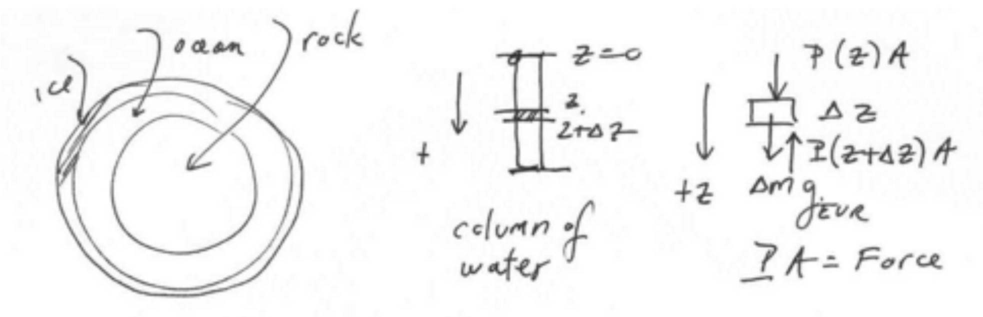
Problem Solving and Estimation
Challenge Problem Solutions

Problem 1: ***Fermi Problem***

One of the moons of Jupiter, Europa, is reported to have its surface covered by an ocean of water which is 100 km deep. The outermost 8 km are frozen as ice. The radius of Europa is approximately 1/4 the radius of the earth. Estimate the pressure at the bottom of Europa's ocean. (Note: there is some speculation that the combination of internal heat and water makes the ocean of Europa the best candidate in the solar system outside the earth for organized life to evolve.

Problem 1 Solution



$$+\hat{z} = p(z)A + \Delta mg - p(z+\Delta z)A = 0 \tag{1.1}$$

$$\Delta mg = (p(z+\Delta z) - p(z))A \tag{1.2}$$

$$\Delta m = \rho_{H_2O} A \Delta z \tag{1.3}$$

Equation (1.2):

$$\rho g A \Delta z = \left(p\left(z+\Delta z\right) - p\left(z\right)\right)A \tag{1.4}$$

$$\rho g = \frac{d\rho}{dz} \tag{1.5}$$

Integrate:

$$\int_{z=0}^{z} \rho g \, dz' = \int_{P_o \cong 0}^{P(z)} dP \tag{1.6}$$

$$\rho g z = P(z) - P_o \tag{1.7}$$

Pressure decreases with depth z:

$$F_{grav,earth} = \frac{GM_E m}{R_E^2}$$

$$F_{grav,europa} = \frac{GmM_{eur}}{R_{eur}^2}$$

$$P(z) = \rho g z \qquad (1.8)$$

$$\frac{F_{grav,earth}}{F_{grav,eur}} = \frac{\frac{GM_E m}{R_E^2}}{\frac{GM_{eur} m}{R_{eur}^2}} = \frac{{}^{1}R_{eur}^2}{R_{earth}^2}\frac{m_E}{m_{eur}} \qquad (1.9)$$

$$g_{eur} = \frac{R_{eur}}{R_{earth}} g_{earth} \simeq \frac{1}{4} g_{earth} \qquad (1.10)$$

$$\frac{g_{earth}}{g_{eur}} \cong \frac{R_{earth}}{R_{eur}} \qquad (1.11)$$

assume $4\ R_{eur} \cong R_{earth}$

$$P(z) \cong \rho_{water} g_{eur} z \cong \rho_{water} \frac{1}{4} g_{earth} z \qquad (1.12)$$

$$P(100km) = \left(\frac{10^3 kg}{m^3}\right)\left(\frac{1}{4}\right)\left(90\frac{m}{s^2}\right)\left(10^5 m\right) \simeq 2.5 \times 10^8 Pa \qquad (1.13)$$

$$\left(P_{atm}\right)_{earth} \simeq \frac{14lb}{in^2} = \frac{14lb}{in^2}\left(\frac{1kg}{2.2lb}\right)\left(9.8\frac{m}{s^2}\right)\left(\frac{in^2}{\left(2.54 \times 10^{-2} m\right)^2}\right) \qquad (1.14)$$

