Mechanical Energy and Simple Harmonic Oscillator

Simple Harmonic Motion

Hooke's Law

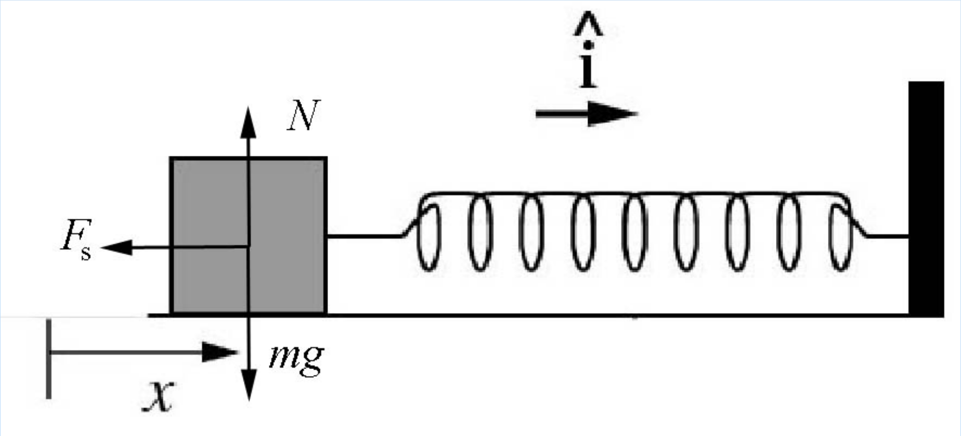
Define system, choose coordinate system.

Draw free-body diagram.

Hooke's Law

$$\vec{\mathbf{F}}_{\text{spring}} = -kx\,\hat{\mathbf{i}}$$

$$-kx = m\frac{d^2x}{dt^2}$$



Checkpoint Problem

Which of the following functions *x*(*t*) has a second derivative which is proportional to the negative of the function

$$\frac{d^2x}{dt^2} \propto -x?$$

1. $x(t) = \frac{1}{2}at^2$
2. $x(t) = Ae^{t/T}$
3. $x(t) = Ae^{-t/T}$
4. $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$

Period and Angular Frequency

Equation of Motion:

$$-kx = m\frac{d^2x}{dt^2}$$

Solution: Oscillatory with Period T

$$x = A\cos\left(\frac{2\pi}{T}t\right) + B\sin\left(\frac{2\pi}{T}t\right)$$

x -component of velocity:

$$v_x \equiv \frac{dx}{dt} = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right) + \frac{2\pi}{T}B\cos\left(\frac{2\pi}{T}t\right)$$

x -component of acceleration:

$$a_x \equiv \frac{d^2x}{dt^2} = -\left(\frac{2\pi}{T}\right)^2 A\cos\left(\frac{2\pi}{T}t\right) - \left(\frac{2\pi}{T}\right)^2 B\sin\left(\frac{2\pi}{T}t\right) = -\left(\frac{2\pi}{T}\right)^2 x$$

