Simple Harmonic Oscillator
Challenge Problems

Problem 1: ***Dimensional Analysis, Estimation and Concepts***

Imagine that one drilled a hole with smooth sides straight through the center of the earth, of radius $R_e = 6.4\times10^6\ \mathrm{m}$. If the air is removed from this tube (and the tube doesn't fill up with water, liquid rock or iron from the core), an object dropped into one end will have enough energy to just exit the other end after an interval of time. Use dimensional analysis to estimate that interval of time. Let $g = 9.8\ \mathrm{m\cdot s^{-2}}$ be the gravitational constant.

Problem 1 Solutions:

The only combination of the given parameters that has dimensions of time is

$$\sqrt{\frac{R_e}{g}} = 810\,\mathrm{s} = 13.5\,\mathrm{min}. \tag{1.1}$$

If such a hole could be made, and the density of the earth were constant, the motion would be simple harmonic. The force on an object would be its greatest at the surface of the earth, decrease to zero as the object approached the center of the earth, and reverse direction (that is, remain directed towards the center of the earth) after passing through the center. For uniform density, the force would be proportional to the distance from the center. If we model this motion as the object being subject to a restoring force with "effective spring constant" k_{eff}, we have $k_{\mathrm{eff}} R_e = mg$, and the period of this motion would be $T = 2\pi\sqrt{R_e / g} = 87\,\mathrm{min}$. It's not hard to show (but not part of this problems) that this is the period of a satellite in low orbit about the earth.
Equation (1.1) might be recognized as (within the factor of 2π, and with small amplitude) as the period of a simple pendulum with length equal to the radius of the earth. If anyone can construct such an item, many of us would pay to see it.

Problem 2: Periodic Motion:

The motion of an object moving in one dimension is given by the function

$$x(t) = A\cos\left(\frac{2\pi}{T}t\right).$$

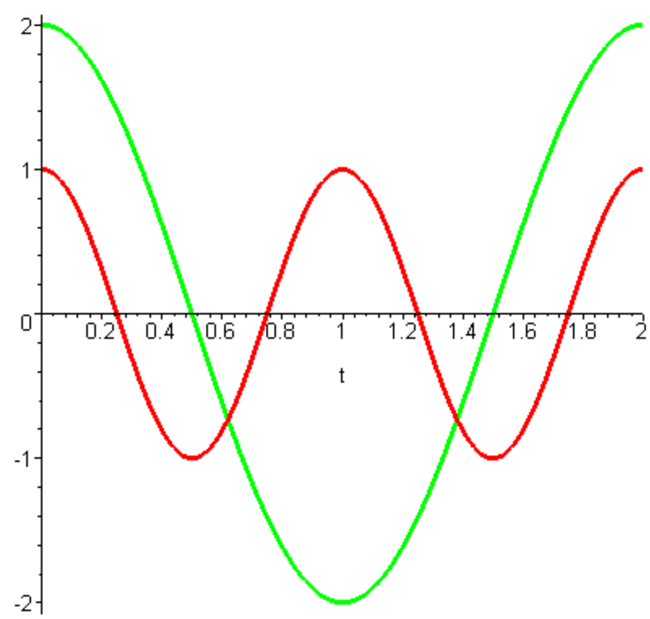
a) In your own words, describe the meaning of the constants T and A that appears in the above equation.

b) Find the velocity and acceleration of the object as functions of time.

c) Graph the position, velocity, and acceleration as functions of time. Be sure to indicate clearly on your graph the constants T and A.

Problem 2 Solution

a) The figure below shows two plots of $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$, one scaled to $A=1$, $T=1$, the other with $A=2$, $T=2$.



The multiplying factor A is known as the *amplitude* (the amplitude is, strictly speaking, is $|A|$). The function $x(t)$ assumes values between $-|A|$ and $|A|$.
The constant T has dimensions of time, and is known at the *period* of the oscillation.

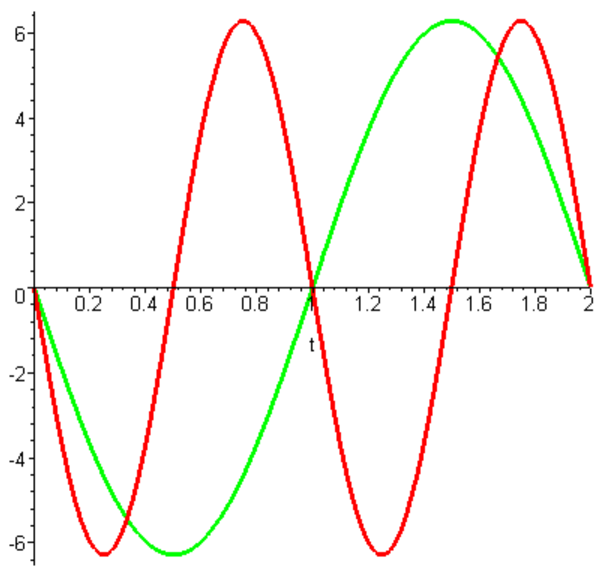
For the purposes of this problem (and many similar problems), a function $x(t)$ has *periodicity* T if

$$x(t) = x(t+T) .$$

b) & c) Let $u(t) = 2\pi\, t / T$, so that $x(t) = A\cos(u(t))$. The velocity of the object is

$$v_x = \frac{d}{dt}x(t) = \left(\frac{d}{du}x\right)\frac{du}{dt} = \left(A\sin(u)\right)\frac{2\pi}{T} = \frac{2\pi}{T}A\left(\sin\left(\frac{2\pi}{T}t\right)\right).$$

Plots of the velocity corresponding to the expressions for $x(t)$ are shown below. Note that the maximum and minimum of both plots are the same, since for the chosen scaling the ration A/T is the same for both plots.



