Simple Harmonic Motion
Concept Questions

Question 1 Which of the following functions $x(t)$ has a second derivative which is proportional to the negative of the function

$$\frac{d^2x}{dt^2} \propto -x?$$

1. $x(t) = \frac{1}{2}at^2$
2. $x(t) = Ae^{t/T}$
3. $x(t) = Ae^{-t/T}$
4. $x(t) = A\sin\left(\frac{2\pi}{T}t\right)$
5. $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$
6. None of the above
7. Two of the above

Solution 7. By direct calculation, when

$$x(t) = A\sin\left(\frac{2\pi}{T}t\right),$$

$$\frac{dx(t)}{dt} = \left(\frac{2\pi}{T}\right)A\cos\left(\frac{2\pi}{T}t\right)$$

$$\frac{d^2x(t)}{dt^2} = -\left(\frac{2\pi}{T}\right)^2 A\sin\left(\frac{2\pi}{T}t\right) = -\left(\frac{2\pi}{T}\right)^2 x(t) .$$

Similarly

$$x(t) = A\cos\left(\frac{2\pi}{T}t\right),$$

$$\frac{dx(t)}{dt} = -\left(\frac{2\pi}{T}\right)A\sin\left(\frac{2\pi}{T}t\right)$$

$$\frac{d^2x(t)}{dt^2} = -\left(\frac{2\pi}{T}\right)^2 A\cos\left(\frac{2\pi}{T}t\right) = -\left(\frac{2\pi}{T}\right)^2 x(t) .$$

Note that

$$x(t) = \frac{1}{2}at^2 ,$$

$$\frac{dx(t)}{dt} = at$$

$$\frac{d^2x(t)}{dt^2} = a \neq -x(t)$$

Also when

$$x(t) = Ae^{t/T} ,$$

$$\frac{dx(t)}{dt} = \frac{1}{T}Ae^{t/T}$$

$$\frac{d^2x(t)}{dt^2} = \left(\frac{1}{T}\right)^2 Ae^{t/T} = \left(\frac{1}{T}\right)^2 x(t) \neq -x(t)$$

And when

$$x(t) = Ae^{-t/T} ,$$

$$\frac{dx(t)}{dt} = -\frac{1}{T}Ae^{-t/T}$$

$$\frac{d^2x(t)}{dt^2} = \left(-\frac{1}{T}\right)^2 Ae^{-t/T} = \left(\frac{1}{T}\right)^2 x(t) \neq -x(t) .$$

Question 2: Simple Harmonic Motion

A block of mass m is attached to a spring with spring constant k is free to slide along a horizontal frictionless surface. At $t = 0$ the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest. What is the x -component of the velocity of the block when it first comes back to the equilibrium?

1. $v_x = -x_0 \dfrac{T}{4}$

2. $v_x = x_0 \dfrac{T}{4}$

3. $v_x = -\sqrt{\dfrac{k}{m}} x_0$

4. $v_x = \sqrt{\dfrac{k}{m}} x_0$

5. None of the above.

Solution 3. The particle starts with potential energy $U_0 = (1/2)kx_0^2$. When it first returns to equilibrium it now has only kinetic energy $K_1 = (1/2)mv_x^2$. Since the energy of the block-spring system is constant, $K_1 = U_0$ and so

$$(1/2)mv_x^2 = (1/2)kx_0^2$$

We can solve for the x-component of the velocity

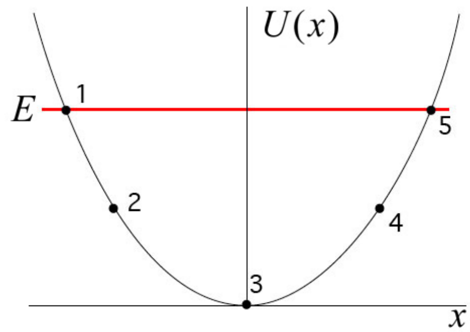
$$v_x = \pm\sqrt{\frac{k}{m}x_0^2} \; .$$

Since the object is moving in the negative x-direction when it first returns to equilibrium so we must take the negative square root,

$$v_x = -\sqrt{\frac{k}{m}x_0^2}$$

Question 3

The potential energy function $U(x)$ for a particle with total mechanical energy E is shown below.



