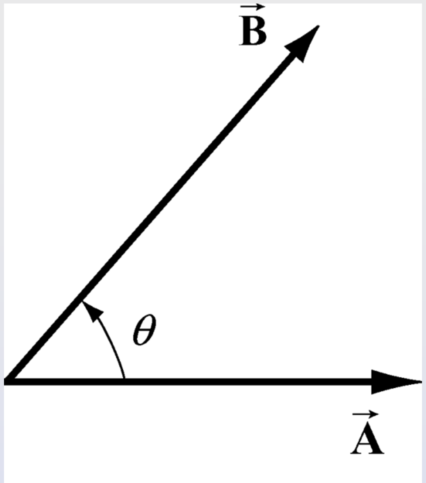
Work and the Dot Product

Dot Product

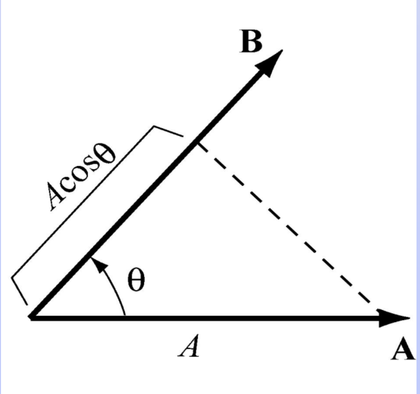


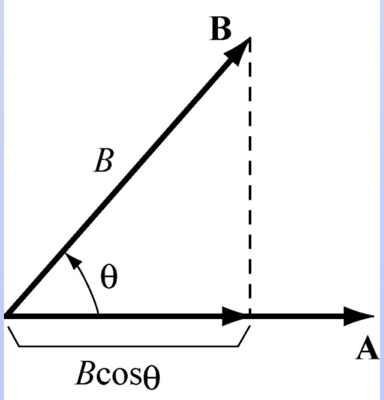
A scalar quantity

Magnitude: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \cos\theta$

The dot product can be positive, zero, or negative

Two types of projections: the dot product is the parallel component of one vector with respect to the second vector times the magnitude of the second vector





$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = A\ \left|\vec{\mathbf{B}}\right|$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right| B$$

Dot Product Properties

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$$

$$c\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = c(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})$$

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \cdot \vec{\mathrm{C}} = \vec{\mathbf{A}} \cdot \vec{\mathrm{C}} + \vec{\mathbf{B}} \cdot \vec{\mathrm{C}}$$

Dot Product in Cartesian Coordinates

With unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}|\cos(0) = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\cos(\pi/2) = 0$$

Example:

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \qquad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

Kinetic Energy

- Velocity

$$\vec{\mathbf{v}} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$$

- Kinetic Energy:
- $$K = \frac{1}{2} m(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = \frac{1}{2} m(v_x^2 + v_y^2 + v_z^2) \geq 0$$
- Change in kinetic energy:

$$\Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2 = \frac{1}{2} m(\vec{\mathbf{v}}_f \cdot \vec{\mathbf{v}}_f) - \frac{1}{2} m(\vec{\mathbf{v}}_0 \cdot \vec{\mathbf{v}}_0)$$

$$\frac{1}{2} m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m(v_{x,0}^2 + v_{y,0}^2 + v_{z,0}^2)$$

Work Done by a Constant Force

Definition: Work

The work done by a constant force $\vec{\mathbf{F}}$ on an object is equal to the component of the force in the direction of the displacement times the magnitude of the displacement:

$$W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}} = \left|\vec{\mathbf{F}}\right|\left|\Delta\vec{\mathbf{r}}\right|\cos\theta = \left|\vec{\mathbf{F}}\right|\cos\theta\left|\Delta\vec{\mathbf{r}}\right| = F\ \left|\Delta\vec{\mathbf{r}}\right|$$

Note that the component of the force in the direction of the displacement can be positive, zero, or negative so the work may be positive, zero, or negative