Problem 3:

Imagine that one drilled a hole with smooth sides straight through the center of the earth. If the air is removed from this tube (and the tube doesn't fill up with water, liquid rock or iron from the core), an object dropped into one end will have enough energy to just exit the other end after an interval of time. Your goal is to find that interval of time. The steps outlined below show a way of finding this time interval. Make the assumption that the earth has uniform mass density.

a) The gravitational force on an object of mass m located inside the earth a distance $r < r_e$ from the center (r_e is the radius of the earth) is due only to the mass of the earth that lies within a solid sphere of radius r. What is the gravitational force as a function of the distance r from the center of the earth? Express your answer in terms of the gravitational acceleration at the surface of the earth g and r_e Note: you do not need the mass of the earth m_e or the universal gravitation constant G to answer this question but you will need to find an expression relating m_e and G to g and r_e. You only need to assume that the earth is of uniform mass density. (You can neglect the amount of mass you drilled out.)

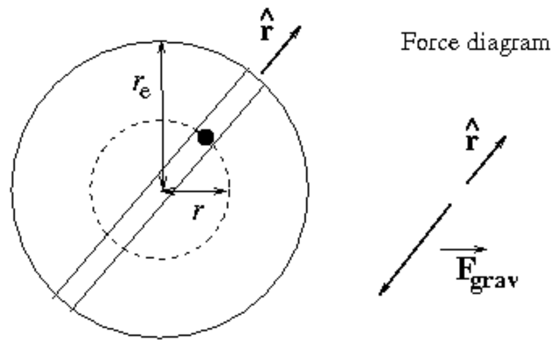
b) Use your result of part a) to explain why the object of mass m should oscillate (analogous to an object attached to a spring). In particular, how long would it take for this object to reach the other side of the earth?

c) What is the potential energy inside the earth as a function of r for the object-earth system? Can you think of a natural point to choose a zero point for the potential energy? Be careful because you will need to do a work integral to determine the change in potential energy when the object moves inside the earth and the gravitation force is no longer an inverse square when the object is inside the earth. Use energy considerations to find the velocity of the object when it passes through the center of the earth.

Problem 3 Solutions:

a) Choose a radial coordinate with unit vector $\hat{\mathbf{r}}$ pointing outward from the center of the earth. The gravitational force on an object of mass m at the surface of the earth is given by two expressions

$$\vec{\mathbf{F}}_{grav} = -\frac{Gmm_e}{r_e^2}\hat{\mathbf{r}} = -mg\hat{\mathbf{r}}\,. \qquad \textbf{(3.1)}$$



Therefore we can solve for the gravitational constant at the surface of the earth,

$$g = \frac{Gm_{\mathrm{e}}}{r_{\mathrm{e}}^{2}}. \qquad \textbf{(3.2)}$$

When the object is a distance r from the center of the earth, the mass of the earth that lies outside the sphere of radius r does not contribute to the gravitational force. The only contribution to the gravitational force is due to the mass enclosed in the sphere of radius r. In terms of the (uniform) mass density,

$$m_{\mathrm{enclosed}} = \rho \frac{4}{3}\pi r^{3}. \qquad \textbf{(3.3)}$$

The mass density is given by

$$\rho = \frac{m_{\mathrm{e}}}{(4/3)\pi r_{\mathrm{e}}^{3}} \qquad \textbf{(3.4)}$$

And the mass enclosed is

$$m_{\mathrm{enclosed}} = \frac{m_{\mathrm{e}}}{(4/3)\pi r_{\mathrm{e}}^{3}}(4/3)\pi r^{3} = \frac{m_{\mathrm{e}} r^{3}}{r_{\mathrm{e}}^{3}}. \qquad \textbf{(3.5)}$$

Therefore the gravitational force on the object of mass m when it is a distance r from the center of the earth is given by

$$\vec{\mathbf{F}}_{\mathrm{grav}} = -G\frac{m\, m_{\mathrm{enclosed}}}{r^{2}}\hat{\mathbf{r}} = -G\frac{m\, m_{\mathrm{e}} r^{3}}{r^{2} r_{\mathrm{e}}^{3}}\hat{\mathbf{r}} = -G\frac{m\, m_{\mathrm{e}}}{r_{\mathrm{e}}^{3}} r\,\hat{\mathbf{r}}. \qquad \textbf{(3.6)}$$

We can use our expression for $g = G\, m_{\mathrm{e}} / r_{\mathrm{e}}^{2}$ to find that the gravitational force on a mass m at a distance r from the center of the earth is given by

$$\vec{\mathbf{F}}_{\text{grav}} = -\frac{mg}{r_{\text{e}}} r\,\hat{\mathbf{r}}\,. \tag{3.7}$$

b) The minus sign indicates that the force is always directed towards the center of the earth (restoring force) and proportional to the distance from the center of the earth. This is analogous to the restoring force of a spring,

$$\vec{\mathbf{F}}_{\text{spring}} = -kx\,\hat{\mathbf{i}} \tag{3.8}$$

where the "spring constant" for gravitation is given by

$$k_{\text{grav}} = \frac{mg}{r_{\text{e}}}\,. \tag{3.9}$$

Note that in Equation (3.9), the combination $k_{\text{grav}} = mg / r_{\text{e}}$ has dimensions of force divided by length, as do spring constants.

