Module 8: Newton's Laws of Motion

8.1 Newton's First Law

The First Law of Motion, commonly called the "Principle of Inertia," was first realized by Galileo. (Newton did not acknowledge Galileo's contribution.) Newton was particularly concerned with how to phrase the First Law in Latin, but after many rewrites Newton perfected the following expression for the First Law (in English translation):

> *Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*
>
> *Projectiles continue in their motions, so far as they are not retarded by the resistance of air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by air. The greater bodies of planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.*

The first law is an experimental statement about the motions of bodies. When a body moves with constant velocity, there are either no forces present or there are forces acting in opposite directions that cancel out. If the body changes its velocity, then there must be an acceleration, and hence a total non-zero force must be present. We note that velocity can change in two ways. The first way is to change the magnitude of the velocity; the second way is to change its direction.

After a bus or train starts, the acceleration is often so small we can barely perceive it. We are often startled because it seems as if the station is moving in the opposite direction while we seem to be still. Newton's First Law states that there is no physical way to distinguish between whether we are moving or the station is, because there is essentially no total force present to change the state of motion. Once we reach a constant velocity, our minds dismiss the idea that the ground is moving backwards because we think it is impossible, but there is no actual way for us to distinguish whether the train is moving or the ground is moving.

8.2 Relatively Inertial Reference Frames

Suppose you choose two different reference frames moving with a relative velocity $\vec{\mathbf{V}}$ with respect to each other. If the relative velocity between the two reference frames is constant, then the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}}. \qquad (8.2.1)$$

When two reference frames are moving with a constant velocity relative to each other as above, the reference frames are considered to be *relatively inertial reference frames*.

We can reinterpret Newton's First Law (see Section 3.1)

Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

as the Principle of Relativity:

In relatively inertial reference frames, if there is no net force impressed on an object at rest in frame S, then there is also no net force impressed on the object in frame S′ .

8.3 Newton's Second Law

Newton's Second Law is possibly the most important experimental statement about motion in physics.

Law II: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

Suppose that a force is applied to a body for a time interval Δt. The impressed force or impulse produces a change in the *momentum* of the body,

$$\vec{\bar{\mathbf{I}}} = \overline{\vec{\mathbf{F}}}\Delta t = \Delta \vec{\mathbf{p}}\,, \tag{8.3.1}$$

where the momentum of a point object, discussed formally in Module ?, is $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$.

From the commentary to the second law, Newton also considered forces that were applied continually to a body instead of impulsively. The instantaneous action of the total force acting on a body at a time t is defining by taking the mathematical limit as the time interval Δt becomes smaller and smaller,

$$\vec{\mathbf{F}}^{\text{total}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{p}}}{\Delta t} \equiv \frac{d\vec{\mathbf{p}}}{dt}\,. \tag{8.3.2}$$

When the mass remains constant in time, the Second Law can be recast in its more familiar form,

$$\vec{\mathbf{F}}^{\text{total}} = m\frac{d\vec{\mathbf{v}}}{dt}. \tag{8.3.3}$$

Because the derivative of velocity is the acceleration, the force is the product of mass and acceleration,

$$\vec{\mathbf{F}}^{\text{total}} = m\vec{\mathbf{a}}. \tag{8.3.4}$$

8.4 Newton's Third Law: Action-Reaction Pairs

Newton realized that when two bodies interact via a force, then the force on one body is equal in magnitude and opposite in direction to the force acting on the other body.

> *Law III: To every action there is always opposed an equal reaction: or, the mutual action of two bodies upon each other are always equal, and directed to contrary parts.*
>
> *Whatever draws or presses another is as much drawn or pressed by that other. If you press on a stone with your finger, the finger is also pressed by the stone.*

The Third Law, commonly known as the "action-reaction" law, is the most surprising of the three laws. Newton's great discovery was that when two objects interact, they each exert the same magnitude of force on each other.

Consider two bodies engaged in a mutual interaction. Label the bodies 1 and 2 respectively. Let $\vec{\mathbf{F}}_{1,2}$ be the force on body 1 due to the interaction with body 2, and $\vec{\mathbf{F}}_{2,1}$ be the force on body 2 due to the interaction with body 1. These forces are depicted in Figure 8.3.

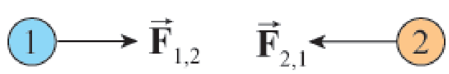


Figure 8.3 Action-reaction pair of forces

These two vector forces are equal in magnitude and opposite in direction,

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}. \tag{8.4.1}$$

Example: Free Fall

In Chapter 4, sections 4.6 and 4.8, we considered the motion of a body interacting with the earth's gravitational field near the surface of the earth. We used the experimental observation that the acceleration is directed downwards and has a magnitude $g = 9.8\ \text{m}\cdot\text{s}^{-2}$. We will now use Newton's Second Law and the force law for the gravitational interaction to "predict" the acceleration.

We shall again consider an object that is released at time $t = 0$ with an initial velocity $\vec{\mathbf{v}}_0$ at a height h above the ground. We will neglect all influences on the object except for the influence of gravity. Choose coordinates with the y-axis in the vertical direction with $\hat{\mathbf{j}}$ pointing upwards and the x-axis in the horizontal direction with $\hat{\mathbf{i}}$ pointing in the direction that the object is moving horizontally. Choose the origin to be at the point where the object is released. Figure 8.4 shows our coordinate system with the position of the object at time t and the coordinate functions $x(t)$ and $y(t)$.

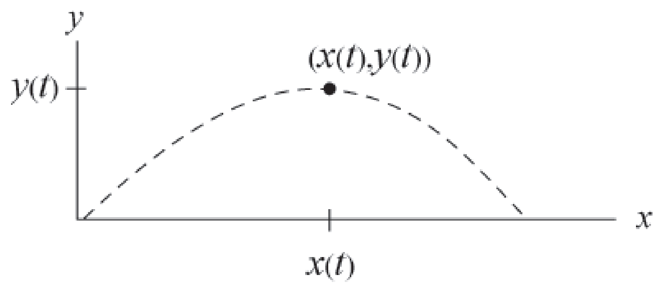


Figure 8.4 A coordinate sketch for parabolic motion.

Force Diagram

The only force acting on the object is the gravitational interaction between the object and the earth. This force acts downward with magnitude mg, where m is the mass of the object and $g = 9.8\ \text{m}\cdot\text{s}^{-2}$. Figure 8.5 shows the force diagram on the object.

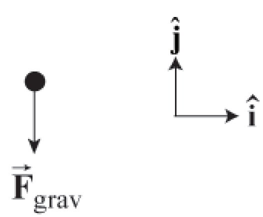


Figure 8.5 A force diagram on the object with the action of gravity

The vector decomposition of the force is

$$\vec{\mathbf{F}}_{\text{grav}} = -mg\,\hat{\mathbf{j}}. \qquad (8.4.2)$$

Should you include the force that gave the object its initial velocity on the force diagram? No! We are only interested in the forces acting on the object once the object has been released. It is a separate problem (and a very hard one at that) to determine exactly what forces acted on the object before the body was released. (This issue obscured the understanding of projectile motion for centuries.)

Second Law Analysis

The force diagram reminds us that the only force is acting in the y-direction. Newton's Second Law states that the total vector force $\vec{\mathbf{F}}^{\text{total}}$ acting on the object is equal to the product of the mass m and the acceleration vector $\vec{\mathbf{a}}$,

$$\vec{\mathbf{F}}^{\text{total}} = m\vec{\mathbf{a}}\,. \tag{8.4.3}$$

This is a vector equation; the components are equated separately:

$$F_y^{\text{total}} = m a_y\,, \tag{8.4.4}$$

$$F_x^{\text{total}} = m a_x\,. \tag{8.4.5}$$

Thus the vector equation in the y-direction becomes

$$-mg = m a_y\,. \tag{8.4.6}$$

Therefore the y-component of the acceleration is

$$a_y = -g. \tag{8.4.7}$$

We see that the acceleration is a constant and is independent of the mass of the object. Notice that $a_y < 0$. This is because we chose our positive y-direction to point upwards. The sign of the y-component of acceleration is determined by how we chose our coordinate system.

