell is the number of moles n_0 times the Avogadro constant, N_A,

$$N_0 = n_0 N_A = \frac{\rho V_0 N_A}{M_{\text{molar}}} \tag{1.5.10}$$

The density of the crystal is related to the mass m of the crystal divided by the volume V of the crystal,

$$\rho = m / V \tag{1.5.11}$$

So the number of atoms in the unit cell can be expressed as

$$N_0 = \frac{m V_0 N_A}{V M_{\text{molar}}} \tag{1.5.12}$$

So the mass of the crystal is

$$m = \frac{M_{\text{molar}}}{N_A} \frac{V}{V_0} N_0 \tag{1.5.13}$$

The molar mass, unit cell volume and volume of the crystal can all be measured directly. Notice that M_{molar} / N_A is the mass of a single atom, and $(V / V_0) N_0$ is the number of atoms in the volume. This approach is therefore reduced to the problem of measuring the Avogadro constant, N_A, with a relative uncertainty of 1 part in 10^8, which is equivalent to the uncertainty in the present definition of the kilogram.

1.6 Radians and Steradians

Radians

Consider the triangle drawn in Figure 1.6.1

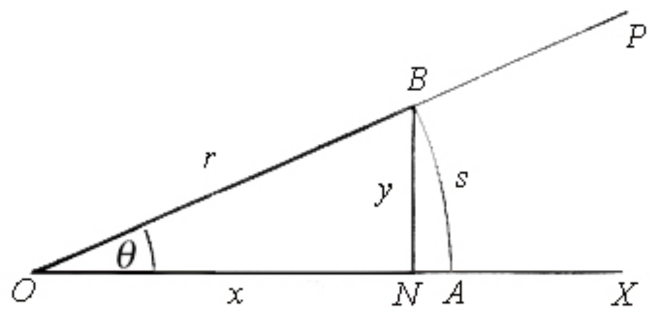

Figure 1.2 Trigonometric relations

You know the basic trigonometric functions of an angle θ in a right-angled triangle ONB:

$$\sin(\theta) = y / r , \quad (1.6.1)$$

$$\cos(\theta) = x / r , \quad (1.6.2)$$

$$\tan(\theta) = y / x \quad (1.6.3)$$

It is very important to become familiar with using the measure of the angle θ itself as expressed in *radians* [rad]. Let θ be the angle between two straight lines OX and OP. If we draw a circle of any radius r centered at O, the lines OP and OX cut the circle at the points A and B where $OA = OB = r$. If the length of the arc AB is s, the radian measure of θ is given by

$$\theta = s / r ,$$

and is the same for circles of all radii centered at O -- just as the ratios y / r and y / x are the same for all right triangles with the angle θ at O. As θ approaches 360°, s approaches the complete circumference $2\pi r$ of the circle, so that $360^{\circ} = 2\pi$ rad .

Let's compare the behavior of $\sin(\theta)$, $\tan(\theta)$ and θ itself for small angles. One can see from the diagram that $s / r > y / r$. It is less obvious that $y / x > \theta$. It is very instructive to plot $\sin(\theta)$, $\tan(\theta)$, and θ as functions of θ [rad] between 0 and $\pi / 2$ on the same graph (see Figure 1.3).

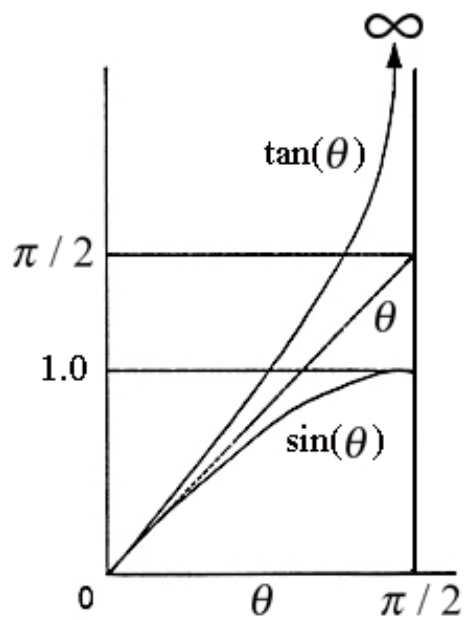

Figure 1.3 Radians compared to trigonometric functions.