Comparison of Equations (3.7) and (3.8) indicates that the object would undergo simple harmonic motion as if it were attached to a spring with the spring constant $k_{\text{grav}} = mg / r_{\text{e}}$.

The radial component of Newton's Second Law, $\vec{\mathbf{F}} = m\mathbf{a}$, becomes

$$-\frac{mg}{r_{\text{e}}} r = m\frac{d^2 r}{dt^2} \tag{3.10}$$

This is another example of the simple harmonic oscillation equation. Recall Newton's Second Law applied to a simple spring-block resulted in an equation of the form

$$-k\,x = m\frac{d^2 x}{dt^2}\,. \tag{3.11}$$

Although the dependent variable has changed from $x \to r$, and the spring constant from $k \to k_{\text{grav}} = mg / r_{\text{e}}$, the form of the equation is the same.

By analogy, the period is given by

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k_{grav} / m}} = \frac{2\pi}{\sqrt{g / r_e}} = 2\pi\sqrt{\frac{r_e}{g}}\,. \tag{3.12}$$

where $\omega_0 = \sqrt{k_{grav} / m} = \sqrt{g / r_e}$ is the angular frequency of oscillation. The object takes half the time T in Equation (3.12) to reach the other side of the earth,

$$t_1 = \frac{T}{2} = \pi\sqrt{\frac{r_e}{g}}\,. \tag{3.13}$$

If the object is released from rest at the surface of the earth, the position of the object up to the time it reaches the center of the earth (remember, the radius r cannot be negative) is given by

$$r(t) = r_{\mathrm{e}} \cos\left(\frac{2\pi}{T}t\right). \tag{3.14}$$

where T is the period of oscillation as given in Equation (3.12). The radial component of the velocity of the object during the same time is

$$v_r(t) = -\frac{2\pi}{T} r_{\mathrm{e}} \sin\left(\frac{2\pi}{T}t\right). \tag{3.15}$$

c) We can define a potential energy function for the gravitation force inside the earth analogous to the spring potential energy function with zero-point for potential energy chosen at the center of the earth,

$$U(r) = \frac{1}{2}k_{\mathrm{grav}}r^2 = \frac{1}{2}\frac{mg}{r_{\mathrm{e}}}r^2. \tag{3.16}$$

To see this, recall that the definition of the change in potential energy between two points a distance r_0 from the center of the earth with $r_0 < r_{\mathrm{e}}$ and a distance r_f from the center of the Earth with $r_f < r_{\mathrm{e}}$ is

$$\begin{aligned}\Delta U_{system} &= -W_c = -\int_A^B \vec{\mathbf{F}}_{grav} \cdot d\vec{\mathbf{r}} = -\int_{r_0}^{r_f} -\frac{mg}{r_{\mathrm{e}}} r\,\hat{\mathbf{r}} \cdot dr\hat{\mathbf{r}} \\ &= \int_{r_0}^{r_f} \frac{mg}{r_{\mathrm{e}}} r\,dr = \frac{1}{2}\frac{mg}{r_{\mathrm{e}}}(r_f^{\,2} - r_0^{\,2})\end{aligned}\,. \tag{3.17}$$

Choose $r_0 = 0$ as a zero reference point for the potential energy, $U(r_0 = 0) = 0$, and let $r_f = r$ represent any point a distance r from the center of the earth with $r < r_{\mathrm{e}}$. The above change in potential energy between these two points is then given by

$$U(r) = \frac{1}{2}\frac{mg}{r_{\mathrm{e}}}r^2. \tag{3.18}$$

If we release the object from rest at the surface of the earth, the initial mechanical energy is all potential energy and is given by

$$E_i = U(r_e) = \frac{1}{2}\frac{mg}{r_e}r_e^{\,2} = \frac{1}{2}mg\,r_e \,. \qquad \textbf{(3.19)}$$

When the object reaches the center of the earth, the mechanical energy is all kinetic energy,

$$E_f = K_f = \frac{1}{2}m\,v_{\text{center}}^2 \,. \qquad \textbf{(3.20)}$$

Since there are no external forces acting on the system, there is no external work done on (or by the system), the mechanical energy is constant,

$$E_i = E_f \qquad \textbf{(3.21)}$$

and

$$\frac{1}{2}mg\,r_e = \frac{1}{2}mv_{\text{center}}^2 \,. \qquad \textbf{(3.22)}$$

The radial component of velocity $v_{r,center}$ just before the object reaches the center of the earth is then

$$v_{r,center} = -\sqrt{gr_e} \,. \qquad \textbf{(3.23)}$$

Note we choose "just before" so that the velocity is radially inward in polar coordinates; there is no well-defined radial direction when the object is located at the origin.

