Module 28: The Kepler Problem: Planetary Mechanics

28.1 Introduction Kepler's Laws:[1]

1. Each planet moves in an ellipse with the sun at one focus.

2. The radius vector from the sun to a planet sweeps out equal areas in equal time.

3. The period of revolution T of a planet about the sun is related to the major axis A of the ellipse by

$$T^2 = k\,A^3 \tag{28.1.1}$$

   where k is the same for all planets.

28.2 Planetary Orbits: The Kepler Problem

Since Johannes Kepler first formulated the laws that describe planetary motion, scientists endeavored to solve for the equation of motion of the planets. In his honor, this problem has been named *The Kepler Problem*.

When there are more than two bodies, the problem becomes impossible to solve exactly. The most important "three-body problem" at the time involved finding the motion of the moon, since the moon interacts gravitationally with both the sun and the earth. Newton realized that if the exact position of the moon were known, the longitude of any observer on the earth could be determined by measuring the moon's position with respect to the stars.

In the eighteenth century, Leonhard Euler and other mathematicians spent many years trying to solve the three-body problem, and they raised a deeper question. Do the small contributions from the gravitational interactions of all the planets make the planetary system unstable over long periods of time? At the end of 18th century, Pierre Simon Laplace and others found a series solution to this stability question, but it was unknown whether or not the series solution converged after a long period of time. Henri Poincaré proved that the series actually diverged.

Poincaré went on to invent new mathematical methods that produced the modern fields of differential geometry and topology in order to answer the stability question using geometric arguments, rather than analytic methods. Poincaré and others did manage to show that the three-body problem was indeed stable, due to the existence of periodic Solution. Just as in the time of Newton and Leibniz and the invention of calculus,

[1] As stated in *An Introduction to Mechanics*, Daniel Kleppner and Robert Kolenkow, McGraw-Hill, 1973, p 401.

unsolved problems in celestial mechanics became the experimental laboratory for the discovery of new mathematics.

28.3 Reducing the Two-Body Problem into a One-Body Problem

We shall begin our solution of the two-body problem by showing how the motion of two bodies interacting via a gravitational force (two-body problem) is mathematically equivalent to the motion of a single body with a *reduced mass* given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (28.3.1)$$

that is acted on by an external central gravitational force. Once we solve for the motion of the reduced body in this *equivalent one-body problem*, we can then return to the real two-body problem and solve for the actual motion of the two original bodies.

The reduced mass was introduced in Section 10.7 of these notes. That section used similar but different notation from that used in this chapter.

Consider the gravitational force between two bodies with masses m_1 and m_2 as shown in Figure 28.1.

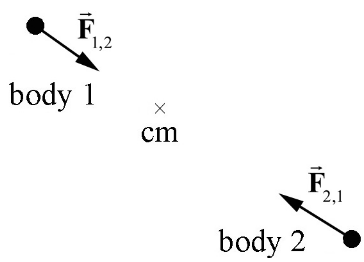


Figure 28.1 Gravitational force between two bodies.

Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 28.2). The *relative position vector* $\vec{\mathbf{r}}$ pointing from body 2 to body 1 is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. We denote the magnitude of $\vec{\mathbf{r}}$ by $|\vec{\mathbf{r}}| = r$, where r is the distance between the bodies, and $\hat{\mathbf{r}}$ is the unit vector pointing from body 2 to body 1, so that

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}} \qquad (28.3.2)$$

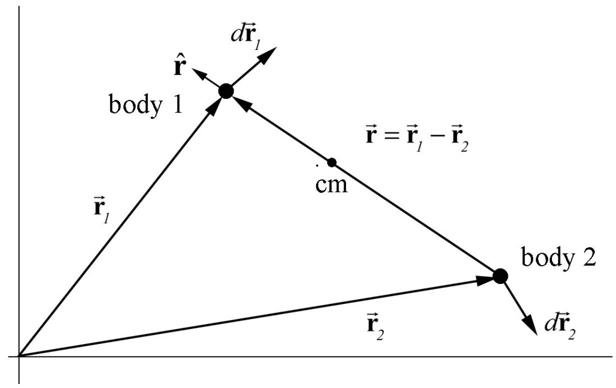


Figure 28.2 Coordinate system for the two-body problem.