For small θ, the values of all three functions are almost equal. But how small is "small"? An acceptable condition is for $\theta << 1$ in radians. We can show this with a few examples.

Example 4: Small Angle Approximation

Since $360^{\circ} = 2\pi \text{ rad}$, $57.3^{\circ} = 1\text{ rad}$, so an angle $6^{\circ} \cong (6^{\circ})(2\pi \text{ rad} / 360^{o}) \cong 0.1 \text{ rad}$ when expressed in radians. Use your pocket calculator to verify the following values of $\sin(\theta)$ and $\tan\theta$ to 4-digit accuracy for $\theta \cong 0.1\text{rad}$:

$$\begin{aligned} \theta\,[\text{rad}] &= \quad 0.1000 \\ \sin(\theta) &= \quad 0.0998 \\ \tan(\theta) &= \quad 0.1003. \end{aligned}$$

So the spread of values in this case is less than $\pm 0.3\%$. Again using your calculator, fill in the blanks below for $\theta = 15^{\circ}$, which is about equal to 0.25 rad:

$$\begin{aligned} \theta\,[\text{rad}] &= \quad 0.2618 \\ \sin(\theta) &= \\ \tan(\theta) &= \qquad\qquad . \end{aligned}$$

You see that provided θ is not too large

$$\sin(\theta) \simeq \tan(\theta) \simeq \theta \qquad (1.6.4)$$

can be used almost interchangeably, within some small percentage error. This is the basis of many useful approximations in physics calculations.

Steradians

The steradian (sr) is the unit of solid angle that, having its vertex in the center of a sphere, cuts off an area of the surface of the sphere equal to that of a square with sides of length equal to the radius of the sphere. The conventional symbol for steradian measure is Ω the uppercase greek "Omega." The total solid angle Ω_{sphere} of a sphere is then found by dividing the surface area of the sphere by the square of the radius,

$$\Omega_{\text{sphere}} = 4\pi r^2 / r^2 = 4\pi \tag{1.6.5}$$

Note that this result is independent of the radius of the sphere. Note also that it was implied that the solid angle was measured from the center of the sphere (the radius r is constant). It turns out that the above result does not depend on the position of the vertex as long as the vertex is inside the sphere.

"The SI unit, *candela*, is the luminous intensity of a source that emits monochromatic radiation of frequency $540 \times 10^{12}\ \text{s}^{-1}$, in a given direction, and that has a radiant intensity in that direction of 1/683 watts per steradian." Note that "in a given direction" cannot be taken too literally. The intensity is measured per steradian of spread, so if the radiation has no spread of directions, the luminous intensity would be infinite.

1.7 Dimensions of Commonly Encountered Quantities

Many physical quantities are derived from the base quantities by set of algebraic relations defining the physical relation between these quantities. The dimension of the derived quantity is written as a power of the dimensions of the base quantitites,

For example velocity is a derived quantity and the dimension is given by the relationship

$$\text{dim velocity} = \text{(length)/(time)} = \text{L} \cdot \text{T}^{-1}. \tag{1.6.6}$$

where $\text{L} \equiv \text{length}$, $\text{T} \equiv \text{time}$.

Force is also a derived quantity and has dimension

$$\text{dim force} = \frac{\text{(mass)(dim velocity)}}{\text{(time)}} \ . \tag{1.6.7}$$

where $M \equiv \text{mass}$. We could express force in terms of mass, length, and time by the relationship

$$\text{dim force} = \frac{(\text{mass})(\text{length})}{(\text{time})^2} = \text{M} \cdot \text{L} \cdot \text{T}^{-2}. \qquad (1.6.8)$$

The derived dimension of kinetic energy is

$$\text{dim kinetic energy} = (\text{mass})(\text{dim velocity})^2, \qquad (1.6.9)$$

which in terms of mass, length, and time is

$$\text{dim kinetic energy} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^2} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} \qquad (1.6.10)$$

The derived dimension of work is

$$\text{dim work} = (\text{dim force})(\text{length}), \qquad (1.6.11)$$

which in terms of our fundamental dimensions is

$$\text{dim work} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^2} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} \qquad (1.6.12)$$

So work and kinetic energy have the same dimensions.