As a check, the velocity of the object while falling is, recalling Equation (3.15),

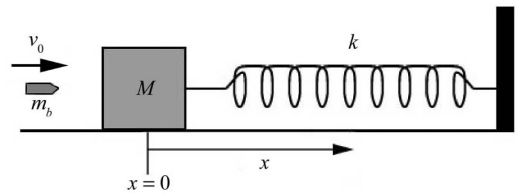
$$v_r(t) = -\frac{2\pi}{T}r_e \sin\left(\frac{2\pi}{T}t\right). \qquad \textbf{(3.24)}$$

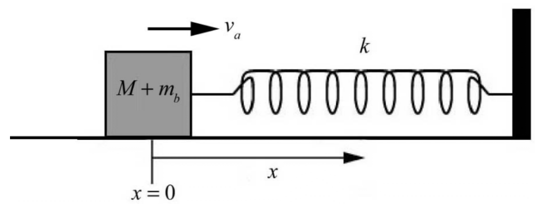
When $t = T/4$, $\cos\big((2\pi/T)t\big) = \cos(\pi/2) = 0$, and from Equation (3.14) the object is at the center of the earth. Also, $\sin\big((2\pi/T)t\big) = \sin(\pi/2) = 1$ and from Equation **(3.24)** the radial component of the velocity at the center of the earth is given by

$$v_{r,\text{center}} = v_r(T/4) = -\sqrt{\frac{g}{r_e}}\,r_e = \sqrt{g\,r_e} \,. \qquad \textbf{(3.25)}$$

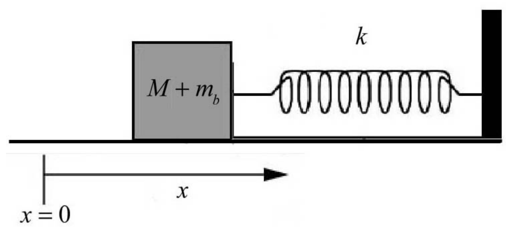
Problem 4

A massless spring with spring constant k is attached at one end of a block of mass M that is resting on a frictionless horizontal table. The other end of the spring is fixed to a wall. A bullet of mass m_b is fired into the block from the left with a speed v_0 and comes to rest in the block. (Assume that this happens instantaneously). The block and bullet are moving immediately after the bullet comes to rest with speed $v_a = m_b v_0 / (m_b + M)$.





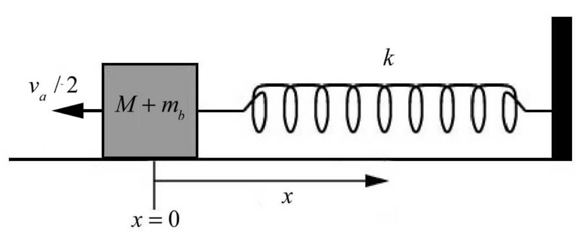
The resulting motion of the block and bullet is simple harmonic motion.



a) Find the amplitude of the resulting simple harmonic motion.

b) How long does it take the block to first return to the position $x = 0$?

c) Now suppose that instead of sliding on a frictionless table during the resulting motion, the block is acted on by the spring and a weak friction force of constant magnitude f. Suppose that when the block first returned to the position $x = 0$, the speed of the block was found to be $v_f = m_b v_0 / 2(m_b + M)$. How far did the block travel?



Problem 4 Solution:

The energy of the spring-object system is constant since there are no external work done on the system (no fricition), therefore

$$\frac{1}{2}(m_b + M)v_a^{\ 2} = \frac{1}{2}kx_{\max}^2 \tag{4.1}$$

where the maximum displacement

$$x_{\max} = \sqrt{\frac{(m_b + M)}{k}}\, v_a = \sqrt{\frac{1}{k(m_b + M)}}\, m_b v_0 \tag{4.2}$$

is the amplitude of the simple harmonic motion.

Alternatively, the position of the system is given by the solution to the simple harmonic equation:

$$x(t) = A\cos\omega t + B\sin\omega t \tag{4.3}$$

where the angular frequency is given by

$$\omega = \sqrt{\frac{k}{m_b + M}} \tag{4.4}$$

The x-component of the velocity is given by

$$v_x(t) = -\omega A\sin\omega t + \omega B\cos\omega t \tag{4.5}$$

At $t = 0$, $x(t = 0) = A = 0$, and $v_x(t = 0) = \omega B = \dfrac{m_b}{(m_b + M)}v_0$, so

$$B = \frac{1}{\omega}\frac{m_b}{(m_b + M)}v_0 = \sqrt{\frac{m_b + M}{k}}\frac{m_b}{(m_b + M)}v_0 = \sqrt{\frac{1}{k(m_b + M)}}\, m_b v_0 \tag{4.6}$$

The system reaches maximum amplitude when $\omega t = \pi / 2$. Thus from eq. (4.3)

$$x(t=\pi/2\omega)=B\sin(\pi/2)=B=\sqrt{\frac{1}{k(m_b+M)}}\;m_b v_0\,. \qquad (4.7)$$