$$\left(P_{atm}\right)_{earth} \simeq 1.0 \times 10^5 Pa \qquad (1.15)$$

Problem 2: Estimating Distances

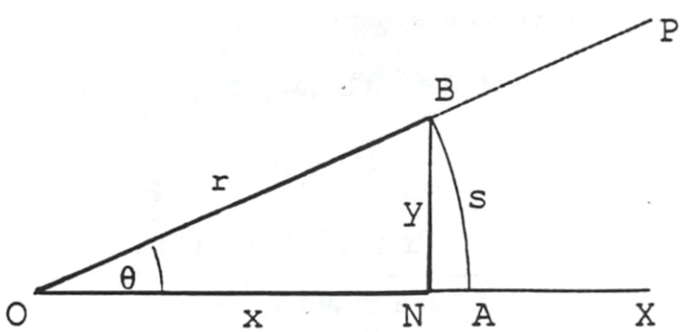
a) Hold a dime at arm's length. What angle in radians is subtended by the diameter of the dime?

b) Using your result from part a), estimate the length of the infinite corridor at MIT. In order to do this, choose a reference height at one end of the corridor and estimate its height. Then go to the other end of the corridor and measure what fraction of the diameter of the dime corresponds to your reference height. You can now calculate the length of the corridor by using similar triangles. The length is published by MIT. Can you find the published value?

c) Now use your dime and wait until the moon is out (the moon is full on Sept. 10) to try an estimate the angular diameter of the moon. Once you have this estimate, what additional information would you need in order to estimate the mass of the moon? Make some estimates regarding these additional quantities and then estimate the mass of the moon. Look up the actual value and compare it with your estimate. How did you do?

Problem 2 Solutions:

a) When I hold a dime at arm's length, the distance from my eye to my fingertips is $x \approx 71\,\text{cm}$ (see figure below.) The diameter of the dime (greatly exaggerated in the figure) is $y \cong 1.8\,\text{cm}$.



The radian measure of � is given by

$$�= \frac{s}{r} \approx \frac{y}{x} = \frac{1.8\,\text{cm}}{71\,\text{cm}} = 2.5 \times 10^{-2}\,\text{rad}\,. \tag{2.1}$$

b) I used a doorway at the end of the infinite corridor (building 8) as a reference height which I measured to be $h = 9.0\,\text{ft} = 2.7\,\text{m}$. I measured the size of this height from the other end of the corridor compared to the dime and found it was approximately $1/2$ of the diameter of the dime or $y \approx 1\,\mathrm{cm}$. Therefore the angular diameter of the door is approximately

$$\theta \cong \frac{y}{x} = \frac{1\,\mathrm{cm}}{71\,\mathrm{cm}} = 1.4 \times 10^{-2}\,\mathrm{rad} \approx 1 \times 10^{-2}\,\mathrm{rad} \tag{2.2}$$

to the one significant figure allowed by the estimated subtended part of the dime.

The length of the infinite corridor is then

$$d \approx \frac{h}{\theta} \approx \frac{2.7\,\mathrm{m}}{1 \times 10^{-2}\,\mathrm{rad}} \approx 2.7 \times 10^{2}\,\mathrm{m}\,. \tag{2.3}$$

In the MIT Bulletin 1999-2000, Page 10, the published result for the length of the infinite corridor is $8.25 \times 10^{2}\,\mathrm{ft} = 2.51 \times 10^{2}\,\mathrm{m}$; our estimate is

$$\frac{2.7 \times 10^{2}\,\mathrm{m}}{2.5 \times 10^{2}\,\mathrm{m}} \cong 1.1 \tag{2.4}$$

of the actual value.
If you don't have an old copy of the MIT Bulletin, try MIT Course Catalogue: Overview.