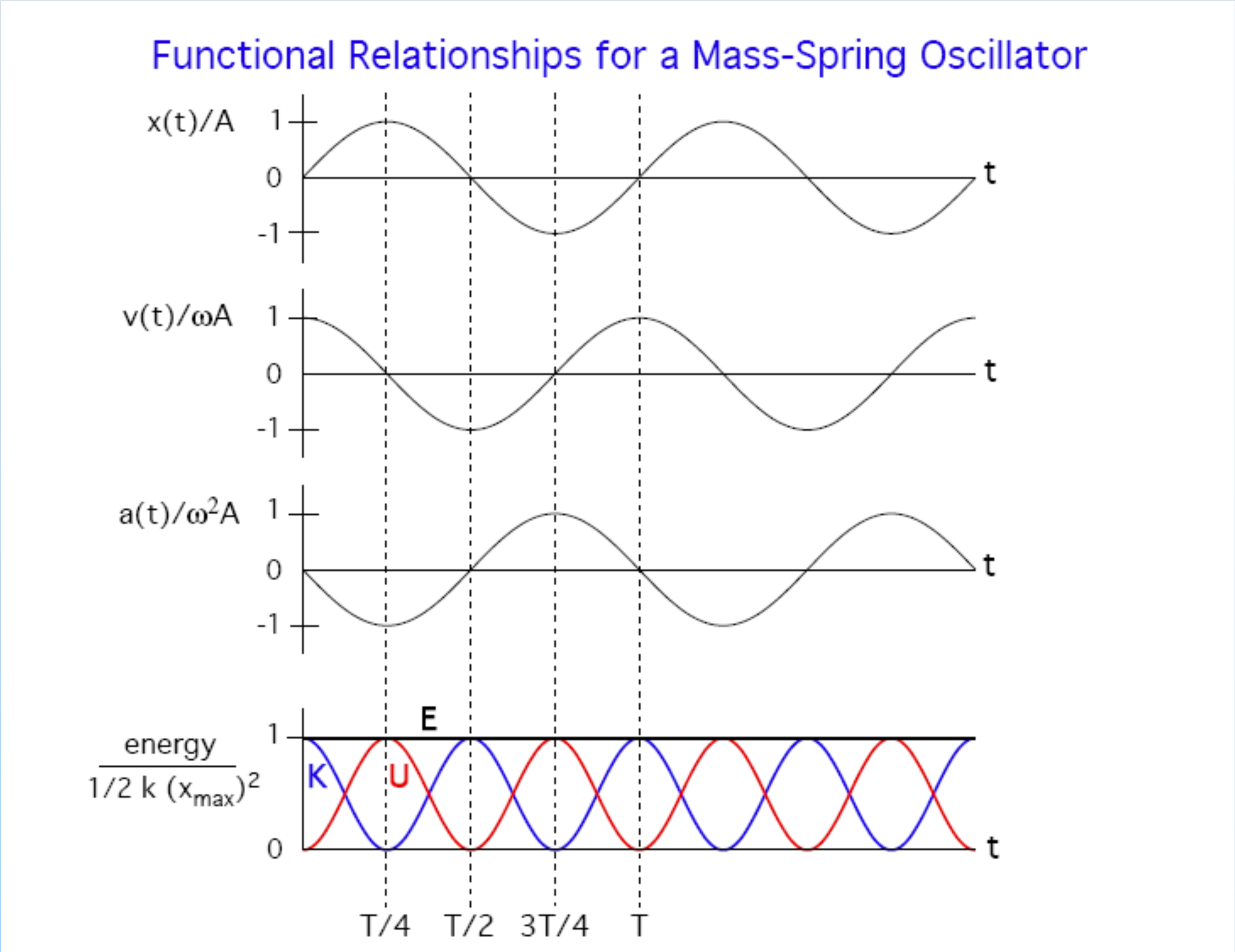
Period:

$$-kx = m\frac{d^2x}{dt^2} = -m\left(\frac{2\pi}{T}\right)^2 x \Rightarrow k = m\left(\frac{2\pi}{T}\right)^2 \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

Angular frequency

$$\omega \equiv \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

Source



Functional Relationships for a Mass-Spring Oscillator
x(t)/A
1
0
-1
t
v(t)/ωA
1
0
-1
t
a(t)/ω²A
1
0
-1
t
energy
1/2 k (x_max)²
E
K
U
1
0
t
T/4
T/2
3T/4
T

Simple Harmonic Motion: Initial Conditions

Equation of Motion: $$-kx = m\frac{d^2x}{dt^2}$$

Solution: Oscillatory with Period $$T = 2\pi\sqrt{\frac{m}{k}}$$

Position: $$x = A\cos\left(\frac{2\pi}{T}t\right) + B\sin\left(\frac{2\pi}{T}t\right)$$

Velocity: $$v_x = \frac{dx}{dt} = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right) + \frac{2\pi}{T}B\cos\left(\frac{2\pi}{T}t\right)$$

Initial Position at *t* = 0: $$x_0 \equiv x(t=0) = A$$

Initial Velocity at *t* = 0: $$v_{x,0} \equiv v_x(t=0) = \frac{2\pi}{T}B$$

General Solution: $$x = x_0\cos\left(\frac{2\pi}{T}t\right) + \frac{T}{2\pi}v_{x,0}\sin\left(\frac{2\pi}{T}t\right)$$