Since there are no horizontal forces acting on the object,

$$F_x^{\text{total}} = 0\,, \tag{8.4.8}$$

and we conclude that the acceleration in the horizontal direction is also zero,

$$a_x = 0\,. \tag{8.4.9}$$

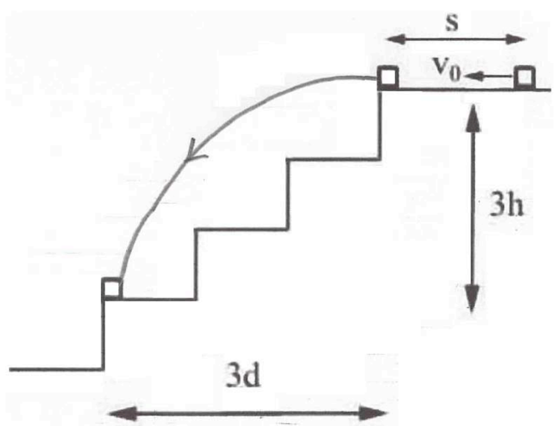
This tells us that the x-component of the velocity remains unchanged throughout the flight of the object.

Newton's Second Law provides an analysis that determines that the acceleration in the vertical direction is constant for all bodies independent of the mass of the object, thus confirming Galileo's Law of Freeing Falling Bodies. Notice that the force law Equation (8.4.2) generalizes the experimental observation that objects fall with constant acceleration. Our prediction is predicated on this force law and if subsequent observations show the acceleration is not constant they we either must include additional forces (for example, air resistance) or modify the force law (for objects that are no longer near the surface of the earth).

8.5 Worked Examples

Example 8.5.1: Staircase

An object of given mass m starts with a given velocity v_0 and slides an unknown distance s along a floor and then off the top of a staircase. The goal of this problem is to find the distance s. The coefficient of kinetic friction between the object and the floor is given by μ_k. The object strikes at the far end of the third stair. Each stair has a given rise of h and a given run of d. Neglect air resistance and use g for the gravitational constant.



a) Briefly describe how you intend to model the motion of the object. What are the given quantities in this problem?

b) What is the distance s that the object slides along the floor? Express your answer in terms of the given quantities only.

Answer:

a) There are two distinct stages to the object's motion, the initial horizontal motion (the floor must be assumed horizontal) and the motion in free fall. The given final position of the object, at the far end of the third stair, will determine the horizontal component of the velocity at the instant the object left the top of the stair. This in turn determined the time the object decelerated, and the deceleration while on the floor determined the distance traveled on the floor.

The given quantities are m , v_0 , μ_k , g , h and d .

b) From the top of the stair to the far end of the third stair, the object is in free fall. Take the positive $\hat{\mathbf{i}}$ -direction to be horizontal, directed to the left in the figure, take the positive $\hat{\mathbf{j}}$ -direction to be vertical (up) and take the origin at the top of the stair, where the object first goes into free fall. The components of acceleration are $a_x = 0$, $a_y = -g$, the initial x -component of velocity will be denoted $v_{x,0}$, the initial y -component of velocity is $v_{y,0} = 0$, the initial x -position is $x_0 = 0$ and the initial y -position is $y_0 = 0$. The equations describing the object's motion as a function of time t are then

$$x(t) = x_0 + v_{x,0}\,t + \frac{1}{2}a_x\,t^2 = v_{x,0}\,t \tag{8.5.1}$$

$$y(t) = y_0 + v_{y,0}\,t + \frac{1}{2}a_y\,t^2 = -\frac{1}{2}gt^2. \tag{8.5.2}$$

It's crucial to see that in this notation and that given in the problem, $v_{x,0} < v_0$.

In the above equations, Eq. (8.5.1) may be solved for t to give

$$t = \frac{x(t)}{v_{x,0}}. \tag{8.5.3}$$

Substituting Eq. (8.5.3) into Eq. (8.5.2) and eliminating the variable t ,

$$y(t) = -\frac{1}{2}g\frac{x^2(t)}{{v_{x,0}}^2}. \tag{8.5.4}$$

Eq. (8.5.4) can now be solved for the square of the horizontal component of the velocity,

$${v_{x,0}}^2 = -\frac{1}{2}g\frac{x^2(t)}{y(t)}. \tag{8.5.5}$$

At the far end of the third stair, $x = 3d$ and $y = -3h$; substitution into Eq. (8.5.5) gives

$${v_{x,0}}^2 = \frac{3gd^2}{2h}. \tag{8.5.6}$$

For the horizontal motion, use the same coordinates with the origin at the edge of the landing. The forces on the object are gravity $m\vec{\mathbf{g}} = -mg\,\hat{\mathbf{j}}$, the normal force $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}}$ and the kinetic friction force $\vec{\mathbf{f}}_{\mathrm{k}} = -f_{\mathrm{k}}\,\hat{\mathbf{i}}$. The components of the vectors in Newton's Second Law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, are

$$\begin{aligned} -f_{\mathrm{k}} &= m\,a_x \\ N - m\,g &= m\,a_y. \end{aligned} \qquad (8.5.7)$$

The object does not move in the y-direction; $a_y = 0$ and thus from the second expression in (8.5.7) $N = m\,g$. The magnitude of the frictional force is then $f_{\mathrm{k}} = \mu_{\mathrm{k}} N = \mu_{\mathrm{k}} mg$, and the first expression in (8.5.7) gives the x-component of acceleration as $a_x = -\mu_{\mathrm{k}} g$. Since the acceleration is constant the x-component of the velocity is given by

$$v_x(t) = v_0 + a_x\,t \qquad (8.5.8)$$

where v_0 is the x-component of the velocity of the object when it just started sliding (this is a different from the x-component of the velocity when it just leaves the top landing, a quantity we denoted by earlier by $v_{x,0}$). The displacement is given by

$$x(t) - x_0 = v_0\,t + \frac{1}{2}a_x\,t^2. \qquad (8.5.9)$$

Denote the time the block is decelerating as t_1. The initial conditions needed to solve the kinematic problem are the given v_0 (assuming we've "reset" our clocks appropriately), $v_x(t_1) = v_{x,0}$ as found above, $x_0 = -s$ and $x(t_1) = 0$. (Note that in the conditions for the initial and final position, we've kept the origin at the top of the stair. This is not necessary, but it works. The important matter is that the x-coordinate increases from right to left.) During this time, the block's x-component of velocity decreases from v_0 to $v_x(t_1) = v_{x,0}$ with constant acceleration $a_x = -\mu_{\mathrm{k}} g$, Using the initial and final conditions, and the value of the acceleration, Eq. (8.5.9) becomes

$$s = v_0\,t_1 - \frac{1}{2}\mu_k g\,t_1^{\,2} \qquad (8.5.10)$$

and we can solve Eq. (8.5.8) for the time the block reaches the edge of the landing,

$$t_1 = \frac{v_{x,0} - v_0}{-\mu_{\mathrm{k}} g} = \frac{v_0 - v_{x,0}}{\mu_{\mathrm{k}} g}. \qquad (8.5.11)$$

Substituting Eq. (8.5.11) into Eq. (8.5.10) yields

$$s = v_0\left(\frac{v_0 - v_{x,0}}{\mu_k g}\right) - \frac{1}{2}\mu_k g\left(\frac{v_0 - v_{x,0}}{\mu_k g}\right)^2 \qquad (8.5.12)$$

and after some algebra, we can rewrite Eq. (8.5.12) as

$$s = v_0\left(\frac{v_0 - v_{x,0}}{\mu_k g}\right) - \frac{1}{2}\mu_k g\left(\frac{v_0 - v_{x,0}}{\mu_k g}\right)^2 = \frac{{v_0}^2}{2\mu_k g} - \frac{{v_{x,0}}^2}{2\mu_k g}. \qquad (8.5.13)$$

Substituting Eq. (8.5.6) for $v_{x,0}^2$ into Eq. (8.5.13), yields the distance the object traveled on the landing,

$$s = \frac{v_0^2 - \frac{3}{2}g\frac{d^2}{h}}{2\mu_k g}. \qquad (8.5.14)$$

8.5.2 Example: Cart Moving on a Track

Consider a cart that free to move along a horizontal track (Figure 8.6). A force is applied to the cart via a string that is attached to a force sensor mounted on the cart, wrapped around a pulley an attached to a block on the other end. When the block is released the cart will begin to accelerate. The force sensor and cart together have a total mass m_1, and the suspended block has mass m_2. You may neglect the small mass of the string and pulley, and assume the string is inextensible. The frictional force is modeled as a coefficient of kinetic friction μ_k between the cart and the track (almost all of the friction is in the wheel bearings, and the model is quite good). What is the acceleration of the cart? What is the tension in the string?