c) I measured the moon and found the diameter of the moon corresponded to approximately $1/2$ of the diameter of the dime or $y \approx 1\,\mathrm{cm}$. Note that this means that if you stand at one end of the infinite corridor and look at the sun or full moon it will entirely fill the doorway. Try it out. This is best done on the second or third floor. There are two days a year when the sun sets exactly in front of the infinite corridor. See MIT Infinite Corridor Astromony - MIThenge.
The angular diameter of the moon is then the same as found in Equation (2.2),

$$\theta \cong \frac{y}{x} = \frac{1\,\mathrm{cm}}{71\,\mathrm{cm}} = 1.4 \times 10^{-2}\,\mathrm{rad} \approx 1 \times 10^{-2}\,\mathrm{rad}\,.$$

Estimating the distance from the earth to the moon to be approximately $R_{\mathrm{e,m}} \cong 3 \times 10^{8}\,\mathrm{m}$, the diameter of the moon is estimated as

$$D = R_{\mathrm{e,m}}\theta \cong \left(3 \times 10^{8}\,\mathrm{m}\right)\left(1.4 \times 10^{-2}\,\mathrm{rad}\right) = 4 \times 10^{6}\,\mathrm{m}\,. \tag{2.5}$$

The volume of the moon is

$$V_{\mathrm{m}} = \frac{4}{3}\pi\left(\frac{D}{2}\right)^{3} = \frac{\pi}{6}D^{3} \approx \frac{\pi}{6}\left(4 \times 10^{6}\,\mathrm{m}\right)^{3} \approx 3 \times 10^{19}\,\mathrm{m}^{3}\,. \tag{2.6}$$

Estimate the density of the moon to be $\rho_{\mathrm{m}} \approx 5\times10^{3}\ \mathrm{kg}\cdot\mathrm{m}^{-3}$, similar to the density of rocks found on Earth. The mass of the moon is then

$$m_{\mathrm{m}} = \rho_{\mathrm{m}} V_{\mathrm{m}} \approx \left(5\times10^{3}\ \mathrm{kg}\cdot\mathrm{m}^{-3}\right)\left(3\times10^{19}\ \mathrm{m}^{3}\right) = 1.5\times10^{23}\ \mathrm{kg}\,. \tag{2.7}$$

The accepted standard value for the mass of the moon is $7.35\times10^{22}\ \mathrm{kg}$, so our estimate is about twice the standard value.

Problem 3: ***Ice Age***

In a recent publication of Nature, Australian scientists determined that during the last ice age (22,000 to 19,000 years ago) the sea level reached its low point, 425 to 440 feet below the present level due to the change of seawater to glacial ice. What approximate volume of ice would this correspond to?

Please do not immediately search for the answer on the web (the problem will not be graded by the answer but by your approach). Outline a strategy for your estimation i.e. describe any assumptions that you make or estimations, identify any relevant quantities that you may need to make a reasonable estimate. The goal here is to base your estimates only on information that you already know without having to look anything up on the web or in some textbook. Then calculate your result. Decide by comparison to other volumes whether you think your estimate is reasonable.

Problem 3 Solution:

Our strategy will be to start off by modeling the volume of the dropped sea as a spherical shell. We then add two corrections. We estimate how much of the area of the earth is covered by seas and finally we estimate the expansion factor due to the fact that ice is less dense than seawater.

In order to estimate the volume of ice a drop in sea level corresponds to we must first estimate how the volume of seawater that is changed to ice. Since the radius of Earth $R_e = 6.4 \times 10^6 \text{ m}$ is much greater than the change in height of $420 \sim 440\,\text{ft}$ (which corresponds to approximately $\Delta h \sim 130 \pm 3\text{m}$), the volume of a spherical shell of seawater is approximately equal to the area of the shell, $A_{\text{shell}} = 4\pi R_e^{\,2}$, times the thickness and is given by

$$V_{shell} \cong A_{\text{shell}}\,\Delta h = 4\pi R_e^{\,2} \Delta h\,. \tag{3.1}$$

We also need to estimate how much of the Earth's surface is covered by seas. The sea surface area is approximately 70% of the total surface of the earth,

$$A_{\text{sea}} \cong (0.7) 4\pi\, R_{\text{earth}}^2 = 3.6 \times 10^{14}\text{ m}^2 \tag{3.2}$$