Demo slide: spray paint oscillator C4

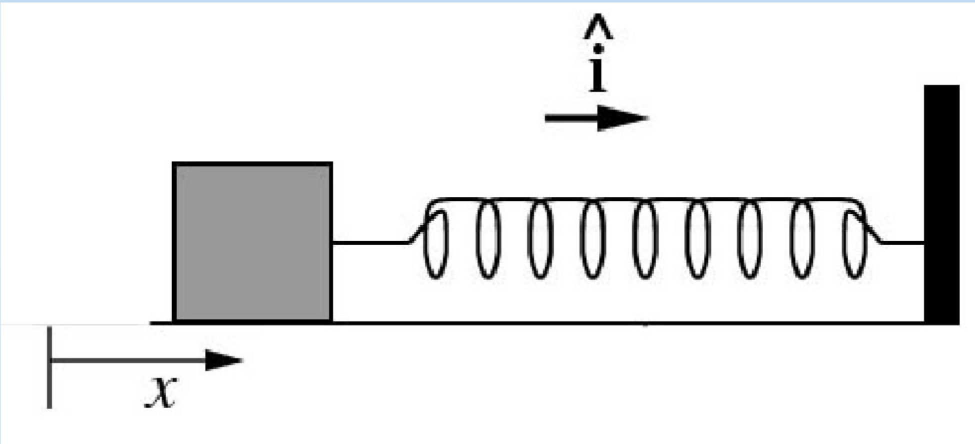
1. Illustrating choice of $t = 0$

Strategy:

1. Recognizing SHO equation
2. Remembering solutions
3. Using initial conditions

Worked Example: Block-Spring System with No Friction

A block of mass m slides along a frictionless horizontal surface with speed $v_{x,0}$. At $t = 0$ it hits a spring with spring constant k and begins to slow down. How far is the spring compressed when the block has first come momentarily to rest?



Initial and Final Conditions

Initial state: $x_0 = A = 0$ and $v_{x,0} = \dfrac{2\pi}{T} B$

$$x(t) = \frac{T}{2\pi} v_{x,0} \sin\left(\frac{2\pi}{T} t\right) \qquad v_x(t) = v_{x,0} \cos\left(\frac{2\pi}{T} t\right)$$

First comes to rest when $v_x(t_f) = 0 \Rightarrow \dfrac{2\pi}{T} t_f = \dfrac{\pi}{2}$

Since at time $t_f = T/4$

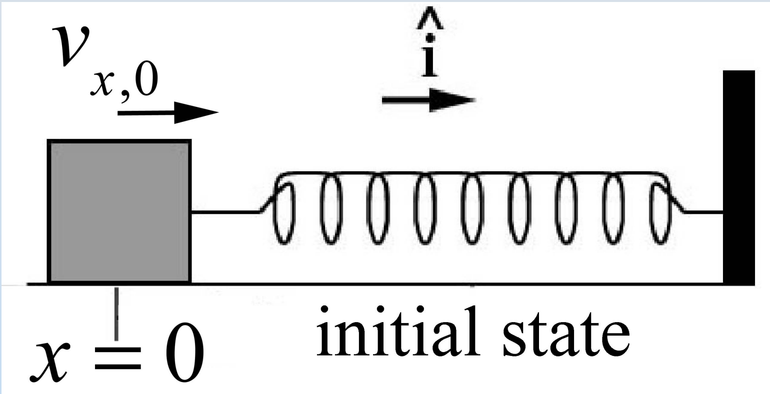
$$\sin\left(\frac{2\pi}{T} t_f\right) = \sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow$$