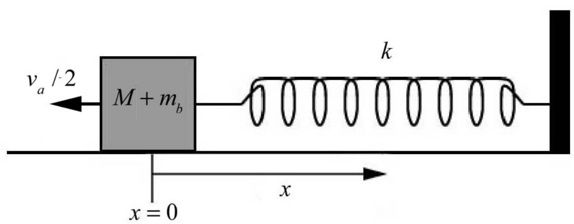
(b) It takes the block half a period to return to the position $x=0$. The period is

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m_b+M}{k}} \qquad (4.8)$$

So the block returns to $x=0$ at time

$$t_1=\frac{T}{2}=\frac{\pi}{\omega}=\pi\sqrt{\frac{m_b+M}{k}} \qquad (4.9)$$

d)



The work done by the friction force is given by

$$W=-fd \qquad (4.10)$$

where d is the distance traveled by the system. This work is equal to the change in kinetic energy of the spring-bullet-block system,

$$W=E_1-E_a \qquad (4.11)$$

The energy when the block returns to $x=0$ is

$$E_1=\frac{1}{2}(m_b+M)\left(\frac{v_a}{2}\right)^2=\frac{1}{4}\left(\frac{1}{2}(m_b+M)v_a^{\,2}\right)=\frac{1}{4}E_a \qquad (4.12)$$

So Eq. (4.11) becomes

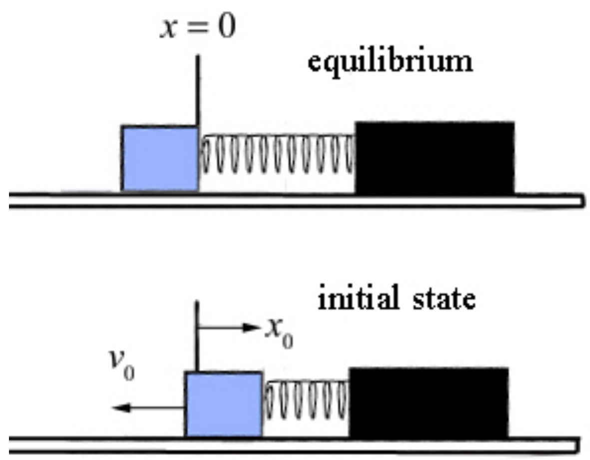
$$-fd=\frac{1}{4}E_a-E_a=-\frac{3}{4}E_a=-\frac{3}{4}\left(\frac{1}{2}(m_b+M)v_a^{\,2}\right)=-\frac{3}{8}\frac{m_b^{\,2}}{(m_b+M)}v_0^{\,2} \qquad (4.13)$$

We can now solve for the distance traveled by the system

$$d = \frac{3}{8}\frac{{m_b}^2}{(m_b + M)f}{v_0}^2 \quad (4.14)$$

Problem 5 ***Simple Harmonic Motion***

Consider an ideal spring with spring constant k. The spring is attached to an object of mass m that lies on a horizontal frictionless surface. The spring-mass system is compressed a distance x_0 from equilibrium and then released with an initial speed v_0 toward the equilibrium position.



a) What is the period of oscillation for this system?

b) How long will it take for the object to first return to the equilibrium position?

c) What is the magnitude of the velocity of the object when it first returns to the equilibrium position?

d) Draw a graph of the position and velocity of the mass as a function of time. Carefully label your axes and clearly specify any special values.

Problem 5 Solutions:

Choose an origin at the equilibrium position and positive x-direction pointing in the stretched direction. Then the initial position $-x_0 < 0$ (note that x_0 is a distance and hence positive) and $v_0 > 0$. Newton's Second Law is

$$-k\,x = m\frac{d^2x}{dt^2} \quad \textbf{(5.1)}$$

The general solution to Eq. (5.1) is given by

$$x(t) = A\cos(\omega_0\,t + \phi)\,, \quad \textbf{(5.2)}$$

where $\omega_0 = \sqrt{k / m}$. The period of oscillation is therefore

$$T = 2\pi / \omega_0 = 2\pi\sqrt{m / k}\,. \tag{5.3}$$

The coefficients A and ϕ depend on the given set of initial conditions $-x_0 \equiv x(t=0)$ and $v_0 \equiv v_x(t=0)$ where x_0 and v_0 are positive constants.

The x-component of the velocity of the object at time t is obtained by differentiating the position function,

$$v_x(t) = dx / dt = -\omega_0 A\sin(\omega_0\, t + \phi)\,. \tag{5.4}$$

To find the constants A and ϕ, substitute $t = 0$ into the Equations (5.2) and (5.4), yielding

$$-x_0 \equiv x(t=0) = A\cos\phi\,, \tag{5.5}$$

and

$$v_0 = v_x(t=0) = -\omega_0 A\sin(\phi)\,. \tag{5.6}$$

We can rewrite Eq. (5.6) as

$$\frac{v_0}{-\omega_0} = A\sin(\phi)\,. \tag{5.7}$$

To find the angle ϕ, divide Eq. (5.7) by Eq. (5.5) and taking the inverse tangent to find that

$$\phi = \tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right). \tag{5.8}$$