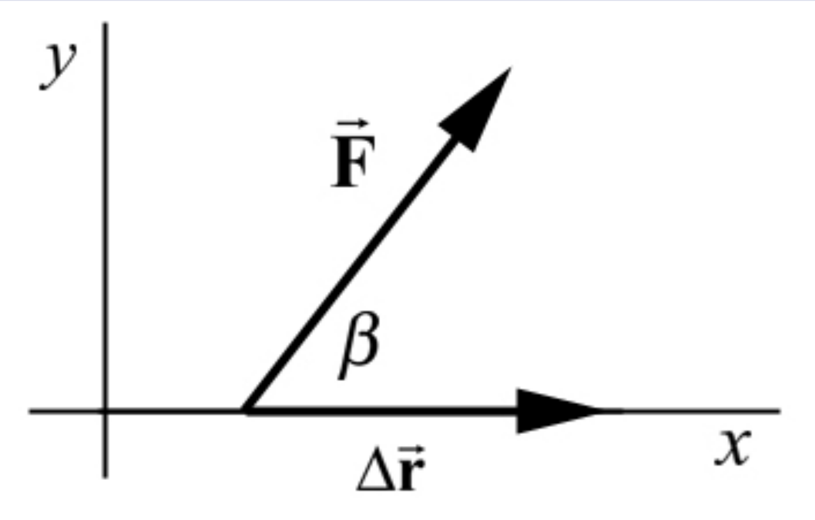
Work as a Dot Product

Let the force exerted on an object be

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$$

$$F_x = F\cos\beta$$

$$F_y = F\sin\beta$$



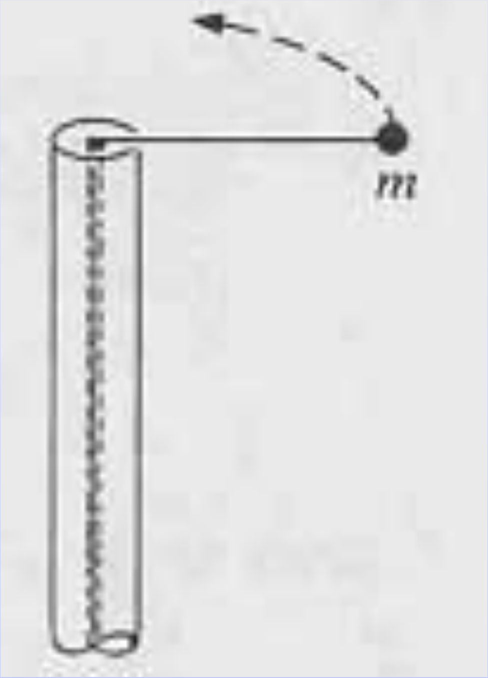
Displacement: $\Delta\vec{\mathbf{r}} = \Delta x\,\hat{\mathbf{i}}$

$$W = \vec{\mathbf{F}}\cdot\Delta\vec{\mathbf{r}} = F\,\Delta x\cos\beta$$

$$= (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}})\cdot(\Delta x\,\hat{\mathbf{i}}) = F_x\,\Delta x$$

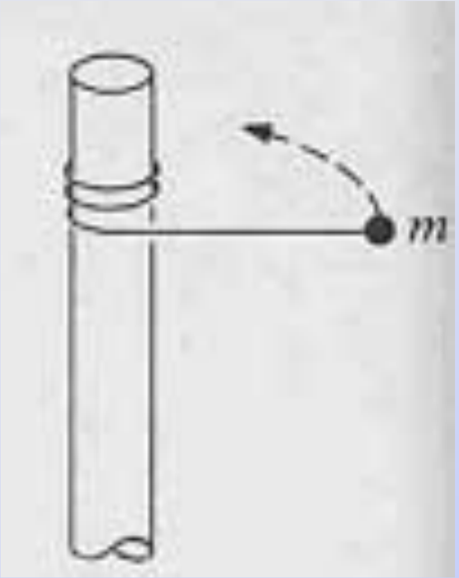
Checkpoint Problem: Is Kinetic Energy Constant Part 1?

A tetherball of mass m is attached to a post of radius by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0 . The string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. Ignore gravity and any dissipative forces. Until the ball hits the post, does the kinetic energy of the ball change or remains constant. Explain your answer.



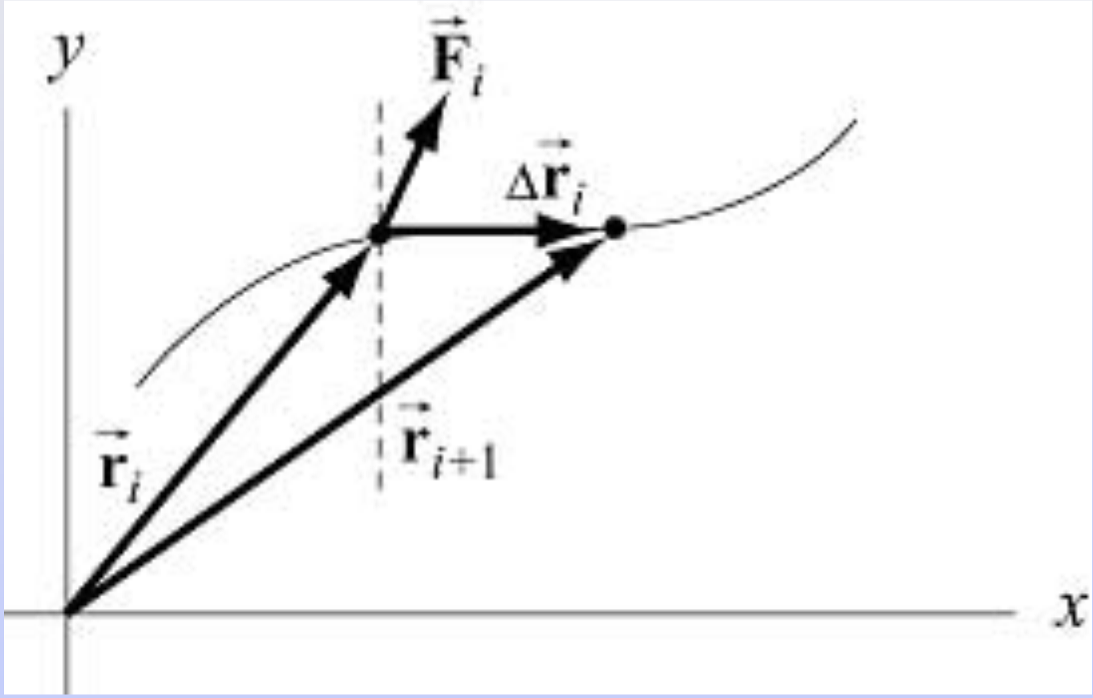
Checkpoint Problem: Is Kinetic Energy Constant Part 2?