The force on body 1 (due to the interaction of the two bodies) can be described as

$$\vec{\mathbf{F}}_{1,2} = -F_{1,2}\,\hat{\mathbf{r}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}\,. \tag{28.3.3}$$

Recall that Newton's Third Law requires that the force on body 2 is equal in magnitude and opposite in direction to the force on body 1,

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}\,. \tag{28.3.4}$$

Newton's Second Law can be applied individually to the two bodies:

$$\vec{\mathbf{F}}_{1,2} = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2}\,, \tag{28.3.5}$$

$$\vec{\mathbf{F}}_{2,1} = m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2}\,. \tag{28.3.6}$$

Dividing through by the mass in each of Equations (28.3.5) and (28.3.6) yields

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2}\,, \tag{28.3.7}$$

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2}\,. \tag{28.3.8}$$

Subtracting the expression in Equation (28.3.8) from that in Equation (28.3.7) gives

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_1} - \frac{\vec{\mathbf{F}}_{2,1}}{m_2} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}}{dt^2}. \qquad (28.3.9)$$

Using Newton's Third Law as given in Equation (28.3.4), Equation (28.3.9) becomes

$$\vec{\mathbf{F}}_{1,2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2\vec{\mathbf{r}}}{dt^2}. \qquad (28.3.10)$$

Using the *reduced mass* μ , as defined in Equation (28.3.1) ,

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \qquad (28.3.11)$$

Equation (28.3.10) becomes

$$\begin{aligned} \frac{\vec{\mathbf{F}}_{1,2}}{\mu} &= \frac{d^2\vec{\mathbf{r}}}{dt^2} \\ \vec{\mathbf{F}}_{1,2} &= \mu\frac{d^2\vec{\mathbf{r}}}{dt^2} \end{aligned} \qquad (28.3.12)$$

where $\vec{\mathbf{F}}_{1,2}$ is given by Equation (28.3.3).

Our result has a special interpretation using Newton's Second Law. Let μ be the reduced mass of a *reduced body* with position vector $\vec{\mathbf{r}} = r\hat{\mathbf{r}}$ with respect to an origin O, where $\hat{\mathbf{r}}$ is the unit vector pointing from the origin O to the reduced body. Then the equation of motion, Equation (28.3.12), implies that the body of reduced mass μ is under the influence of an attractive gravitational force pointing toward the origin. So, the original two-body gravitational problem has now been reduced to an equivalent one-body problem, involving a reduced body with reduced mass μ under the influence of a central force $-\mathrm{F}_{1,2}\,\hat{\mathbf{r}}$. Note that in this reformulation, there is no body located at the central point (the origin O). The parameter r in the two-body problem is the relative distance between the original two bodies, while the same parameter r in the one-body problem is the distance between the reduced body and the central point.

28.4 Energy and Angular Momentum, Constants of the Motion

Consider the reduced body with reduced mass given by Equation (28.3.1), orbiting about a central point under the influence of a radially attractive force given by Equation (28.3.3). The equivalent one-body problem has two constants of the motion, energy E and the angular momentum L about the origin O. Energy is a constant because there are no external forces acting on the reduced body, and angular momentum is constant about the origin because the only force is directed towards the origin, and hence the torque about the origin due to that force is zero (the vector from the origin to the reduced body is anti-parallel to the force vector and $\sin\pi = 0$). Since angular momentum is constant, the orbit of the reduced body lies in a plane with the angular momentum vector pointing perpendicular to this plane.