To find A, add the square of Eq. (5.7) to the square of Eq. (5.5) and take the square root yielding

$$A = \sqrt{{x_0}^2 + (v_0 / \omega_0)^2}\,, \tag{5.9}$$

Then the position of the object-spring system is given by

$$x(t) = \sqrt{x_0^{\ 2} + (v_0 / \omega_0)^2} \cos(\omega_0 \, t + \tan^{-1} \frac{v_0}{\omega_0 x_0}). \qquad \textbf{(5.10)}$$

and the x-component of the velocity of the spring-mass system is

$$v_x(t) = -\omega_0 \sqrt{x_0^{\ 2} + (v_0 / \omega_0)^2} \sin(\omega_0 \, t + \tan^{-1} \frac{v_0}{\omega_0 x_0}). \qquad \textbf{(5.11)}$$

a) The period of oscillation is given by

$$T = 2\pi / \omega_0 = 2\pi / \sqrt{k / m}. \qquad \textbf{(5.12)}$$

b) The spring first reaches equilibrium at time $t = t_1$ where $x(t_1) = 0$. This is satisfied when

$$\omega_0 \, t_1 + \phi = \pi / 2 \qquad \textbf{(5.13)}$$

Solving for t_1 yields

$$t_1 = (\pi / 2 - \phi) / \omega_0 \qquad \textbf{(5.14)}$$

c) The object is first completely extended when the velocity is zero. This occurs when

$$0 = v_x(t_2) = -\omega_0 x_0 \sin(\omega_0 \, t_2 + \phi) \qquad \textbf{(5.15)}$$

This is satisfied when

$$\omega_0 \, t_2 + \phi = \pi. \qquad \textbf{(5.16)}$$

Solving for t_2 yields

$$t_2 = (\pi - \phi) / \omega_0 \qquad \textbf{(5.17)}$$

Alternatively, the general solution to Eq. **(5.1)** is given by

$$x(t) = A\cos(\omega_0 \, t) + B\sin(\omega_0 \, t), \qquad \textbf{(5.18)}$$

where $\omega_0 = \sqrt{k/m}$ the coefficients and A and B depend on a given set of initial conditions $-x_0 \equiv x(t=0)$ and $v_0 \equiv v(t=0)$ where x_0 and v_0 are positive constants.

The velocity of the object at time t is then obtained by differentiating the position function,

$$v(t) = dx/dt = -\omega_0 A \sin(\omega_0\, t) + \omega_0 B \cos(\omega_0\, t). \tag{5.19}$$

To find the constants A and B, substitute $t = 0$ into the Equations (5.2) and (5.4) Since $\cos(0) = 1$ and $\sin(0) = 0$, the initial position at time $t = 0$ is

$$-x_0 \equiv x(t=0) = A. \tag{5.20}$$

The velocity at time $t = 0$ is

$$v_0 = v(t=0) = -\omega_0 A \sin(0) + \omega_0 B \cos(0) = \omega_0 B. \tag{5.21}$$

Thus

$$A = -x_0 \text{ and } B = \frac{v_0}{\omega_0}. \tag{5.22}$$

Then the position of the object-spring system is given by

$$x(t) = -x_0 \cos(\omega_0\, t) + \frac{v_0}{\omega_0} \sin(\omega_0\, t). \tag{5.23}$$

and the velocity of the spring-mass system is

$$v(t) = \omega_0 x_0 \sin(\sqrt{k/m}\; t) + v_0 \cos(\omega_0\, t). \tag{5.24}$$

The spring first reaches equilibrium at time $t = t_1$

$$0 = x(t_1) = -x_0 \cos(\omega_0\, t_1) + \frac{v_0}{\omega_0} \sin(\omega_0\, t_1) \tag{5.25}$$

This can be rewritten as

$$\tan(\omega_0\, t_1) = \frac{\sin(\omega_0\, t_1)}{\cos(\omega_0\, t_1)} = \frac{x_0 \omega_0}{v_0} \tag{5.26}$$

So we can solve this for the time the object first reaches equilibrium.

$$t_1 = \frac{1}{\omega_0}\tan^{-1}\left(\frac{x_0\omega_0}{v_0}\right) \quad \textbf{(5.27)}$$

Although Eq. (5.27) and Eq. (5.14) do not appear equal, rewrite Eq. (5.14) as

$$\omega_0 t_1 = \pi / 2 - \phi \quad \textbf{(5.28)}$$

Then take the sin of both sides

$$\begin{aligned}
\sin(\omega_0 t_1) &= \sin\left(\pi / 2 - (\tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)\right) \\
&= \sin(\pi / 2)\cos\left(\tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)\right) - \cos(\pi / 2)\sin\left(\tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)\right) \\
&= \cos\left(\tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)\right) = \sin\left(\tan^{-1}\left(\frac{\omega_0 x_0}{v_0}\right)\right)
\end{aligned} \quad \textbf{(5.29)}$$

Thus

$$t_1 = \frac{1}{\omega_0}\tan^{-1}\left(\frac{\omega_0 x_0}{v_0}\right) \quad \textbf{(5.30)}$$

in agreement with Eq.(5.27).