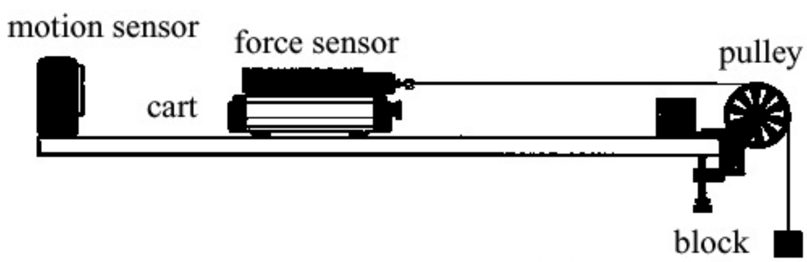


Figure 8.6 A falling block will accelerate a cart on a track via the pulling force of the string. The force sensor measures the tension in the string.

In general, we would like to draw free-body diagrams on all the individual bodies (cart, sensor, pulley, rope, and block) but we can also choose a system consisting of two (or more) bodies knowing that the forces of interaction between any two bodies will cancel in pairs by Newton's Third Law. In this example, we shall choose the sensor/cart as one free-body, and the block as the other free-body. The free-body force diagram for the sensor/cart is shown in Figure 8.7. There are three forces acting on the sensor/cart: the gravitational force, the pulling force $\vec{\mathbf{T}}_{s,\text{string}}$ of the string on the force sensor, and the contact force between the track and the cart. In Figure 8.7, we decompose the contact force into its two components, the kinetic friction force $\vec{\mathbf{f}}_k = -f_k\hat{\mathbf{i}}$ and the normal force, $\vec{\mathbf{N}}_1 = N_1\,\hat{\mathbf{j}}$.

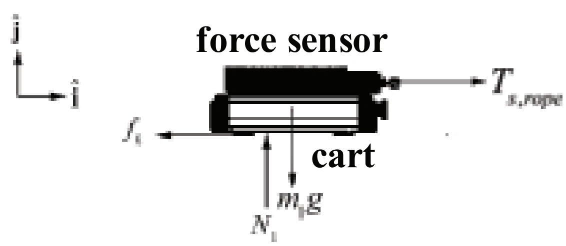


Figure 8.7 Force diagram on sensor/cart with a vector decomposition of the contact force into horizontal and vertical components

The cart is only accelerating in the horizontal direction with $\vec{\mathbf{a}}_1 = a_{x,1}\,\hat{\mathbf{i}}$, so the total component of the force in the vertical direction must be zero, $a_{y,1} = 0$. We can now apply Newton's Second Law in the horizontal and vertical directions and find that

$$\begin{aligned}\hat{\mathbf{i}}:&\ T_{s,\text{string}} - f_{\text{k}} = m_1 a_{x,1}\\ \hat{\mathbf{j}}:&\ N_1 - m_1 g = 0.\end{aligned} \tag{8.5.15}$$

From the second equation in (8.5.15), we conclude that the normal component is

$$N_1 = m_1 g\,. \tag{8.5.16}$$

We use Equation 3.5.8 for the kinetic friction force law, and Equation (8.5.16) for the normal force to find that the magnitude of the kinetic friction force is

$$f_{\text{kinetic}} = \mu_{\text{k}} N_1 = m_1 g\,. \tag{8.5.17}$$

The first equation in (8.5.15) then becomes

$$T_{s,\text{string}} - \mu_{\text{k}} m_1 g = m_1 a_{x,1}\,. \tag{8.5.18}$$

The force diagram for the block is shown in Figure 8.8. The two forces acting on the block are the pulling force $\vec{\mathbf{T}}_{B,\text{string}}$ of the string and the gravitational force $m_2\vec{\mathbf{g}}$.

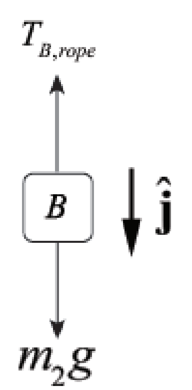


Figure 8.8 Forces acting on the block

We now apply Newton's Second Law to the block and find that

$$\hat{\mathbf{j}}: m_2 g - T_{B,\text{string}} = m_2 a_{y,B}\,. \tag{8.5.19}$$

In Equation (8.5.19), the symbol $a_{y,B}$ represents the component of the acceleration with sign determined by our choice of direction for the unit vector. Note that we made a different choice of direction for the unit vector in the vertical direction in the free-body diagram for the block shown in Figure 8.8 than in the free-body diagram for the sensor/cart shown in Figure 8.7. Each free-body diagram has an independent set of unit vectors that define a sign convention for vector decomposition of the forces acting on the free-body and the acceleration of the free-body. In our example, with the unit vector pointing downwards in Figure 8.8, if we solve for the component of the acceleration and it is positive, then we know that the direction of the acceleration is downwards.

There is a second subtle way that signs are introduced with respect to the forces acting on a free-body. In our example, the force between the string and the block acting on the block points upwards, so in the vector decomposition of the forces acting on the block that appears on the left-hand side of Equation (8.5.19), this force has a minus sign and the quantity $T_{B,\text{string}}$ is assumed positive. However, if for some reason we were uncertain about the direction of the force between the string and the block acting on the block, and drew the arrow downwards, then when we solved the problem we would discover that $T_{B,\text{string}}$ is negative, indicating that the force points in a direction opposite the direction of the arrow on the free-body diagram.

Our assumption that the mass of the rope and the mass of the pulley are negligible enables us (see the Appendix to this chapter) to assert that the tension in the rope is uniform and equal in magnitude to the forces at each end of the rope,

$$T_{s,\text{string}} = T_{B,\text{string}} = T\,. \tag{8.5.20}$$

We also assumed that the string is inextensible (does not stretch). This implies that the rope, block, and sensor/cart all have the same magnitude of acceleration,

$$a_{x,1} = a_{y,B} \equiv a\,. \tag{8.5.21}$$

Using Equations (8.5.20) and (8.5.21), we can now rewrite the equation of motion for the sensor/cart, Equation (8.5.18), as

$$T - \mu_k m_1 g = m_1 a\,, \tag{8.5.22}$$

and the equation of motion (8.5.19) for the block as

$$m_2 g - T = m_2 a\,. \tag{8.5.23}$$

Since we have only two unknowns T and a, we can now solve the two equations (8.5.22) and (8.5.23) simultaneously for the acceleration of the sensor/cart and the tension in the rope. We first solve Equation (8.5.22) for the tension

$$T = \mu_k m_1 g + m_1 a \tag{8.5.24}$$

and then substitute Equation (8.5.24) into Equation (8.5.23) and find that

$$m_2 g - (\mu_k m_1 g + m_1 a) = m_2 a\,. \tag{8.5.25}$$

We can now solve Equation (8.5.25) for the acceleration,

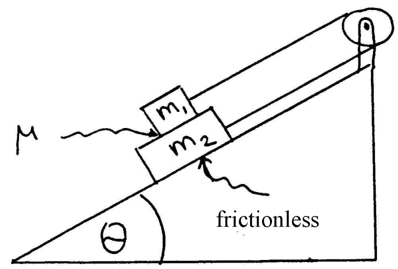
$$a = \frac{m_2 g - \mu_k m_1 g}{m_2 + m_1}\,. \tag{8.5.26}$$

Substitution of Equation (8.5.26) this into Equation (8.5.24) gives the tension in the string,

$$\begin{aligned} T &= \mu_k m_1 g + m_1 a \\ &= \mu_k m_1 g + m_1 \frac{m_2 g - \mu_k m_1 g}{m_2 + m_1} \\ &= (\mu_k + 1)\frac{m_2 m_1}{m_2 + m_1} g. \end{aligned} \tag{8.5.27}$$

In this example, we applied Newton's Second Law to two free bodies, one a composite body consisting of the sensor and the cart, and the other the block. We analyzed the forces acting on each body and also any constraints imposed on the acceleration of each body. We used the force laws for kinetic friction and gravitation on each free body system. The three equations of motion enable us to determine the forces that depend on the parameters in the example: the tension in the rope, the acceleration of the bodies, and normal force between the cart and the table.