Choose polar coordinates (r,θ) for the reduced body (see Figure 28.3), where r is the distance of the reduced body from the central point that is now taken as the origin, θ is the angle that the reduced body makes with respect to a chosen direction, and which increases positively in the counterclockwise direction, and $\vec{\mathbf{F}}_{grav} = -\mathrm{F}_{1,2}\,\hat{\mathbf{r}}$.

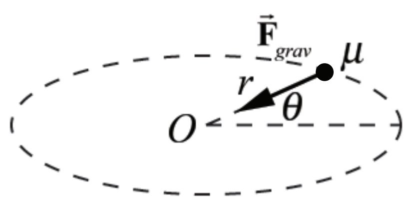


Figure 28.3 Coordinate system for the orbit of the reduced body.

Since the force is conservative, the potential energy with choice of zero reference point $U(\infty) = 0$ is given by

$$U(r) = -\frac{G m_1 m_2}{r}. \tag{28.4.1}$$

The total energy E is constant, and the sum of the kinetic energy and the potential energy is

$$E = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}. \tag{28.4.2}$$

The kinetic energy term, $\mu v^2/2$, has the reduced mass and the relative speed v of the two bodies. The velocity is cylindrical coordinates is given by (add link)

$$\begin{aligned} \vec{\mathbf{v}} &= v_{\text{rad}}\,\hat{\mathbf{r}} + v_{\tan}\hat{\boldsymbol{\theta}}, \\ v^2 &= v_{\text{rad}}{}^2 + v_{\tan}{}^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 \end{aligned} \tag{28.4.3}$$

where $v_{\text{rad}} = dr/dt$ and $v_{\tan} = r(d\theta/dt)$. Equation (28.A.2) then becomes

$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{G\,m_1\,m_2}{r}. \qquad (28.4.4)$$

The magnitude of the angular momentum with respect to the center of mass is

$$L = \mu\, r\, v_{\tan} = \mu\, r^2 \frac{d\theta}{dt}. \qquad (28.4.5)$$

28.5 The Orbit Equation for the Reduced Body

There are two approaches to describing the motion of the reduced body. We can try to find both the distance from the origin, $r(t)$ and the angle, $\theta(t)$, as functions of the parameter time, but in most cases explicit functions can't be found analytically. We can also find the distance from the origin, $r(\theta)$, as a function of the angle θ. This second approach offers a spatial description of the motion of the reduced body.

We begin with Newton's Second Law

$$\vec{\mathbf{F}} = \mu\,\vec{\mathbf{a}}$$
$$-G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} = \mu\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}}. \qquad (28.5.1)$$

Note that the motion is no longer circular, the radius is not constant so there is a term in the radial acceleration $d^2 r / dt^2$ along with the usual centripetal acceleration term $-r(d\theta / dt)^2$. Setting the components equal, using the constant of motion $L = \mu r^2 (d\theta / dt)$ and rearranging,

$$\mu\frac{d^2 r}{dt^2} = \frac{L^2}{\mu\, r^3} - \frac{G m_1 m_2}{r^2}. \qquad (28.5.2)$$

What we will do is use the same substitution $u = 1/r$ and change the independent variable from t to r, using the chain rule twice, since Equation (28.5.2) is a second-order equation. That is, the first time derivative is

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}. \qquad (28.5.3)$$

From $r = 1/u$ we have $dr/du = -1/u^2$. Combining with the angular velocity $d\theta / dt = L / \mu r^2$ in terms of L and u, $d\theta / dt = L u^2 / \mu$, Equation (28.5.3) becomes

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{Lu^2}{\mu} = -\frac{du}{d\theta}\frac{L}{\mu}, \qquad (28.5.4)$$

Taking the second derivative with respect to t,

$$\begin{aligned} \frac{d^2r}{dt^2} &= \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{d\theta}\left(\frac{dr}{dt}\right)\frac{d\theta}{dt} \\ &= -\frac{d^2u}{d\theta^2}\frac{L}{\mu}\left(\frac{L}{\mu}u^2\right) \\ &= -\frac{d^2u}{d\theta^2}u^2\frac{L^2}{\mu^2}. \end{aligned} \qquad (28.5.5)$$