c) The object is first completely extended when the velocity is zero. This occurs when

$$0 = v(t_2) = \omega_0 x_0 \sin(\omega_0\ t_2) + v_0 \cos(\omega_0\ t_2) \quad \textbf{(5.31)}$$

This can be rewritten as

$$\tan(\omega_0\ t_2) = \frac{\sin(\omega_0\ t_2)}{\cos(\omega_0\ t_2)} = -\frac{v_0}{\omega_0 x_0} \quad \textbf{(5.32)}$$

So we can solve for the time when it reaches maximal stretch

$$t_2 = \frac{1}{\omega_0}\tan^{-1}\left(-\frac{v_0}{\omega_0 x_0}\right) \quad \textbf{(5.33)}$$

Although Eq. (5.33) and Eq. (5.17) do not appear equal, rewrite Eq. (5.17) as

$$\omega_0 t_2 = \pi - \phi \qquad \textbf{(5.34)}$$

Then take the tan of both sides

$$\tan(\omega_0 t_2) = \tan\left(\pi - (\tan^{-1}\left(\frac{v_0}{\omega_0 x_0}\right)\right) = -\frac{v_0}{\omega_0 x_0} \qquad \textbf{(5.35)}$$

Thus

$$t_2 = \frac{1}{\omega_0}\tan^{-1}\left(-\frac{v_0}{\omega_0 x_0}\right) \qquad \textbf{(5.36)}$$

in agreement with Eq. (5.33).

Although the question did not ask for this we can use the fact that the mechanical energy is constant to find the amplitude at maximal stretch. Initially the mechanical energy is

$$E_0 = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2 \qquad \textbf{(5.37)}$$

The mechanical energy when the spring is fully extended is

$$E_2 = \frac{1}{2}kx_{\max}^2 \qquad \textbf{(5.38)}$$

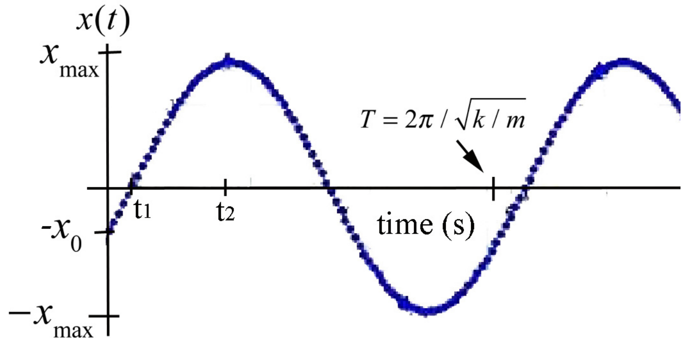
So conservation of mechanical energy implies that

$$E_0 = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2 = \frac{1}{2}kx_{\max}^2 = E_2 \qquad \textbf{(5.39)}$$

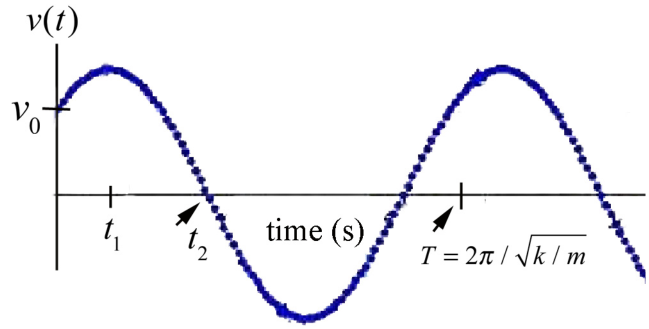
which we can solve for the maximal stretch

$$x_{\max} = \sqrt{x_0^2 + \frac{m}{k}v_0^2} \qquad \textbf{(5.40)}$$

d) A graph of the position of the object as a function looks like

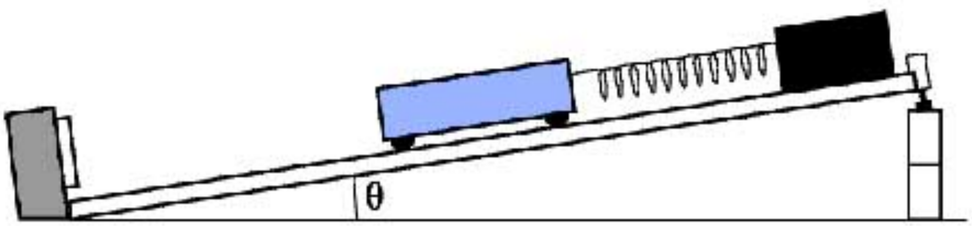


A graph of the velocity as a function of time looks like



Problem 6:

Consider an ideal spring that has an unstretched length l_0 and spring constant k. Suppose the spring is attached to a cart of mass m that lies on a frictionless plane that is inclined at an angle θ from the horizontal. The given quantities in this problem are l_0, m,k, θ and the gravitational constant g.



