Work and the Dot Product
Challenge Problems

Problem 1:

A ball is thrown upward in a strong but steady wind, blowing towards the east. It rises a height h, during which time it moves eastward by a distance l. Assume that the wind exerts a steady force on the ball of magnitude F, toward the east. How much work does the wind do on the ball, from the time it is thrown to when it reaches its maximum height? You can write your answer without explanation.

Problem 1 Solution: In this case the force of the wind is constant, but the motion is not one-dimensional.

The work done by the force is given by $W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}}$, where $\Delta\vec{\mathbf{r}}$ is the displacement vector. If we adopt a coordinate system with the x axis pointing to the east, and the y axis pointing vertically, then $\vec{\mathbf{F}} = F\,\hat{\mathbf{i}}$ and $\Delta\vec{\mathbf{r}} = l\,\hat{\mathbf{i}} + h\,\hat{\mathbf{j}}$. Hence

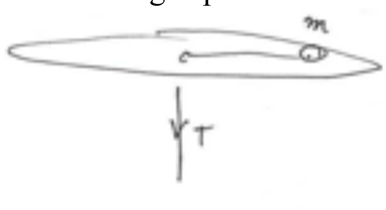
$$W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}} = (F\,\hat{\mathbf{i}}) \cdot (l\,\hat{\mathbf{i}} + h\,\hat{\mathbf{j}}) = dW = F_x \Delta r_x + F_y \Delta r_y = Fl$$

Problem 2: Conservation of Energy and Newton's Second Law: Tetherball

A body of mass m whirls around on a string which passes through a fixed ring located at the center of the circular motion. The string is held by a person who pulls the string downward with a constant velocity of magnitude V so that the radial distance to the body decreases from an initial distance r_0 to a final distance r_f from the center. The body has an initial angular velocity ω_0. You may neglect the effect of gravity. Show that the work done in pulling the string equals the increase in kinetic energy of the body.

Problem 2 Solution:

Assume string is pulled inward so that $\dot{r} = v = \text{constant}$



$$\hat{r}: \; -T = -mr\omega^2 \tag{2.1}$$

$$\hat{\theta}: \; 0 = m\left(2\dot{r}\omega + r\dot{\omega}\right) \tag{2.2}$$

$$\omega^{n.c.} = \int_{r_0}^{r_f} \vec{T} \cdot d\vec{r} = -\int_{r_0}^{r_f} T\, dr$$

$$\omega^{n.c.} = -\int_{r_0}^{r_f} mr\omega^2\, dr \tag{2.3}$$

Now ω is a function of r. To see this, consider the tangential force equation

$$0 = 2\dot{r}\omega + r\dot{\omega}$$

$$\Rightarrow 2\frac{dr}{dt}\omega = -r\frac{d\omega}{dt} \Rightarrow \frac{2dr}{r} = -\frac{d\omega}{\omega} \tag{2.4}$$

Equation (2.4) can be integrated

$$\int_{r_0}^{r} \frac{2dr'}{r'} = -\int_{\omega_0}^{\omega} \frac{d\omega'}{\omega'}$$

$$2\ln\left(\frac{r}{r_0}\right) = \ln\left(\frac{\omega_0}{\omega}\right) \Rightarrow \left(\frac{r}{r_0}\right)^2 = \frac{\omega_0}{\omega}$$

$$\Rightarrow r^2\omega = r_0^2\omega_0 \qquad \textbf{(2.5)}$$

Equation (2.5) shows that the quantity $r^2\omega = \text{constant}$ or $= \frac{r_0^2\omega_0}{r^2}$

So equation (2.3) becomes

$$\omega^{n.c} = -\int_{r_0}^{r_f} (mr')\left(\frac{r_0^4\omega_0^2}{r^4}\right) dr'$$

$$\omega^{n.c} = -\int_{r_0}^{r_f} mr_0^4\omega_0^2 \frac{dr'}{r^{3'}} = \frac{mr_0^4\omega_0^2}{2r^2}\Bigg|_{r_0}^{r_f}$$

$$= \frac{mr_0^4\omega_0^2}{2r_f^2} - \frac{mr_0^2\omega_0^2}{2} \qquad \textbf{(2.6)}$$

From equation (2.5): $r_f^2\omega_f = r_0^2\omega_0$

So equation (2.6) becomes:

$$\omega^{n.c} = \frac{1}{2}m\frac{r_f^4\omega_f^2}{r_f^2} - \frac{1}{2}mr_0^2\omega_0^2 \qquad \textbf{(2.7)}$$

$$\omega^{n.c.} = \frac{1}{2}mr_f^2\omega_f^2$$

The $\vec{v} = \dot{r}\hat{r} + r\omega\dot{\theta}$ with $v^2 = \left(\dot{r}^2 + r^2\omega^2\right)$

So the initial kinetic energy and final kinetic energy are:

$$K_o = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr_0^2\omega_0^2$$

$$K_f = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr_f^2\omega_f^2$$

Thus

$$K_f - K_0 = \frac{1}{2} m r_f^2 \omega_f^2 - \frac{1}{2} m r_0^2 \omega_0^2 = \omega^{n.c.}$$