Substituting into Equation (28.5.2), and using $r = 1/u$,

$$-\frac{d^2u}{d\theta^2}u^2\frac{L^2}{\mu} = \frac{L^2}{\mu}u^3 - Gm_1m_2u^2. \qquad (28.5.6)$$

Canceling the common factor of u^2 and rearranging,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu G m_1 m_2}{L^2}. \qquad (28.5.7)$$

Equation (28.5.7) is mathematically equivalent to the Harmonic Oscillator Equation with a constant term. The solution consists of two parts: the angle-independent solution

$$u_0 = \frac{\mu G m_1 m_2}{L^2} \qquad (28.5.8)$$

and a sinusoidally varying term of the form

$$u_{\mathrm{H}}(\theta) = A\cos(\theta - \theta_0), \qquad (28.5.9)$$

where A and θ_0 are constants determined by the form of the orbit. The expression in Equation (28.5.8) is the *inhomogeneous solution* and represents a circular orbit. The expression in Equation (28.5.9) is the *homogeneous solution* (as hinted by the subscript) and must have two independent constants. Define

$$r_0 \equiv L^2 / \mu G\, m_1\, m_2. \qquad (28.5.10)$$

This is called the *semilatus rectum.* Then $u_0 = 1/r_0$ and we can write the solution to Equation (28.5.7) as the sum of the inhomogeneous (Eq. (28.5.8)) and homogenous (Eq. (28.5.9)) pieces

$$u = u_0 + u_{\mathrm{H}} = \frac{1}{r_0}(1 + r_0 A(\cos(\theta - \theta_0)))$$
$$r = \frac{r_0}{1 + r_0 A\cos(\theta - \theta_0)}. \tag{28.5.11}$$

We have two constants to choose. Define $A \equiv \varepsilon / r_0$ where

$$\varepsilon = \sqrt{1 + 2\mu E r_0^{\,2} / L^2} = \sqrt{1 + 2EL^2 / \mu(G\, m_1\, m_2)^2} \tag{28.5.12}$$

a quantity called *eccentricity*. Choose $\theta_0 = \pi$ then $\cos(\theta - \pi) = -\cos(\theta)$. Thus Equation (28.5.11) can be written in the form

$$r = \frac{r_0}{1 - \varepsilon\cos\theta}. \tag{28.5.13}$$

The two constants of the motion in terms of r_0 and ε are

$$L = (\mu\, G\, m_1\, m_2\, r_0)^{\frac{1}{2}}$$
$$E = \frac{G\, m_1\, m_2(\varepsilon^2 - 1)}{2\, r_0}. \tag{28.5.14}$$

The orbit equation as given in Equation (28.5.13) is a general *conic section* and is perhaps somewhat more familiar in Cartesian coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$, with $r^2 = x^2 + y^2$. The orbit equation can be rewritten as

$$r = r_0 + \varepsilon\, r\cos\theta. \tag{28.5.15}$$

Using the Cartesian substitutions for x and y, rewrite Equation (28.5.15) as

$$(x^2 + y^2)^{1/2} = r_0 + \varepsilon\, x. \tag{28.5.16}$$

Squaring both sides of Equation (28.5.16),

$$x^2 + y^2 = r_0^{\,2} + 2\,\varepsilon\, x\, r_0 + \varepsilon^2 x^2. \tag{28.5.17}$$

After rearranging terms, Equation (28.5.17) is the general expression of a conic section with axis on the x-axis,

$$x^2(1 - \varepsilon^2) - 2\,\varepsilon\, x\, r_0 + y^2 = r_0^{\,2} \tag{28.5.18}$$

(we now see that the dotted axis in Figure 28.3 can be taken to be the x-axis).

For a given $r_0 > 0$, corresponding to a given nonzero angular momentum as in Equation (28.5.10), there are four cases determined by the value of the eccentricity.