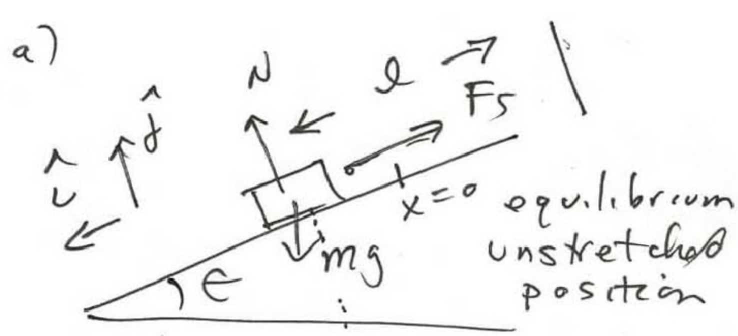
a) The spring stretches slightly to a new length $l > l_0$ to hold the cart in equilibrium. Find the length l in terms of the given quantities.

b) Now move the cart up along the ramp so that the spring is compressed a distance x_0 from the unstretched length l_0. Then the cart is released from rest. What is the speed of the cart when the spring has first returned to its unstretched length l_0?

c) What is the period of oscillation of the cart?

Problem 6 Solutions:

The force diagram is shown below. As indicated in the diagram, the positive x-direction is taken to be down the incline, and the positive y-direction is taken to be perpendicular to the ramp, with positive vertical component. The origin of the x-axis is at the unstretched length of the spring, a distance l_0 from the attachment point.



The x-components of the forces are $mg\sin\theta$ and the spring force $-k\left(l-l_0\right)$, with the minus sign indicating that at equilibrium, the spring force is directed up the ramp to hold the cart in place. The net force must be zero at equilibrium, so

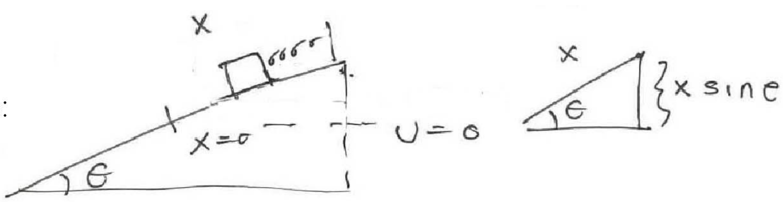
$$mg\sin\theta - k\left(l-l_0\right) = 0; \qquad (6.1)$$

solving for l gives

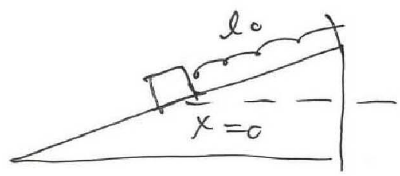
$$l = l_0 + \frac{mg\sin\theta}{k}. \qquad (6.2)$$

b) The surface is frictionless, so mechanical energy is conserved. We have the choice of where to take the zero of potential energy; we have already taken $x = 0$ to be the zero of the spring potential energy, and it turns out that it's most convenient to use this as the zero of potential energy as well.

Initial state:



Final state:



Conservation of mechanical energy is expressed as

$$\Delta K + \Delta U = 0. \qquad (6.3)$$

The initial kinetic energy is zero (the cart is released from rest) and the final kinetic energy is $\left(1/2\right)m v_f^{\,2}$, where v_f is the final speed which we wish to find. The initial potential energy is the sum of the gravitational and spring potential energies. With our choice of the common zero of these potential energies at $x = 0$, the initial potential energy is

$$U_i = mg\,x_0 \sin\theta + \frac{1}{2}k\,x_0^{\,2} \qquad (6.4)$$

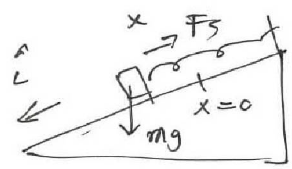
and the final potential energy is $U_f = 0$. Equaton (6.3) then becomes

$$\left(\left(1/2\right)m v_f^{\,2}-0\right)+\left(0-\left(mg\,x_0\sin\theta+\frac{1}{2}k\,x_0^{\,2}\right)\right)=0\,. \qquad (6.5)$$

Solving for the final speed v_f yields

$$v_f=\sqrt{2\,g\,x_0\sin\theta+\frac{k}{m}x_0^{\,2}}\ . \qquad (6.6)$$

c) The free body force diagram on the cart is shown in the figure below.



If the cart is not in equilibrium Equation (6.1) becomes

$$mg\sin\theta-k\,x=m\frac{d^2x}{dt^2} \qquad (6.7)$$

(this is of course Newton's Second Law). There are many ways to find the solution to Equation (6.7). The following method uses what we know from part a) and our intuition, in that the cart will oscillate about its equilibrium position $l=l_0+\dfrac{mg\sin\theta}{k}$. We measure x from the unstrecthed length l_0, so introduce the *auxiliary variable*

$$z=x-\left(l-l_0\right)=x-\frac{mg\sin\theta}{k}\,. \qquad (6.8)$$

(Make sure you do not confuse the auxiliary variable z with the third Cartesian coordinate.) We have $\dfrac{dz^2}{dt^2}=\dfrac{dx^2}{dt^2}$, so Equation (6.7) becomes

$$-k\,z=m\frac{dz^2}{dt^2}\,. \qquad (6.9)$$

This is the standard harmonic oscillator equation, with well-known solutions, all of which have period

$$T=2\pi\sqrt{\frac{m}{k}}\ . \qquad (6.10)$$