This corresponds to a volume

$$A_{\text{sea}}\,\Delta h = 4.6 \times 10^{16}\text{ m}^3\,. \tag{3.3}$$

We finally note that the density of ice is 0.92 that of seawater, and so the corresponding volume of ice is

$$V_{ice} \cong 5.0 \times 10^{16}\ \mathrm{m}^3 \tag{3.4}$$

Problem 4: ***Estimation: average density of matter in galaxy***

Estimate the average density of matter (average mass per unit volume) in the Milky Way Galaxy. Please do not immediately search for the answer on the web (the problem will not be graded by the answer but by your approach). Outline a strategy for your estimation, describe any assumptions that you make, estimate or look up or search any relevant quantities that you may need, and calculate your result. Then search the web for a current estimate and compare with your estimate. How accurate were you? In particular try to identify how accurate your estimate was of each of your relevant quantities.

Problem 4 Solution:

The average density of matter in the Milky Way galaxy as the total mass divided by the volume,

$$\bar{\rho} = \frac{total\ mass}{volume} \tag{4.1}$$

In order to estimate the density we need to make two estimates. We must first estimate the amount of mass is the galaxy and then we must estimate the volume of the galaxy.

Although you may not know this beforehand, the Milky Way is actually considered a giant sized spiral galaxy that is composed of stars, gas, and dust (called the luminous matter) and is surrounded by a halo of dark matter. There are approximately 200 billion to as many as 400 billion stars ($N_{stars} \simeq 2\times 10^{11}$ stars). Our sun is an average sized star with a mass that is approximately $m_{sun} \simeq 2\times 10^{30}$ kg (this is a nice quantity to know but you may want to look it up either in your textbook or on the web.) Most stars in the galaxy are smaller than the sun. We can estimate the amount of luminous matter contained in the stars, gas, and dust by multiplying the number of stars by the solar mass. Therefore we can estimate the amount of luminous matter to be $m_{luminous} \simeq N_{stars}\, m_{sun} = 2\times 10^{11}\, m_{sun}$. Current estimates for the mass of the luminous matter are about 3 to 5 times this amount or

$$mass\ of\ luminous\ mattter \simeq 5N_{stars}\, m_{sun} = 10^{12}\, m_{sun} \tag{4.2}$$

The mass of the dark matter in the dark halo is estimated to be about 10 to 20 times the mass of the luminous matter, so we estimate the mass of the Milky Way to be

$$\begin{aligned} mass\ of\ galaxy &\simeq 10\times 10^{12}\, m_{sun} = 10^{13}\, m_{sun} \\ &= (10^{13})(2\times 10^{30}\ \text{kg}) = 2\times 10^{43}\ \text{kg} \end{aligned} \tag{4.3}$$

The Milky Way galaxy consists of a spiral disk with a radius of about 60,000 light years, a galactic bulge consisting of older stars at the center of radius 6,000 light years, a galactic halo of radius 65,000 light years, and a dark matter spherical halo of radius approximately 300,000 to 400,000 light years.

If we estimate the volume of the spherical halo we get

$$\begin{aligned} \textit{Volume of dark halo} &\simeq \frac{4}{3}\pi R_{halo}{}^{3} = \frac{4}{3}\pi(3\times10^{5}\ \mathrm{ly})^{3}(1\times10^{16}\,\mathrm{m\ ly^{-1}})^{3} \\ &\simeq 1\times10^{65}\ \mathrm{m}^{3} \end{aligned} \tag{4.4}$$

Note that this is about 10,000 times the volume of the disk and bulge.
So we can estimate the average density of matter as

$$\bar{\rho} = \frac{\textit{total mass}}{\textit{volume}} \simeq \frac{2\times10^{43}\,\mathrm{kg}}{1\times10^{65}\ \mathrm{m}^{3}} = 2\times10^{-18}\,\mathrm{kg\ m^{-3}} \tag{4.5}$$