Case 1: When $\varepsilon = 0$, $E = E_{\text{min}} < 0$ and $r = r_0$. Equation (28.5.18) is the equation for a circle,

$$x^2 + y^2 = r_0^{\,2} \tag{28.5.19}$$

Case 2: When $0 < \varepsilon < 1$, $E_{\text{min}} < E < 0$ and Equation (28.5.18) describes an ellipse,

$$y^2 + A\,x^2 - B\,x = k \tag{28.5.20}$$

where $A > 0$ and k is a positive constant. (Appendix 17.C shows how this expression may be expressed in the more traditional form involving the coordinates of the center of the ellipse and the semimajor and semiminor axes.)

Case 3: When $\varepsilon = 1$, $E = 0$ and Equation (28.5.18) describes a parabola,

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2}. \tag{28.5.21}$$

Case 4: When $\varepsilon > 1$, $E > 0$ and Equation (28.5.18) describes a hyperbola,

$$y^2 - A\,x^2 - B\,x = k \tag{28.5.22}$$

where $A > 0$ and k is a positive constant.

28.6 Energy Diagram, Effective Potential Energy, and Orbits

The energy (Equation (28.A.7)) of the reduced body moving in two dimensions can be reinterpreted as the energy of a reduced body moving in one dimension, the radial direction r, in an *effective potential energy* given by two terms,

$$U_{\text{eff}} = \frac{L^2}{2\mu\, r^2} - \frac{G\,m_1\,m_2}{r}. \tag{28.6.1}$$

The total energy is still the same, but our interpretation has changed;

$$E = K_{\text{eff}} + U_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu\, r^2} - \frac{G\,m_1\,m_2}{r}, \tag{28.6.2}$$

where the *effective kinetic energy* K_{eff} associated with the one-dimensional motion is

$$K_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 . \tag{28.6.3}$$

The graph of U_{eff} as a function of $\mathrm{r} = r/r_0$, where r_0 as given in Equation (28.5.10), is shown in Figure 28.4. The upper curve (red, if you can see this in color) is proportional to $L^2/(2\mu r^2) \sim 1/2r^2$. The lower blue curve is proportional to $-Gm_1m_2/r \sim -1/r$. The sum U_{eff} is represented by the green curve. The minimum value of U_{eff} is at $r = r_0$, as will be shown analytically below. The vertical scale is in units of $-U_{\text{eff}}(r_0)$.

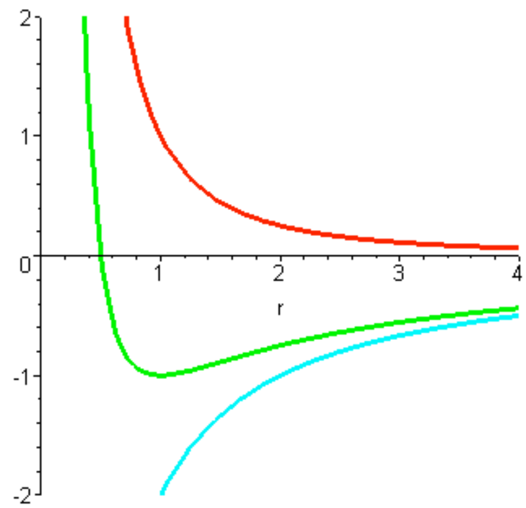


Figure 28.4 Graph of effective potential energy.