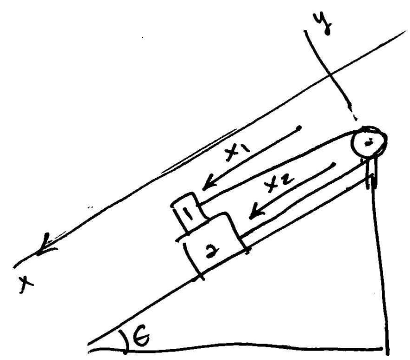
Example 8.5.3 Pulleys and Ropes Two blocks with masses m_1 and m_2 such that $m_1 << m_2$ are connected by a massless inextensible string and a massless pulley as shown in the figure below. The pulley is rigidly connected to the top of a wedge with angle θ. The coefficient of friction between the blocks is μ. The surface between the lower block and the wedge is frictionless. The goal of this problem is to find the magnitude of the acceleration of each block.



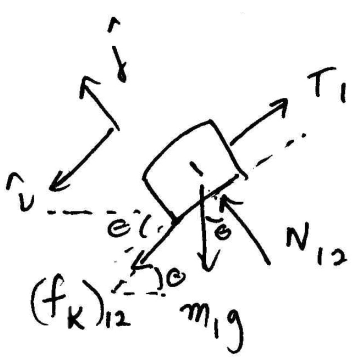
a) Write down a plan for finding the magnitude of the acceleration of each block. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any diagrams or sketches that you plan to use.

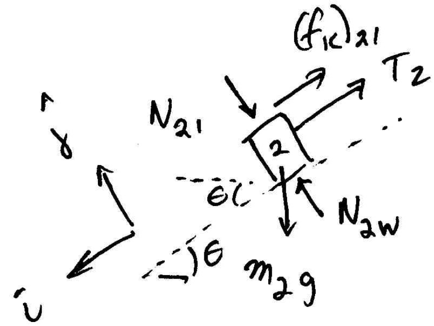
b) What are the magnitudes of the acceleration of the two blocks?

Solution: We shall apply Newton's Second Law to each block. Since the blocks are constrained by the rope, the accelerations of the blocks must be related. We should be able to then simultaneously solve the equations of motion for each block to find the magnitude of the acceleration. We begin by choosing a coordinate system shown in Figure below.



We then draw free body force diagrams for each object. We show these in Figures below.





Newton's Second law on object 1 becomes

$$\hat{\mathbf{j}}: N_{12} - m_1 g \cos\theta = 0 \qquad (8.5.28)$$

$$\hat{\mathbf{i}}: m_1 g \sin\theta + (f_k)_{12} - T_1 = m_1 a_{x1}\,. \tag{8.5.29}$$

Newton's Second law on object 2 becomes

$$\hat{\mathbf{j}}: N_{2W} - N_{21} - m_2 g \cos\theta = 0 \tag{8.5.30}$$

$$\hat{\mathbf{i}}: m_2 g \sin\theta - (f_k)_{21} - T_2 = m_2 a_{x2}\,. \tag{8.5.31}$$

We can use Eq. (8.5.28) to solve for the normal force between the surfaces,

$$N_{12} = m_1 g \cos\theta \tag{8.5.32}$$

We note that because we assumed that the string is massless and inextensible the tension in the string is uniform, so

$$T \equiv T_1 = T_2\,. \tag{8.5.33}$$

The contact force between the blocks form an action-reaction pair. When we decompose this force into its tangent (kinetic friction) and perpendicular (normal force) components, we have that

$$f_k \equiv (f_k)_{12} = (f_k)_{21}\,. \tag{8.5.34}$$

Also we model the kinetic friction by the force law

$$f_k = \mu N_{12}\,. \tag{8.5.35}$$

Using Eq. (8.5.28) for the magnitude of the normal force, the kinetic friction force ahs magnitude

$$f_k = \mu N = \mu m_1 g \cos\theta\,. \tag{8.5.36}$$

Finally we note that the x-components of the accelerations of the two objects are related by

$$a \equiv a_{x2} = -a_{x1}\,. \tag{8.5.37}$$

Note that a is also the magnitude of the acceleration fo the blocks because we expect that $a \equiv a_{x2} > 0$. Thus using Eqs. (8.5.37), (8.5.36), and (8.5.36), we can rewrite Eqs. (8.5.29) as

$$m_1 g \sin\theta + \mu m_1 g \cos\theta - T = -m_1 a \tag{8.5.38}$$

and Eq. (8.5.31) as

$$m_2 g\sin\theta - \mu m_1 g\cos\theta - T = m_2 a\,. \tag{8.5.39}$$

We have two unknowns T and a, and two independent equations so we can algebraically solve for the magnitude of the acceleration of the blocks.

We subtract Eq. (8.5.38) from Eq. (8.5.39) yielding after some simplifications yielding

$$((m_2 - m_1)g\sin\theta - 2\mu m_1\cos\theta)g = (m_1 + m_2)a\,. \tag{8.5.40}$$

Thus we can solve Eq. (8.5.40) for the magnitude of the acceleration of the blocks

$$a = \frac{(m_2 - m_1)\sin\theta - 2\mu m_1\cos\theta}{m_1 + m_2}g\,. \tag{8.5.41}$$

In the following example the accelerations of various moving objects are related by geometric constraint conditions. The key to solving such constraints is to find a relationship between a constant quantity such as the length of a string connecting various moving objects and the positions of the various objects. Then taking two derivatives of this length will produce the constraint relationship between the accelerations, If the constraint condition Eq. (8.5.37) is not immediately obvious, consider that the length of the string is given by

$$l = x_1 + x_2 + \pi R\,, \tag{8.5.42}$$

where R is the radius of the pulley. Since the length of the string is constant we can differentiate Eq. (8.5.42) twice with respect to time and find that

$$.\frac{d^2 l}{dt^2} = \frac{d^2 x_1}{dt^2} + \frac{d^2 x_2}{dt^2} + \pi\frac{d^2 R}{dt^2} \tag{8.5.43}$$

Since R and l are constants

$$\frac{d^2 l}{dt^2} = 0,\ and\ \ \pi\frac{d^2 R}{dt^2} = 0\,. \tag{8.5.44}$$

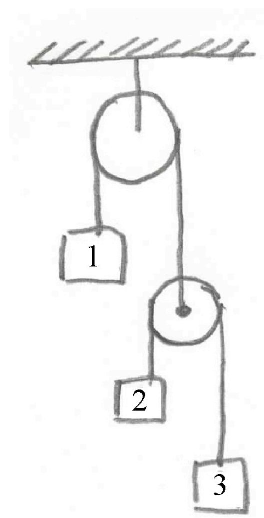
Therefore Eq. (8.5.43) becomes

$$0 = \frac{d^2 x_1}{dt^2} + \frac{d^2 x_2}{dt^2} = a_{x1} + a_{x2} \tag{8.5.45}$$

Thus justifying Eq. (8.5.45).

Example 8.4.4 Pulleys and Ropes Constraint Conditions

Consider the arrangement of pulleys and blocks shown in the figure. The pulleys are assumed massless and frictionless and the connecting strings are massless and unstretchable. Denote the respective masses of the blocks as m_1, m_2 and m_3. The upper pulley in the figure is free to rotate but its center of mass does not move. Both pulleys have the same radius R.

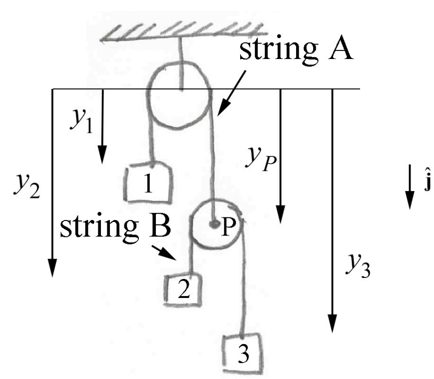


a) How are the accelerations of the objects related?

b) Draw force diagrams on each moving object.

c) Solve for the accelerations of the objects and the tensions in the ropes.