The position of the particle as a function of time is given by

$$x(t) = D\cos(\omega t) + D\sin(\omega t) \qquad (3.1)$$

.
where $D > 0$. The particle first reaches the position 3 when

1. $\omega t = 0$
2. $\omega t = \pi / 4$
3. $\omega t = \pi / 2$
4. $\omega t = 3\pi / 4$
5. $\omega t = \pi$
6. $\omega t = 5\pi / 4$
7. $\omega t = 3\pi / 2$
8. $\omega t = 7\pi / 4$

Answer 4: The initial conditions associated with

$$x(t) = D\cos(\omega_0 t) - D\sin(\omega_0 t)$$

are that $x_0 = D > 0$. Taking a time derivative shows that

$$v_x(t) = -\omega_0 D\sin(\omega_0 t) - \omega_0 D\cos(\omega_0 t)\,,$$

hence $v_{x,0} = -\omega_0 D < 0$. Thus the particle starts out with a positive x- coordinate and the initial x-component of the velocity is negative therefore it is moving toward from the origin. It has both non-zero potential and kinetic energies and hence it I not at it's maximal position so at $t = 0$ it is located at position 4.

Note: It arrives at

$$x(t) = D\cos(\omega_0 t) - D\sin(\omega_0 t) = 0$$

when $\cos(\omega_0 t) = \sin(\omega_0 t)$ which first occurs at $\omega_0 t_1 = \pi / 4$ or $t_1 = \pi / 4\omega_0$.

Question 4: SHO and the Pendulum

Suppose the point-like object of a simple pendulum is pulled out at by an angle θ_0 << 1 rad. Is the angular speed of the point-like object equal to the angular frequency of the pendulum?

1. Yes.
2. No.
3. Only at bottom of the swing.
4. Not sure.

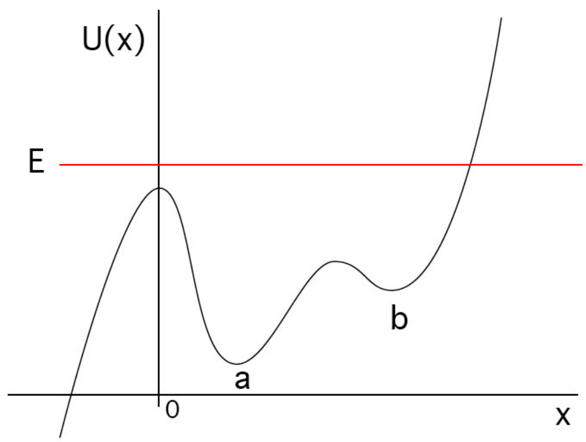
Solution 2:

The angular frequency is a constant of the motion and by definition is $\omega_0 \equiv 2\pi / T$. For small angle the pendulum approximates a simple harmonic oscillator with $\omega_0 \equiv \sqrt{g/l}$ where l is the length of the pendulum. The angular speed by definition is the magnitude of the component of the angular velocity $\omega \equiv d\theta / dt$. Note that sometimes the symbol ω may be used for both quantities. This is a result of the fact that for uniform circular motion, angular frequency and angular speed are equal because the period $T = 2\pi R / v$ and the speed and angular speed are related by $v = R\omega$. Therefore $T = 2\pi R / R\omega = 2\pi / \omega$. So $\omega = \omega_0$ for this special case.

Question 5: Energy Diagram 1

A particle with total mechanical energy E has position $x > 0$ at $t = 0$

1. escapes to infinity in the – x-direction
2. approximates simple harmonic motion
3. oscillates around a
4. oscillates around b
5. periodically revisits a and b
6. not enough information
7. two of the above.

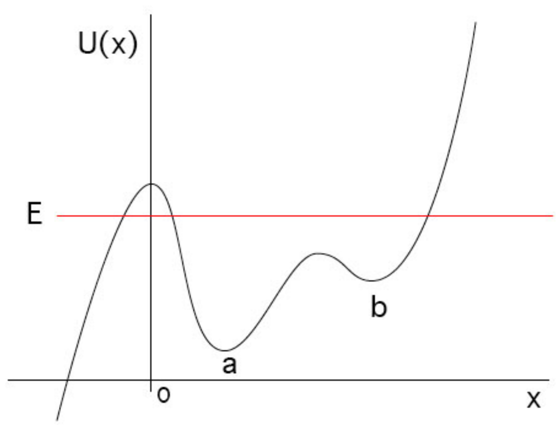


Solution 1. Because the energy is greater than the potential energy for values of x that approach negative infinity (on the left in the above figure), the particle can escape to infinity in the – x-direction with a positive kinetic energy.

Question 6: Energy Diagram 2

A particle with total mechanical energy E has position $x > 0$ at $t = 0$

1. escapes to infinity in the – x-direction
2. approximates simple harmonic motion
3. oscillates around a
4. oscillates around b
5. periodically revisits a and b
6. not enough information
7. two of the above.