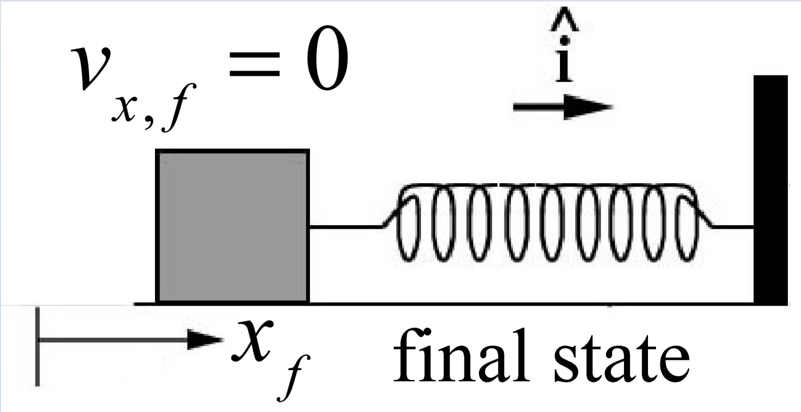
Final position

$$x(t_f) = \frac{T}{2\pi} v_{x,0} = \sqrt{\frac{m}{k}} v_{x,0}$$

Modeling the Motion: Energy

Choose initial and final states:





Change in potential energy: $U(x_f) - U(x_0) = \frac{1}{2}k\left(x_f^2 - x_0^2\right)$

Choose zero point for potential energy: $U(x = 0) = 0$

Potential energy function: $U(x) = \frac{1}{2}kx^2, \quad U(x = 0) = 0$

Mechanical energy is constant $(W_{\text{nc}} = 0)$ $\quad E_{\text{final}}^{\text{mechanical}} = E_{\text{initial}}^{\text{mechanical}}$

Kinetic Energy vs. Potential Energy

State	Kinetic energy	Potential energy	Mechanical energy
Initial $x_0 = 0$ $v_{x,0} > 0$	$K_0 = \frac{1}{2}mv_{x,0}^2$	$U_0 = 0$	$E_0 = \frac{1}{2}mv_{x,0}^2$
Final $x_f > 0$ $v_{x,f} = 0$	$K_f = 0$	$U_f = \frac{1}{2}kx_f^2$	$E_f = \frac{1}{2}kx_f^2$

Conservation of Mechanical Energy

$$E_f = E_0 \quad \Rightarrow \quad \frac{1}{2}kx_f^2 = \frac{1}{2}mv_{x,0}^2$$

The amount the spring has compresses when the object first comes to rest is

$$x_f = \sqrt{\frac{m}{k}}v_{x,0}$$

Checkpoint Problem: Simple Harmonic Motion

A block of mass m is attached to a spring with spring constant k is free to slide along a horizontal frictionless surface.

At $t = 0$ the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest. What is the x -component of the velocity of the block when it first comes back to the equilibrium?

Energy Diagram

Choose zero point for potential energy:

$$U(x = 0) = 0$$

Potential energy function:

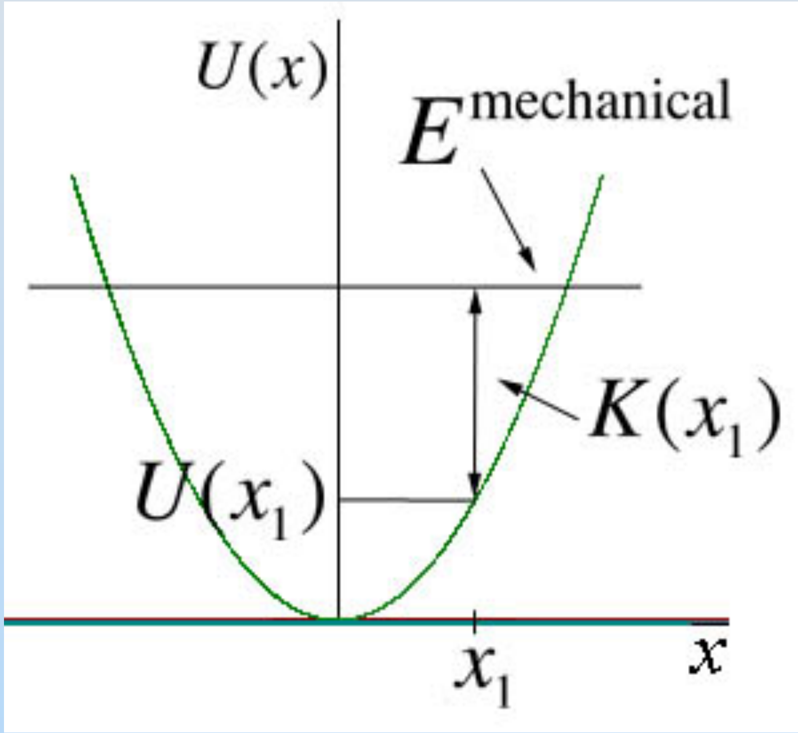
$$U(x) = \frac{1}{2}kx^2, \quad U(x = 0) = 0$$