Solution: Choose an origin at the center of the upper pulley. Introduce coordinate functions for the three moving blocks, y_1, y_2 and y_3. Introduce a coordinate function y_P for the moving pulley (the pulley on the lower right in the figure). Choose downward for positive direction; the coordinate system is shown in the figure below then.



a) The length of string A is given by

$$l_A = y_1 + y_P + \pi R \tag{8.5.46}$$

where πR is the arclength that the rope is in contact with the pulley. This length is constant, and so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_A}{dt^2} = \frac{d^2 y_1}{dt^2} + \frac{d^2 y_P}{dt^2} = a_{y,1} + a_{y,P} . \tag{8.5.47}$$

Thus block 1 and the moving pulley's components of acceleration are equal in magnitude but opposite in sign,

$$a_{y,P} = -a_{y,1} . \tag{8.5.48}$$

The length of string B is given by

$$l_B = (y_3 - y_P) + (y_2 - y_P) + \pi R = y_3 + y_2 - 2y_P + \pi R \tag{8.5.49}$$

where πR is the arclength that the rope is in contact with the pulley. This length is also constant so the second derivative with respect to time is zero,

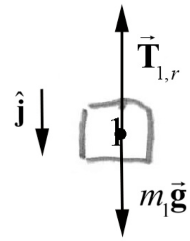
$$0 = \frac{d^2 l_B}{dt^2} = \frac{d^2 y_2}{dt^2} + \frac{d^2 y_3}{dt^2} - 2\frac{d^2 y_P}{dt^2} = a_{y,2} + a_{y,3} - 2a_{y,P} . \tag{8.5.50}$$

We can substitute Equation (8.5.48) for the pulley acceleration into Equation (8.5.50) yielding the *constraint relation* between the components of the acceleration of the three blocks,

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1} . \tag{8.5.51}$$

b) Free Body Force diagrams:

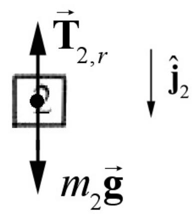
The forces acting on block 1 are: the gravitational force $m_1\vec{\mathbf{g}}$ and the pulling force $\vec{\mathbf{T}}_{1,r}$ of the string acting on the block 1. Since the string is assumed to be massless and the pulley is assumed to be massless and frictionless, the tension T_A in the string is uniform and equal in magnitude to the pulling force of the string on the block. The free body diagram is shown below.



Newton's Second Law applied to block 1 is then

$$\hat{\mathbf{j}}: \; m_1 g - T_A = m_1 \, a_{y,1} \, . \qquad (8.5.52)$$

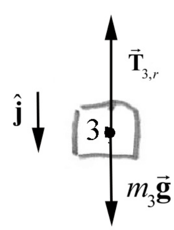
The forces on the block 2 are the gravitational force $m_2\vec{\mathbf{g}}$ and the string holding the block, $\vec{\mathbf{T}}_{2,r}$, with magnitude T_B. The free body diagram for the forces acting on block 2 are shown below.



Newton's second Law applied to block 2 is

$$\hat{\mathbf{j}}: \; m_2 g - T_B = m_2 \, a_{y,2} \, . \qquad (8.5.53)$$

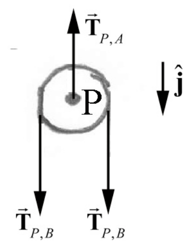
The forces on the block 3 are the gravitational force $m_3\vec{\mathbf{g}}$ and the string holding the block, $\vec{\mathbf{T}}_{3,r}$, with magnitude T_B. The free body diagram for the forces acting on block 3 are shown below.



Newton's second Law applied to block 3 is

$$\hat{\mathbf{j}}: \; m_3 g - T_B = m_3 \, a_{y,3} \, . \qquad (8.5.54)$$

The forces on the moving pulley are the gravitational force $m_P \vec{\mathbf{g}} = \vec{\mathbf{0}}$ (the pulley is assumed massless); string B pulls down on the pulley on each side with a force, $\vec{\mathbf{T}}_{P,B}$, which has magnitude T_B. String A holds the pulley up with a force $\vec{\mathbf{T}}_{P,A}$ with the magnitude T_A equal to the tension in string A. The free body diagram for the forces acting on the moving pulley are shown below.



Newton's second Law applied to the pulley is

$$\hat{\mathbf{j}}:\ 2T_B - T_A = m_P\, a_{y,P} = 0\,. \qquad (8.5.55)$$

Since the pulley is massless we can use this last equation to determine the condition that the tension in the two strings must satisfy,

$$2T_B = T_A \qquad (8.5.56)$$

We are now in position to determine the accelerations of the blocks and the tension in the two strings. We record the relevant equations as a summary.

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1} \qquad (8.5.57)$$

$$m_1 g - T_A = m_1\, a_{y,1} \qquad (8.5.58)$$

$$m_2 g - T_B = m_2\, a_{y,2} \qquad (8.5.59)$$

$$m_3 g - T_B = m_3\, a_{y,3} \qquad (8.5.60)$$

$$2T_B = T_A\,. \qquad (8.5.61)$$

There are five equations with five unknowns, so we can solve this system. We shall first use Equation (8.5.61) to eliminate the tension T_A in Equation (8.5.58), yielding

$$m_1 g - 2T_B = m_1\, a_{y,1}\,. \qquad (8.5.62)$$

We now solve Equations (8.5.59), (8.5.60) and (8.5.62) for the accelerations,

$$a_{y,2} = g - \frac{T_B}{m_2} \qquad (8.5.63)$$

$$a_{y,3} = g - \frac{T_B}{m_3} \quad (8.5.64)$$

$$a_{y,1} = g - \frac{2T_B}{m_1}. \quad (8.5.65)$$

We now substitute these results for the accelerations into the constraint equation, Equation (8.5.57),

$$0 = g - \frac{T_B}{m_2} + g - \frac{T_B}{m_3} + 2g - \frac{4T_B}{m_1} = 4g - T_B\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right). \quad (8.5.66)$$

We can now solve this last equation for the tension in string B,

$$T_B = \frac{4g}{\left(\dfrac{1}{m_2} + \dfrac{1}{m_3} + \dfrac{4}{m_1}\right)} = \frac{4g\,m_1\,m_2\,m_3}{m_1\,m_3 + m_1 m_2 + 4\,m_2\,m_3}. \quad (8.5.67)$$

From Equation (8.5.61), the tension in string A is

$$T_A = 2T_B = \frac{8g\,m_1\,m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3}. \quad (8.5.68)$$

We find the acceleration of block 1 from Equation (8.5.65), using Equation (8.5.67) for the tension in string B,

$$a_{y,1} = g - \frac{2T_B}{m_1} = g - \frac{8g\,m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3} = g\frac{m_1\,m_3 + m_1\,m_2 - 4\,m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3}. \quad (8.5.69)$$

We find the acceleration of block 2 from Equation (8.5.63), using Equation (8.5.67) for the tension in string B,

$$a_{y,2} = g - \frac{T_B}{m_2} = g - \frac{4g\,m_1\,m_3}{m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3} = g\frac{-3\,m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3}. \quad (8.5.70)$$

Similarly, we find the acceleration of block 3 from Equation (8.5.64), using Equation (8.5.67) for the tension in string B,

$$a_{y,3} = g - \frac{T_B}{m_3} = g - \frac{4\,g\,m_1\,m_2}{m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3} = g\frac{m_1\,m_3 - 3\,m_1 m_2 + 4\,m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4\,m_2\,m_3}. \quad (8.5.71)$$

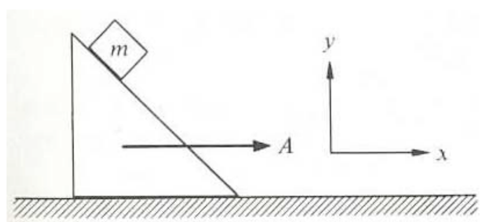
As a check on our algebra we note that

$$2a_{1,y} + a_{2,y} + a_{3,y} =$$

$$2g\frac{m_1 m_3 + m_1 m_2 - 4m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3} + g\frac{-3m_1 m_3 + m_1 m_2 + 4m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3} + g\frac{m_1 m_3 - 3m_1 m_2 + 4m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3} \quad (8.5.72)$$