A tetherball of mass m is attached to a post of radius R by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0. The string wraps around the outside of the post. Ignore gravity and any dissipative forces. Until the ball hits the post, does the kinetic energy of the ball change or remains constant. Explain your answer.



Work Done Along an Arbitrary Path

$$\Delta W_i = \vec{\mathbf{F}}_i \cdot \Delta \vec{\mathbf{r}}_i$$



$$W = \lim_{\substack{N \to \infty \\ |\Delta \vec{\mathbf{r}}_i| \to 0}} \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \cdot \Delta \vec{\mathbf{r}}_i = \int_0^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Work in Three Dimensions

Let the force acting on an object be given by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}$$

The displacement vector for an infinitesimal displacement is

$$\mathrm{d}\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dy\,\hat{\mathbf{k}}$$

The work done by the force for this infinitesimal displacement is

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}) \cdot (dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dy\,\hat{\mathbf{k}})$$

$$dW = F_x\,dx + F_y\,dy + F_z\,dz$$

Integrate to find the total work

$$W = \int_0^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^f F_x dx + \int_0^f F_y dy + \int_0^f F_z dz$$

Checkpoint Problem: Work Done by the Inverse Square Gravitational Force

Consider a magnetic rail gun that shoots an object of mass m radially away from the surface of the earth (mass m_e). When the object leaves the rail gun it is at a distance r_i from the center of the earth moving with speed v_i . What speed of the object as a function of distance from the center of the earth?

Checkpoint Problem: Work Done by by the Inverse Square Gravitational Force

Consider an object of mass m moving towards the sun (mass m_s). Initially the object is at a distance r_0 from the center of the sun. The object moves to a final distance r_f from the center of the sun. How much work does the gravitational force between the sun and the object do on the object during this motion?

Work-Energy Theorem in Three-Dimensions

Newton's Second Law:

$$F_x = ma_x\ ,\ \ F_y = ma_y\ ,\ \ F_z = ma_z$$

Total work:

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_x dx + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_y dy + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z dz$$

becomes

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} ma_x dx + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} ma_y dy + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} ma_z dz$$

Work-Energy Theorem in Three-Dimensions

Recall

$$\int_{x_0}^{x_f} ma_x dx = \int_{x_0}^{x_f} m\frac{dv_x}{dt} dx = \int_{x_0}^{x_f} m\frac{dx}{dt} dv_x = \int_{v_{x,0}}^{v_{x,f}} mv_x dv_x = \frac{1}{2}mv_{x,f}^2 - \frac{1}{2}mv_{x,0}^2$$

Repeat argument for *y*- and *z*-direction

$$\int_{y_0}^{y_f} ma_y dx = \frac{1}{2}mv_{y,f}^2 - \frac{1}{2}mv_{y,0}^2 \qquad \int_{z_0}^{z_f} ma_z dx = \frac{1}{2}mv_{z,f}^2 - \frac{1}{2}mv_{z,0}^2$$

Adding these three results

$$W = \int_{z_0}^{z_f} (ma_x\, dx + ma_y\, dy + ma_z\, dz) dx$$

$$W = \frac{1}{2}m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2}m(v_{x,0}^2 + v_{y,0}^2 + v_{z,0}^2) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$$

$$\boxed{W = \Delta K}$$

Instantaneous Power

For an applied constant force the instantaneous power is

$$P = \frac{dW}{dt} = \frac{d}{dt}(\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}) = \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$$

the time rate of change of the kinetic energy for a body

$$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \right) = m\left(\frac{d}{dt} \vec{\mathbf{v}} \right) \cdot \vec{\mathbf{v}} = m\,\vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = P$$