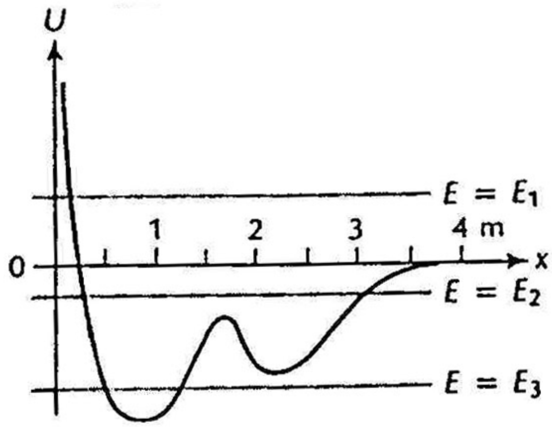
Potential Energy Diagrams
Challenge Problems

Problem 1



A particle moves along the x-axis under the influence of a conservative force with a potential energy $U(x)$. A plot of $U(x)$ vs. x is shown in the figure above. The figure shops several alternative energy levels for the particle: $E = E_1$, $E = E_2$, and $E = E_3$. Assume that the particle is initially at $x = x_0$. For each of the three alternative energy levels describe the motion qualitatively, answering the following questions.

a) Roughly, where are the turning points (right and left)?

b) Where is the speed of the particle maximum? Where is the speed minimum?

c) Is the orbit bound or unbound?

Solution:

a) For $E = E_1$, the left turning point is approximately at $x_{1,L} \simeq 0.3\text{ m}$. The right turning point is at $x_{1,R} = \infty$. For $E = E_2$, the left turning point is approximately at $x_{2,L} \simeq 0.4\text{ m}$. The right turning point is at $x_{2,R} \simeq 3.0\text{ m}$. For $E = E_3$, the left turning point is approximately at $x_{3,L} \simeq 0.5\text{ m}$. The right turning point is at $x_{2,R} \simeq 1.2\text{ m}$.

b)) For $E = E_1$, $E = E_2$, and $E = E_3$, the speed of the particle is maximum at the lowest point in the potential function that occurs at approximately $x_{\min} \simeq 0.8\text{ m}$. The speed is minimum at the finite valued turnaround points for each energy where it is zero.

c) The orbits are bound for $E = E_2$, and $E = E_3$, and unbound for $E = E_1$ because the particle has non-zero kinetic energy at infinity.

Problem 2: The force of interaction between a particle of mass m_1 and a second particle of mass m_2 separated by a distance r is given by an attractive gravitational force and a repulsive force that is proportional to r^{-3}, with a proportionality constant C,

$$\vec{\mathbf{F}}(r) = \left(-\frac{Gm_1m_2}{r^2} + C\frac{1}{r^3} \right)\hat{\mathbf{r}} .$$

a) Choose your zero point for potential energy at infinity. If the masses start off an infinite distance apart and are then moved until they are a distance r apart, what is the potential energy difference $U(r) - U(\infty) = -\int_{\infty}^{r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$?

b) What is the distance r_0 between the two masses when they are in stable equilibrium? What is the value of the potential energy $U(r_0)$ at stable equilibrium?

Solution: a) Because the force is radial symmetric, we can choose a radial path from $\infty \to r$ and choose for the path element $d\vec{\mathbf{s}} = dr\hat{\mathbf{r}}$. Then the dot product

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \left(-\frac{Gm_1m_2}{r^2} + C\frac{1}{r^3} \right)\hat{\mathbf{r}} \cdot dr\hat{\mathbf{r}} = \left(-\frac{Gm_1m_2}{r^2} + C\frac{1}{r^3} \right)dr .$$

The potential energy difference is then the integral

$$U(r) - U(\infty) = -\int_{\infty}^{r} \left(-\frac{Gm_1m_2}{r^2} + C\frac{1}{r^3} \right)dr = -\left.\frac{Gm_1m_2}{r}\right|_{\infty}^{r} + \left.\frac{2C}{r^2}\right|_{\infty}^{r} = -\frac{Gm_1m_2}{r} + \frac{2C}{r^2} .$$

a) Stable equilibrium occurs at $r = r_0$ when the force on the particle is zero. So set

$$-\frac{Gm_1m_2}{r_0^2} + C\frac{1}{r_0^3} = 0 ,$$

and solve for r_0:

$$r_0 = \frac{C}{Gm_1m_2} .$$

The value of the potential energy at $r = r_0$ is then

$$U(r_0) = -\frac{Gm_1m_2}{r_0} + \frac{2C}{r_0{}^2} = r_0 = -\frac{(Gm_1m_2)^2}{C} + \frac{2(Gm_1m_2)^2}{C} = \frac{(Gm_1m_2)^2}{C}$$