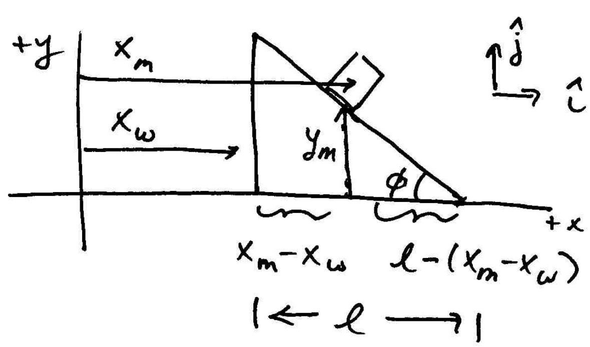
$$= 0.$$

Example 8.5.5 A 45^o wedge is pushed along a table with constant acceleration $\vec{\mathbf{A}} = A\,\hat{\mathbf{i}}$ (see figure for choice of axes and positive directions) according to an observer at rest with respect to the table. A block of mass m slides without friction down the wedge. Find its acceleration with respect to an observer at rest with respect to the table. Write down a plan for finding the magnitude of the acceleration of the block. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any free body force diagrams or sketches that you plan to use.



Solution:

We shall apply Newton's Second Law to the object sliding down the wedge. Since the wedge is accelerating, there is a constraint relation between the x and y components of the acceleration of the object. In order to find that constraint we choose a coordinate system for the wedge and object sliding down the wedge shown in the figure below.



We shall find the constraint relationship between the accelerations by a geometric argument. From the figure above, we see that

$$\tan\phi = \frac{y_m}{l - (x_m - x_w)} \quad (8.5.73)$$

therefore

$$y_m = (l - (x_m - x_w))\tan\phi \;. \qquad (8.5.74)$$

If we differentiate Eq. (8.5.74) twice with respect to time noting that

$$\frac{d^2 l}{dt^2} = 0 \qquad (8.5.75)$$

we have that

$$\frac{d^2 y_m}{dt^2} = -\left(\frac{d^2 x_m}{dt^2} - \frac{d^2 x_w}{dt^2}\right)\tan\phi \;. \qquad (8.5.76)$$

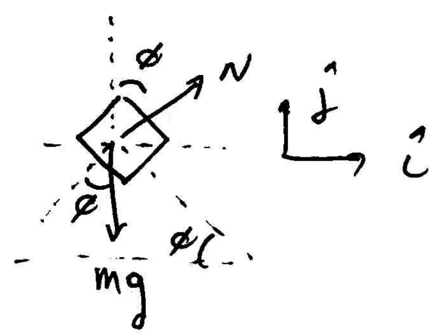
Therefore

$$a_{ym} = -(a_{xm} - A)\tan\phi \qquad (8.5.77)$$

where

$$A = \frac{d^2 x_w}{dt^2} \;. \qquad (8.5.78)$$

We now draw a free body force diagram for the object (figure below)



Newton's Second Law becomes

$$\hat{\mathbf{i}} : N\sin\phi = m a_{xm} \;. \qquad (8.5.79)$$

$$\hat{\mathbf{j}} : N\cos\phi - mg = m a_{ym} \qquad (8.5.80)$$

We can solve for the normal force from Eq. (8.5.79):

$$N = \frac{ma_{xm}}{\sin\phi} \tag{8.5.81}$$

We now substitute Eq. (8.5.77) and Eq. (8.5.81) into Eq. (8.5.80) yielding

$$\frac{ma_{xm}}{\sin\phi}\cos\phi - mg = m(-(a_{xm} - A)\tan\phi)\,. \tag{8.5.82}$$

We now clean this up yielding

$$ma_{xm}(\operatorname{cotan}\phi + \tan\phi) = m(g + A\tan\phi) \tag{8.5.83}$$

Thus the x-component of the acceleration is then

$$a_{xm} = \frac{g + A\tan\phi}{\operatorname{cotan}\phi + \tan\phi} \tag{8.5.84}$$

From Eq. (8.5.77), the y-component of the acceleration is then

$$a_{ym} = -(a_{xm} - A)\tan\phi = -\left(\frac{g + A\tan\phi}{\operatorname{cotan}\phi + \tan\phi} - A\right)\tan\phi\,. \tag{8.5.85}$$

This simplifies to

$$a_{ym} = \frac{A - g\tan\phi}{\operatorname{cotan}\phi + \tan\phi} \tag{8.5.86}$$

When $\phi = 45^\circ$, $\operatorname{cotan} 45^\circ = \tan 45^\circ = 1$, and so Eq. (8.5.84) becomes

$$a_{xm} = \frac{g + A}{2} \tag{8.5.87}$$

and Eq. (8.5.86) becomes

$$a_{ym} = \frac{A - g}{2}\,. \tag{8.5.88}$$

The magnitude of the acceleration is then

$$a = \sqrt{a_{xm}{}^2 + a_{ym}{}^2} = \sqrt{\left(\frac{g + A}{2}\right)^2 + \left(\frac{A - g}{2}\right)^2} \tag{8.5.89}$$

$$a = \sqrt{\left(\frac{g^2 + A^2}{2}\right)}. \tag{8.5.90}$$

8.6 Friction Force as a Linear Function of Velocity (Advanced Topic)

In many physical situations the force on an object will be modeled as depending on the object's velocity. We have already seen static and kinetic friction between surfaces modeled as being independent of the surfaces' relative velocity. Common experience (swimming, throwing a Frisbee) tells us that the friction force between an object and a fluid can be a complicated function of velocity. Indeed, these complicated relations are an important part of such topics as aircraft design.

A reasonable model for the frictional force on an object m moving at low speeds through a viscous medium is

$$\vec{\mathbf{F}}_{\text{friction}} = -\gamma\, m\vec{\mathbf{v}} \tag{8.6.1}$$

where γ is a constant that depends on the properties (density, viscosity) of the medium and the size and shape of the object. Note that γ has dimensions of inverse time,

$$\dim[\gamma] = \frac{\dim\left[\text{Force}\right]}{\dim[\text{mass}]\cdot\dim\left[\text{velocity}\right]} = \frac{\text{M}\cdot\text{L}\cdot\text{T}^{-2}}{\text{M}\cdot\text{L}\cdot\text{T}^{-1}} = \text{T}^{-1}. \tag{8.6.2}$$

The minus sign in Equation (8.6.1) indicates that the frictional force is directed against the object's velocity (relative to the fluid).

In a situation where $\vec{\mathbf{F}}_{\text{friction}}$ is the net force, Newton's Second Law becomes

$$-\gamma\, m\vec{\mathbf{v}} = m\vec{\mathbf{a}} \tag{8.6.3}$$

and so the acceleration is

$$\vec{\mathbf{a}} = -\gamma\,\vec{\mathbf{v}}\,. \tag{8.6.4}$$

The acceleration has no component perpendicular to the velocity, and in the absence of other forces will move in a straight line, but with varying speed. Denote the direction of this motion as the x-direction, so that Equation (8.6.4) becomes

$$a_x = \frac{d\,v_x}{dt} = -\gamma\, v_x\,. \tag{8.6.5}$$

Equation (8.6.5) is now a differential equation. For our purposes, we'll create an initial-condition problem by specifying that the initial x-component of velocity is $v(t = 0) = v_{x0}$.