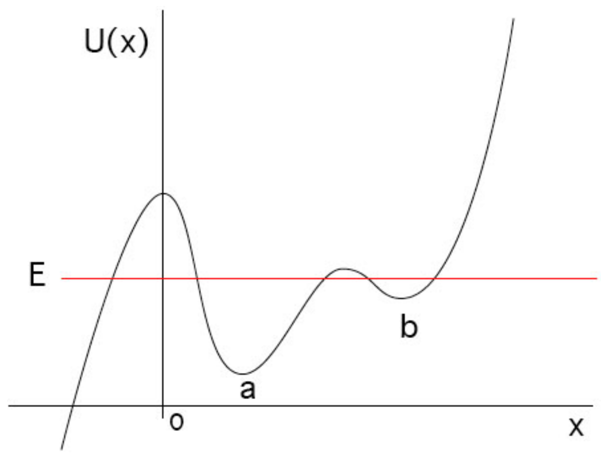


Solution 5. Now the range of motion for the particle is limited to the regions in which the kinetic energy is either zero or positive, so the particle is confined to the regions where the potential energy is less than the energy. Hence the particle periodically revisits a and b.

Question 7: Energy Diagram 3

A particle with total mechanical energy E has position $x > 0$ at $t = 0$

1. escapes to infinity in the – x-direction
2. approximates simple harmonic motion
3. oscillates around a
4. oscillates around b
5. periodically revisits a and b
6. not enough information
7. two of the above

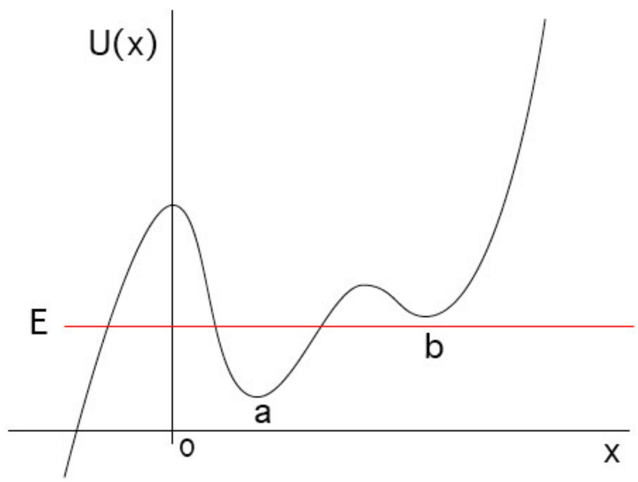


Solution 6. Now the range of motion for the particle is limited to the regions in which the kinetic energy is either zero or positive, so the particle is confined to either the region around a and or the region around b but since we do not know where the particle has started, we do not have enough information to state where the particle will be.

Question 8: Energy Diagram 4

A particle with total mechanical energy E has position $x > 0$ at $t = 0$

1. escapes to infinity in the – x-direction
2. approximates simple harmonic motion
3. oscillates around a
4. oscillates around b
5. periodically revisits a and b
6. not enough information
7. two of the above

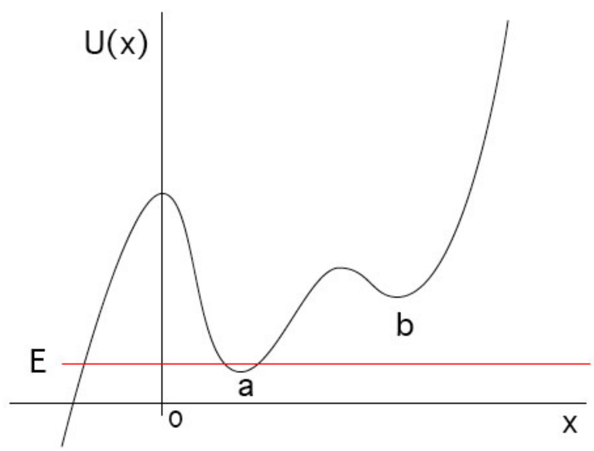


Solution 3. Now the range of motion for the particle is limited to the regions in which the kinetic energy is either zero or positive, so the particle is confined to move around the region surrounding a . The motion will be periodic but not simple harmonic motion because the potential energy function is not a quadratic function and only for quadratic potential energy functions will the motion be simple harmonic. Hence the particle oscillates around a.

Question 9: Energy Diagram 5

A particle with total mechanical energy E has position $x > 0$ at $t = 0$

1. escapes to infinity in the – x-direction
2. approximates simple harmonic motion
3. oscillates around a
4. oscillates around b
5. periodically revisits a and b
6. not enough information
7. two of the above



Solution 2. Now the particle oscillates around the region surrounding a. Since the energy is so close to the minimum of the potential energy, we can approximate the potential energy as a quadratic function and hence the particle motion approximates simple harmonic motion. Mathematically, we, calculate the Taylor series polynomial expansion of the potential energy about the minimum. The coefficient of the linear term is equal to the slope at the minimum which is zero, and the next term in the polynomial that is non-zero is the quadratic term. The higher order polynomial terms are much smaller so we can ignore them. Thus we approximate the potential energy by a quadratic polynomial and hence the motion is approximately simple harmonic oscillator.