Mechanical energy is represented by a horizontal line since it is a constant

$$E^{\text{mechanical}} = K(x) + U(x) = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$

Kinetic energy is difference between mechanical energy and potential energy (independent of choice of zero point)

$$K = E^{\text{mechanical}} - U$$



Graph of Potential energy function $U(x)$ vs. x

Checkpoint Problem: Energy Diagram

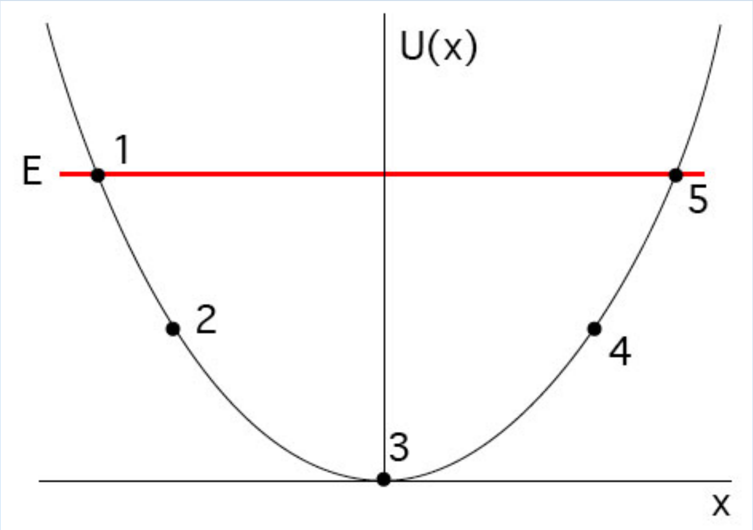
The potential energy function

$$U(x) = (1/2)kx^2$$

for a particle with total mechanical energy *E* is shown in the figure. The position of the particle as a function of time is given by

$$x(t) = A\cos(\omega t - \pi/4)$$

where ω is the angular frequency of oscillation.



a) At what time does the particle first reach position 3.

b) Is it moving in the positive or negative *x*-direction when it first reaches position 3.

Checkpoint Problem: SHO and the Pendulum

Suppose the point-like object of a simple pendulum is pulled out at by an angle θ_0 << 1 rad. Is the angular speed of the point-like object equal to the angular frequency of the pendulum?

1. Yes.
2. No.
3. Only at bottom of the swing.
4. Not sure.

Checkpoint Problem: Simple Pendulum by Energy Method

A simple pendulum consists of a massless string of length l and a point like object of mass m is attached to one end. Suppose the string is fixed at the other end and the object is initially pulled out at an angle of θ_0 from the vertical and released at rest.

1. Use the fact that the energy is constant to find a differential equation describing how the second derivative of the angle θ the object makes with the vertical varies in time.
2. Find an expression for the angular velocity of the object at the bottom of its swing.

Now assume that the initial angle $\theta_0 << 1$ rad and thus you can use the small angle approximation.

3. First use $\sin\theta \cong \theta$ to find a differential equation describing how the second derivative of the angle θ the object makes with the vertical varies in time.
4. Also use the approximation $\cos\theta_0 \cong 1 - \theta_0^2/2$, to find an expression for the angular velocity of the object at the bottom of its swing.

Checkpoint Problem: fluid oscillations in a U-tube

A U-tube open at both ends to atmospheric pressure is filled with an incompressible fluid of density ρ. The cross-sectional area A of the tube is uniform and the total length of the column of fluid is L. A piston is used to depress the height of the liquid column on one side by a distance x, and then is quickly removed. What is the frequency of the ensuing simple harmonic motion? Assume streamline flow and no drag at the walls of the U-tube. The gravitational constant is g.