Note that a proton weights about $1.7\times10^{-27}\ \mathrm{kg}$. So the average density works out to be equivalent to

$$\frac{\bar{\rho}}{m_{proton}} = \frac{2\times10^{-18}\,\mathrm{kg\ m^{-3}}}{1.7\times10^{-27}\ \mathrm{kg}} \simeq 1\times10^{9}\ \mathrm{protons/m^{3}} \tag{4.6}$$

Problem 5: Mass in Universe

The density of luminous matter in the universe is currently estimated to be about $\left(5\times10^{-28}\,kg\cdot m^{-3}\right)$ which is about 1/10 the critical density necessary to keep the universe from expanding indefinitely. Estimate the total mass in the universe.

Problem 5 Solution:

Assume the universe is sphere with radius given by the latest estimates for the radius of the universe, 78 billion light years. This number is larger than 13.7 billion light years due to the expansion of the universe. Assume that the density of mass is the critical mass density $\left(5\times10^{-27}\,kg\cdot m^{-3}\right)$. Then the total mass is

$$\left(5\times10^{-27}\,kg\cdot m^{-3}\right)\frac{4\pi}{3}\left(\left(7.8\times10^{10}\,lyr\right)\left(9.461\text{ x }10^{15}\text{ meters}\cdot lyr^{-1}\right)\right)^{3}=8\times10^{54}\,kg$$

If we assumed no expansion, then

$$\left(5\times10^{-27}\,kg\cdot m^{-3}\right)\frac{4\pi}{3}\left(\left(1.37\times10^{10}\,lyr\right)\left(9.461\text{ x }10^{15}\text{ meters}\cdot lyr^{-1}\right)\right)^{3}=4\times10^{52}\,kg\,.$$

Remark:

Dark matter and dark energy account for the other 90% of mass in the universe.

Problem 6: Circular Motion Estimation

a) Estimate the speed of a MIT student in Room 26-152 undergoing circular motion about the earth's axis of rotation.

b) Estimate the magnitude of that student's acceleration.

c) Estimate the speed (in SI units) of the earth in orbit around the sun.

Problem 6 Solution:

a) The circumference of the earth is $2\pi R_{\mathrm{e}} \approx 4\times10^{7}\,\mathrm{m}$ (this was at one time the definition of a meter, more or less) and the period of the earth's rotation is $T_{\mathrm{e}} = 86,400\,\mathrm{s} \approx 10^{5}\,\mathrm{s}$ (for the purpose of this problem, we certainly won't distinguish between a solar day and a sidereal day), giving an equatorial speed of about $200\,\mathrm{m}\cdot\mathrm{s}^{-1}$. The fact that we're at about 42° north latitude would reduce this figure by about a factor of $3/4$, roughly the same as our overestimate for the period, so $200\,\mathrm{m}\cdot\mathrm{s}^{-1}$ should be an okay estimate.

b) Using $a_c = v^2/R$ would involve a factor of $\sin\left(42^{\circ}\right)$ twice in the numerator and once in the denominator, so we'll ignore that factor and say

$$a_c \approx \frac{\left(200\,\mathrm{m}\cdot\mathrm{s}^{-1}\right)^2}{10^{7}\,\mathrm{m}} = 4\times10^{-3}\,\mathrm{m}\cdot\mathrm{s}^{-2} \qquad (6.1)$$

c) The speed is $2\pi\,\mathrm{AU}\cdot\mathrm{yr}^{-1}$, where $1\,\mathrm{AU}=1.5\times10^{11}\,\mathrm{m}$ is an "astronomical unit," the mean earth-sun distance. In SI units, one year is quite close to
$1\,\mathrm{yr} \approx 10^{7.5}\,\mathrm{s} \approx 3\times10^{7}\,\mathrm{s} \approx \pi\times10^{7}\,\mathrm{s}$, for an orbital speed of $\approx 3\times10^{4}\,\mathrm{m}\cdot\mathrm{s}^{-1}$.
(For the source of these numbers, to far more accuracy than we need in this problem, see http://pdg.lbl.gov/2006/reviews/astrorpp.pdf.)