Problem 3

A particle of mass m moves in one dimension. Its potential energy is given by

$$U(x) = -U_0 e^{-x^2/a^2} ,$$

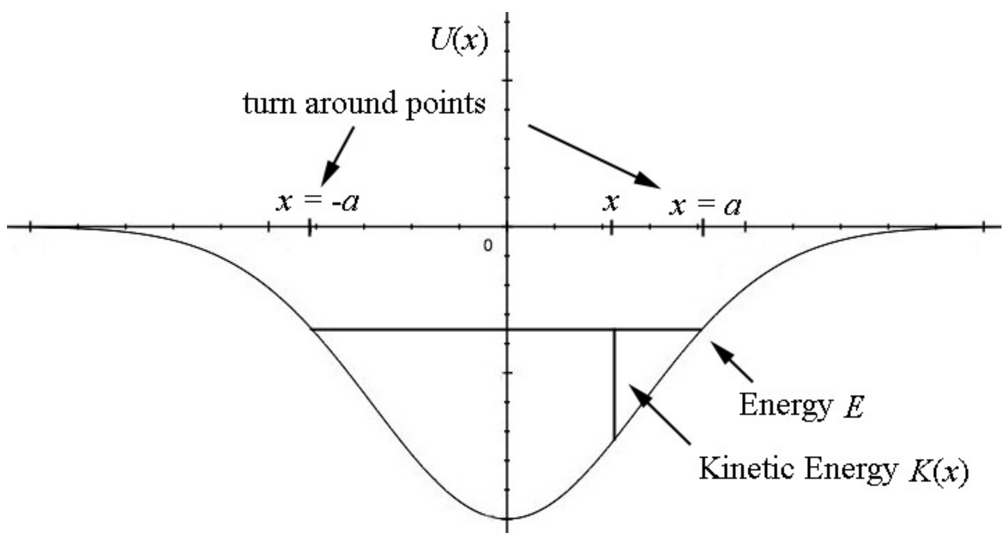
where U_0 and a are constants.

a) Draw an energy diagram showing the potential energy $U(x)$, the kinetic energy $K(x)$, and the total energy $E < 0$ for a motion which is bound between turning points $\pm a$.

b) Find the force on the particle, $F(x)$, as a function of position x.

c) Find the speed at the origin $x = 0$ such that the when the particle reaches the positions $x = \pm a$, it will reverse its motion.

Solution:



The force on the particle is zero at the minimum of the potential which occurs at

$$F_x(x) = -\frac{dU}{dx}(x) = \frac{2x}{a^2} U_0 e^{-x^2/a^2} . \tag{3.1}$$

The minimum value occurs where $F_x(x) = 0$. i.e. when $x = 0$.

The urn around points occur where the kinetic energy is zero hence $E = U$. Thus when $x = \pm a$, the potential energy and hence the energy is given by

$$E = U(x = \pm a) = -U_0 e^{-1} \tag{3.2}$$

At the minimum, $x = 0$, the potential energy $U(0) = -U_0$. Therefore the kinetic energy is given by

$$K(0) = E - U(0) = -U_0 e^{-1} - (-U_0) = U_0(1 - e^{-1}). \tag{3.3}$$

Hence the speed at the origin is given by

$$v = \sqrt{\frac{2K(0)}{m}} = \sqrt{\frac{2U_0(1 - e^{-1})}{m}}. \tag{3.4}$$

Problem 4

The force on a particle is given by

$$\vec{\mathbf{F}}(x) = F_0(e^{-2(x-x_0)/x_0} - e^{-x/x_0})\hat{\mathbf{i}}$$

where F_0 and x_0 are positive and $\hat{\mathbf{i}}$ is a unit vector in the positive x-direction.

a) For what value of x is the force zero?

b) What is $U(x) - U(x_0)$, the potential energy, when the particles are a distance x apart?

c) Sketch $U(x)$ with the choice that $U(x_0) = (F_0 x_0 / 2)(1 - 2e^{-1})$

Solution:

a) $\vec{\mathbf{F}}(x) = F_0(e^{-2(x-x_0)/x_0} - e^{-x/x_0})\hat{\mathbf{i}} = 0$ when $e^{-2(x-x_0)/x_0} = e^{-x/x_0}$ or $e^{x/x_0} = e^2$. Taking the natural logarithm of each side of the above equation yields $x / x_0 = 2$ or $x = 2x_0$.

b) The potential difference $U(x) - U(x_0)$ is the negative of the work done in displacing the particle from x_0 to x. Because the force is not constant we must integrate

$$U(x) - U(x_0) = -\int_{x_0}^{x} \vec{F} \cdot d\vec{r} = -\int_{x_0}^{x} F_0(e^{-2(x-x_0)/x_0} - e^{-x/x_0})\hat{\mathbf{i}} \cdot dx\hat{\mathbf{i}}$$

$$= -\int_{x_0}^{x} F_0(e^{-2(x-x_0)/r_0} - e^{-x/x_0})dx = -F_0 \left.\frac{e^{-2(x-x_0)/x_0}}{(-2/x_0)}\right|_{x_0}^{x} + F_0 \left.\frac{e^{-x/x_0}}{(-1/x_0)}\right|_{x_0}^{x}$$

$$= \left(-F_0 \frac{e^{-2(x-x_0)/x_0}}{(-2/x_0)} - -F_0 \frac{1}{(-2/x_0)}\right) + \left(F_0 \frac{e^{-x/x_0}}{(-1/x_0)} - F_0 \frac{e^{-1}}{(-1/x_0)}\right)$$

$$= \frac{F_0 x_0}{2}\left(e^{-2(x-x_0)/x_0} - 2e^{-x/x_0} - 1 + 2e^{-1}\right)$$

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b) With the choice $U(x_0) = (F_0 x_0 / 2)(1 - 2e^{-1})$,

$$U(x) = \frac{F_0 x_0}{2}\left(e^{-2(x-x_0)/x_0} - 2e^{-x/x_0}\right); \quad U(x_0) = \frac{F_0 x_0}{2}(1 - 2e^{-1}).$$

c) A plot of $U(x)$ vs. x is shown in the figure below for the values $F_0 = 2\ \text{N}$ and $x_0 = 1\ \text{m}$.