The differential equation in (8.6.5) is known as a separable equation, in that the equation may be rewritten as

$$\frac{d\,v_x}{v_x} = -\gamma\,dt\ . \tag{8.6.6}$$

and each side can be separably integrated. The integration in this case is simple, leading to

$$\int_{v_{x0}}^{v_x(t)} \frac{d\,v_x}{v_x} = -\gamma \int_0^t dt\ . \tag{8.6.7}$$

The left hand side is

$$\text{LHS} = \ln(v_x)\Big|_{v_{x0}}^{v_x(t)} = \ln(v_x(t) - \ln(v_{x0}) = \ln(v_x(t)/v_{x0}) \tag{8.6.8}$$

and the right hand side is

$$\text{RHS} = -\gamma\,t \tag{8.6.9}$$

Equating the two sides yields

$$\ln(v_x(t)/v_{x0}) = -\gamma\,t \tag{8.6.10}$$

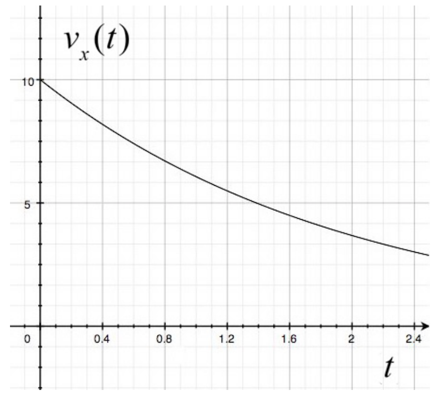
Exponentiate each side of the above equation yielding

$$v_x(t)/v_{x0} = e^{-\gamma\,t} \tag{8.6.11}$$

Thus the x-component of the velocity as a function of time is given by

$$v_x(t) = v_{x0}e^{-\gamma\,t} \tag{8.6.12}$$

A plot of v_x vs. t is shown in the figure below with initial conditions $v_{x0} = 10\text{ m/s}$ and $\gamma = 0.5\text{ N/J}$.



Example: An object moving the x-axis with an initial x-component of the velocity $v_x(t=0) = v_{x0}$ experiences a retarding frictional force whose magnitude is proportional to the square of the speed (a case known as *Newtonian Damping*),

$$\left|\vec{\mathbf{F}}_{\text{friction}}\right| = -\gamma\, m v^2 \tag{8.6.13}$$

Show that the x-component of the velocity of the object as a function of time is given by

$$v_x(t) = v_{x0}\frac{1}{1+t/\tau}, \tag{8.6.14}$$

and find the constant τ.

Solution: Newton's Second Law can be written as

$$-\gamma\, v_x^2 = \frac{d v_x}{dt}. \tag{8.6.15}$$

Differentiating our possible solution yields

$$\frac{d v_x}{dt} = -v_{x0}\frac{1}{\tau}\frac{1}{\left(1+t/\tau\right)^2} = -\frac{1}{v_{x0}\tau}v_x^2 \tag{8.6.16}$$

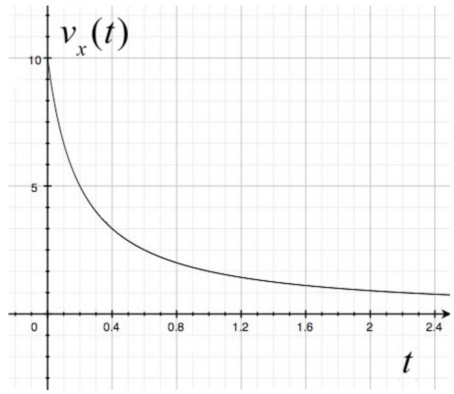
Substituting into the Second Law yields

$$-\gamma\, v_x^2 = -\frac{1}{v_{x0}\tau}v_x^2. \tag{8.6.17}$$

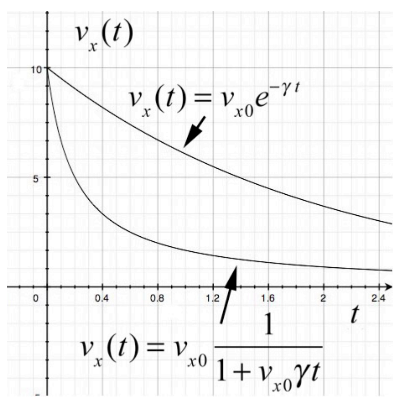
Thus our function is a solution providing that

$$\tau = \frac{1}{v_{x0}\gamma} . \tag{8.6.18}$$

A plot of v_x vs. t is shown in the figure below with initial conditions $v_{x0} = 10 \text{ m/s}$ and $\gamma = 0.5 \text{ N/J}$.



The figure below shows the two x-component of the velocity functions Eq. and Eq. plotted on the same graph .



8.7: Linear Friction with Gravity (Advanced Topic)

A common extension of the above example is to have an object falling through the same viscous medium, subject to gravity but no other forces. Taking the positive y-direction to be downward, the equation of motion becomes

$$a_y = \frac{d v_y}{dt} = g - \gamma v_y . \tag{8.7.1}$$

Note that the expression in Equation (8.7.1) is valid for the vertical velocity directed upwards ($v_y < 0$) or downwards ($v_y > 0$). We expect that in the limit of long times and no other forces, any object would eventually fall straight down. We will use the expectation to simplify our methods of solution, starting with an ansatz that assumes a terminal velocity (actually, a terminal speed) v_{term}; the terminal velocity is that for which the acceleration given by Equation (8.7.1) is zero, $v_{\text{term}} = g/\gamma$ (note that v_{term} has dimensions of velocity). At this point, it helps to rewrite Equation (8.7.1) as

$$\frac{d v_y}{dt} = \gamma \left(v_{\text{term}} - v_y \right) \tag{8.7.2}$$

If we have, as before, an initial-value problem, in this case the initial condition being $v_y(0) = v_{y0}$, our trial solution will be one that has $v_y(0) = v_{y0}$ but which approaches v_{term} for large times. From our previous experience, we suspect that a function involving an exponential will be more likely to lead to success than a rational function. So, our trial function will be

$$v_y = v_{\text{term}} + \left(v_{y0} - v_{\text{term}} \right) e^{-t/\tau} . \tag{8.7.3}$$

Performing the differentiation,

$$\begin{aligned} \frac{d v_y}{dt} &= -\frac{1}{\tau}\left(v_{y0} - v_{\text{term}} \right) e^{-t/\tau} \\ &= -\frac{1}{\tau}\left(v_y - v_{\text{term}} \right) \end{aligned} \tag{8.7.4}$$

and we see that v_y is a solution to the problem with the choice;

$$v_y = v_{\text{term}} + \left(v_{y0} - v_{\text{term}} \right) e^{-\gamma t} . \tag{8.7.5}$$

A plot of the ratio v_y / v_{term} as a function of $t_1 = \gamma t$ is shown in Figure 8.19, for the four different initial conditions

1: $v_{y0} = 2 v_{\text{term}}; v_{y1} = v_{\text{term}}(1 + e^{-\gamma t})$,
2: $v_{y0} = v_{\text{term}}; v_{y2} = v_{\text{term}}$,
3: $v_{y0} = 0; v_{y3} = v_{\text{term}}(1 - e^{-\gamma t})$,
4: $v_{y0} = -v_{\text{term}}; v_{y4} = v_{\text{term}}(1 - 2e^{-\gamma t})$.

Note that the last of these is a situation where the object is initially moving upward.

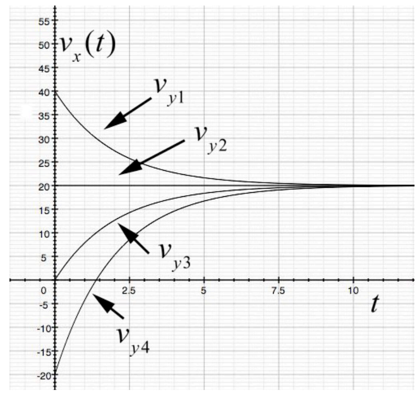


Figure 8.19 Falling objects with friction

The success of our ansatz suggests a more direct technique. In Equation (8.7.2), make the substitution $u = v_y - v_{\text{term}}$. Recognizing that $du/dt = dv_y/dt$, that equation becomes

$$\frac{du}{dt} = -\gamma u \tag{8.7.6}$$

and the initial condition becomes $u(t=0) = v_{y0} - v_{\text{term}}$. This has been reduced to a problem done previous (the previous section) and we can just quote that result;

$$v_y - v_{\text{term}} = \left(v_{y0} - v_{\text{term}}\right)e^{-\gamma t} \tag{8.7.7}$$

and we're done.

In the result, the behavior of the solution should be checked in the large and small limits of time. From the graphs, or from the analytic form, $v_y \to v_{\text{term}}$ as $t \to \infty$. For small times $t \ll 1/\gamma$, consider the solution expressed as

$$\begin{aligned} v_y &= v_{\text{term}}\left(1 - e^{-\gamma t}\right) + v_{y0}\, e^{-\gamma t} \\ &= \frac{g}{\gamma}\left(1 - e^{-\gamma t}\right) + v_{y0}\, e^{-\gamma t} \\ &\simeq \frac{g}{\gamma}\left(\gamma t\right) + v_{y0} \\ &= g t + v_{y0} \end{aligned} \tag{8.7.8}$$

to zero order in γ, as expected.