Power is defined to be the rate of change in time of work so the dimensions are

$$\text{dim power} = \frac{\text{dim work}}{\text{time}} = \frac{(\text{dim force})(\text{length})}{\text{time}} = \frac{(\text{mass})(\text{length})^2}{(\text{time})^3} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-3} \qquad (1.6.13)$$

In Table 1.1 we include the derived dimensions of some common mechanical quantities in terms of mass, length, and time.

Table 1.1 Dimensions of Some Common Mechanical Quantities

$\mathrm{M} \equiv$ mass, $\mathrm{L} \equiv$ length, $\mathrm{T} \equiv$ time

Quantity	Dimension	MKS unit
Angle	dimensionless	Dimensionless = radian
Steradian	dimensionless	Dimensionless = radian^2
Area	L^2	m^2
Volume	L^3	m^3
Frequency	T^{-1}	s^{-1} = hertz = Hz
Velocity	$\mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{m} \cdot \mathrm{s}^{-1}$
Acceleration	$\mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{m} \cdot \mathrm{s}^{-2}$
Angular Velocity	T^{-1}	$\mathrm{rad} \cdot \mathrm{s}^{-1}$
Angular Acceleration	T^{-2}	$\mathrm{rad} \cdot \mathrm{s}^{-2}$
Density	$\mathrm{M} \cdot \mathrm{L}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^{-3}$
Momentum	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$
Angular Momentum	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}$
Force	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$ = newton = N
Work, Energy	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$ = joule = J
Torque	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
Power	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$ = watt = W
Pressure	$\mathrm{M} \cdot \mathrm{L}^{-1} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}$ = pascal = Pa

Dimensional Analysis

There are many phenomena in nature that can be explained by simple relationships between the observed phenomena.

Example 5: Period of a Pendulum

Consider a simple pendulum consisting of a massive bob suspended from a fixed point by a string. Let T_{period} denote the time (period of the pendulum) that it takes the bob to complete one cycle of oscillation. How does the period of the simple pendulum depend on the quantities that define the pendulum and the quantities that determine the motion?

Solution:

What possible quantities are involved? The length of the pendulum l, the mass of the pendulum bob m, the gravitational acceleration g, and the angular amplitude of the bob

θ_0 are all possible quantities that may enter into a relationship for the period of the swing. Have we included every possible quantity? We can never be sure but let's first work with this set and if we need more than we will have to think harder!

Our problem is then to find a function f such that

$$T_{\text{period}} = f\left(l, m, g, \theta_0\right) \tag{1.6.14}$$

We first make a list of the dimensions of our quantities as shown in Table 1.2. Choose the set: mass, length, and time, to use as the base dimensions.

Table 1.2 Dimensions of quantities that may describe the period of pendulum

Name of Quantity	Symbol	Dimensional Formula
Time of swing	t	T
Length of pendulum	l	L
Mass of pendulum	m	M
Gravitational acceleration	g	$\mathrm{L} \cdot \mathrm{T}^{-2}$
Angular amplitude of swing	θ_0	No dimension

Our first observation is that the mass of the bob cannot enter into our relationship, as our final quantity has no dimensions of mass and no other quantity can remove the dimension of the pendulum mass. Let's focus on the length of the string and the gravitational acceleration. In order to eliminate length, these quantities must divide each other in the above expression for T_{period} must divide each other. If we choose the combination l / g, the dimensions are

$$\dim[l / g] = \frac{\text{length}}{\text{length/(time)}^2} = (\text{time})^2 \tag{1.6.15}$$

It appears that the time of swing is proportional to the square root of this ratio. We have an argument that works for our choice of constants, which depend on the units we choose for our fundamental quantities. Thus we have a candidate formula

$$T_{\text{period}} \sim \left(\frac{l}{g}\right)^{1/2} \tag{1.6.16}$$

(in the above expression, the symbol ": " represents a proportionality, not an approximation).

Since the angular amplitude θ_0 is dimensionless, it may or may not appear. We can account for this by introducing some function $y(\theta_0)$ into our relationship, which is beyond the limits of this type of analysis. Then the time of swing is

$$T_{\text{period}} = y(\theta)\left(\frac{l}{g}\right)^{1/2} \tag{1.6.17}$$

We shall discover later on that $y(\theta_0)$ is nearly independent of the angular amplitude θ_0 for very small amplitudes and is equal to $y(\theta_0) = 2\pi$,

$$T_{\text{period}} = 2\pi\left(\frac{l}{g}\right)^{1/2} \tag{1.6.18}$$