8.8: Tension in a Rope Wrapped Around a Pulley (Advanced Topic)

Consider the general case of a string or rope wrapped a quarter-turn about a frictionless pulley, as in Example 8.4.2 in the text. The force diagram for the pulley is shown in Figure 8.8.1(a). The forces acting on the pulley are the downward pulling force $\vec{\mathbf{T}}_{\text{P,rope}}$ of the rope connecting the pulley and the block (called a more general "object" in this appendix, the force $\vec{\mathbf{F}}_{\text{P,support}}$ exerted on the pulley by the support, and the gravitational force $m_{\text{P}}\vec{\mathbf{g}}$ on the pulley. Note that in the limit of a massless pulley, the arrows representing the pulley's weight would be vanishingly small; these vectors are included in the figure for completeness and in anticipation of considering the more general case.

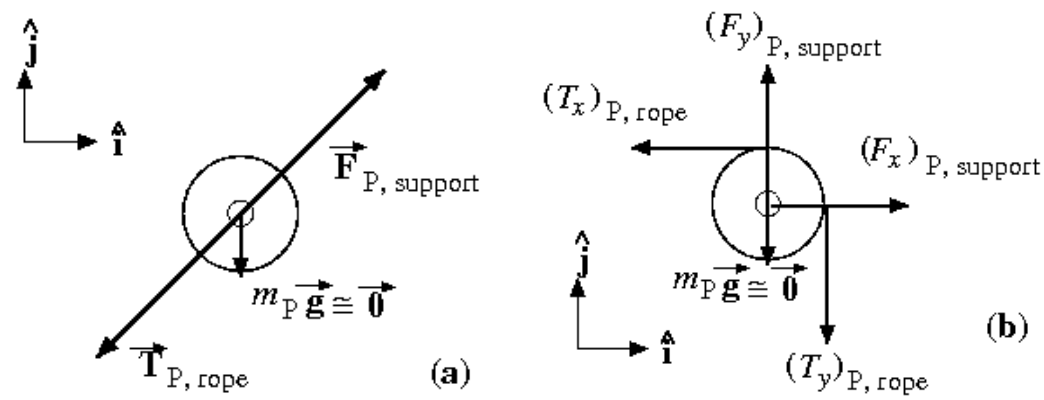


Figure 8.8.1 (a) Force diagram on pulley **(b)** Vector decomposition of forces on pulley

In Figure 8.8.1(b), the support force and the rope force on the pulley are decomposed into vector components. If we assume that the mass m_{P} of the pulley is negligible, we can ignore the gravitational force. The center of mass of the pulley is fixed hence the acceleration of the pulley is zero. We can now apply Newton's Second Law in each direction shown in Figure 8.8.1(b):

$$\hat{\mathbf{i}}: \left(F_x\right)_{\text{P,support}} - \left(T_x\right)_{\text{P,rope}} = 0 \tag{8.8.1}$$

$$\hat{\mathbf{j}}: \left(F_y\right)_{\text{P,support}} - \left(T_y\right)_{\text{P,rope}} \simeq 0\,. \tag{8.8.2}$$

We also assumed that the mass of the rope is negligible. In Module 7, we determined that for forces applied to a rope with negligible mass along a line (one dimension), the force is transmitted uniformly through the rope. When the rope is wrapped around the pulley this is not necessarily the case because the pulley is not in static equilibrium; it is rotating. We shall show in a later module that if we assume that the pulley is massless then the pulley also satisfies our second condition of static equilibrium that the torque about the center of mass of the pulley is zero, hence the forces due to the ropes have equal magnitude,

$$\left(T_y\right)_{\text{P, rope}} = \left(T_x\right)_{\text{P, rope}} \equiv T\,. \tag{8.8.3}$$

We can substitute Equation (8.8.3) into both Equations (8.8.1) and (8.8.2) to conclude that the components of the support force between the support and the pulley are each equal in magnitude to the forces exerted by the ropes,

$$\left(F_y\right)_{\text{P, support}} = \left(F_x\right)_{\text{P, support}} = T\,. \tag{8.8.4}$$

The force diagram on the rope is shown in Figure 8.8.2(a). The pulley exerts a force $\vec{\mathbf{T}}_{\text{P, rope}}$ on the rope, the block exerts a force $\vec{\mathbf{T}}_{\text{rope, object}}$ on the rope, and the sensor exerts a force $\vec{\mathbf{T}}_{\text{rope, sensor}}$ on the rope. (In assuming that mass of the rope is negligible, we neglect the gravitational force acting on the rope).

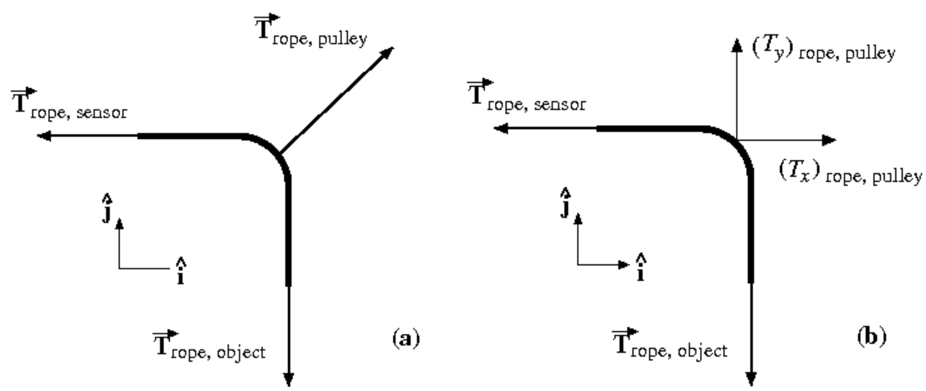


Figure 8.8.2 (a) Forces acting on rope with negligible mass
(b) vector decomposition of forces on rope

The vector decomposition of the forces is shown in Figure 8.8.2(b). Now apply Newton's Second Law in each direction:

$$\hat{\mathbf{i}}: \left(T_x\right)_{\text{rope, pulley}} - T_{\text{rope, sensor}} = m_{\text{rope}} a_{x,\text{rope}} \simeq 0 \tag{8.8.5}$$

$$\hat{\mathbf{j}}: \left(T_y\right)_{\text{rope, pulley}} - T_{\text{rope, object}} = m_{\text{rope}} a_{y,\text{rope}} \simeq 0\,. \tag{8.8.6}$$

Notice that in both Equations (8.8.5) and (8.8.6), the right hand side is zero because we assumed that the mass of the rope was negligible. In general, the rope has non-zero acceleration.

We can solve Equation (8.8.5) for the force that the sensor exerts on the rope

$$T_{\text{rope, sensor}} = \left(T_x\right)_{\text{rope, pulley}}, \tag{8.8.7}$$

and Equation (8.8.6) for the force that the object exerts on the rope,

$$T_{\text{rope, object}} = \left(T_y\right)_{\text{rope, pulley}}. \tag{8.8.8}$$

We can now compare the two results in Equations (8.8.7) and (8.8.8) to Equation (8.8.3) and conclude that the tension in the rope is uniform and equal in magnitude to the forces at each end of the rope,

$$T_{\text{rope, sensor}} = T_{\text{rope, object}} = T. \tag{8.8.9}$$

There are two Newton's Third Law action-reaction pairs: the interaction between the sensor/cart and rope,

$$T_{\text{rope, sensor}} = T_{\text{sensor, rope}} \tag{8.8.10}$$

and the interaction between the object and rope,

$$T_{\text{rope, object}} = T_{\text{object, rope}} \tag{8.8.11}$$

Thus by substituting Equations (8.8.10) and (8.8.11) into Equation (8.8.9), we can conclude that the tension in the rope is equal to the force of the rope on the sensor/cart and the force of the rope on the object,

$$T_{\text{sensor, rope}} = T_{\text{object, rope}} = T. \tag{8.8.12}$$