Whenever the one-dimensional kinetic energy is zero, $K_{\text{eff}} = 0$, the energy is equal to the effective potential energy,

$$E = U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r} . \tag{28.6.4}$$

Recall that the potential energy is defined to be the negative integral of the work done by the force. For our reduction to a one-body problem, using the effective potential, we will introduce an *effective force* such that

$$U_{\text{eff},B} - U_{\text{eff},A} = -\int_A^B \vec{\mathbf{F}}^{\text{eff}} \cdot d\vec{\mathbf{r}} = -\int_A^B F_r^{\text{eff}}\, dr \tag{28.6.5}$$

The fundamental theorem of calculus (for one variable) then states that the integral of the derivative of the effective potential energy function between two points is the effective potential energy difference between those two points,

$$U_{\text{eff},B} - U_{\text{eff},A} = \int_A^B \frac{dU_{\text{eff}}}{dr} dr \tag{28.6.6}$$

Comparing Equation (28.6.6) to Equation (28.6.5) shows that the radial component of the effective force is the negative of the derivative of the effective potential energy,

$$F_r^{\text{eff}} = -\frac{dU_{\text{eff}}}{dr} \tag{28.6.7}$$

The effective potential energy describes the potential energy for a reduced body moving in one dimension. (Note that the effective potential energy is only a function of the variable r and is independent of the variable θ). There are two contributions to the effective potential energy, and the total radial component of the force is

$$F_r^{\text{eff}} = -\frac{d}{dr}U_{\text{eff}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}\right) \tag{28.6.8}$$

Thus there are two "forces" acting on the reduced body,

$$F_r^{\text{eff}} = F_{r,\,\text{centifugal}} + F_{r,\,\text{gravity}}\,, \tag{28.6.9}$$

with an *effective centrifugal force* given by

$$F_{r,\text{centrifugal}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) = \frac{L^2}{\mu r^3} \tag{28.6.10}$$

and the conventional gravitational force

$$F_{r,\,\text{gravity}} = -\frac{G m_1 m_2}{r^2}\,. \tag{28.6.11}$$

With this nomenclature, let's review the four cases presented in Section 17.3.

Case 1: Circular Orbit $E = E_{\text{min}}$

The lowest energy state, $E_{\min}$, corresponds to the minimum of the effective potential energy, $E_{\min} = (U_{\text{eff}})_{\min}$. When this condition is satisfied the effective kinetic energy is zero since $E = K_{\text{eff}} + U_{\text{eff}}$. The condition

$$K_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 = 0 \qquad (28.6.12)$$

implies that the radial velocity is zero, so the distance r from the central point is a constant. This is the condition for a circular orbit. The condition for the minimum of the effective potential energy is

$$0 = \frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{\mu r^3} + \frac{G m_1 m_2}{r^2}. \qquad (28.6.13)$$

We can solve Equation (28.6.13) for r,

$$r \equiv r_0 = \frac{L^2}{G m_1 m_2}, \qquad (28.6.14)$$

reproducing Equation (28.5.10).

Case 2: Elliptic Orbit $E_{\min} < E < 0$

When $K_{\text{eff}} = 0$, the mechanical energy is equal to the effective potential energy, $E = U_{\text{eff}}$, as in Equation (28.6.4). Having $dr/dt = 0$ corresponds to a point of closest or furthest approach as seen in Figure 28.4. This condition corresponds to the minimum and maximum values of r for an elliptic orbit,

$$E = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r} \qquad (28.6.15)$$

Equation (28.6.15) is a quadratic equation for the distance r,

$$r^2 + \frac{G m_1 m_2}{E} r - \frac{L^2}{2\mu E} = 0 \qquad (28.6.16)$$

with two roots

$$r = -\frac{G m_1 m_2}{2E} \pm \left(\left(\frac{G m_1 m_2}{2E}\right)^2 + \frac{L^2}{2\mu E}\right)^{1/2}. \qquad (28.6.17)$$

Equation (28.6.17) may be simplified somewhat as

$$r = -\frac{G m_1 m_2}{2E}\left(1 \pm \left(1 + \frac{2L^2 E}{\mu (G m_1 m_2)^2}\right)^{1/2}\right) \tag{28.6.18}$$

Recall from Equation (28.5.12), the square root is the eccentricity ε, given by

$$\varepsilon = \sqrt{1 + 2\mu E r_0^{\,2} / L^2} = \sqrt{1 + 2EL^2 / \mu (G m_1 m_2)^2} \tag{28.6.19}$$

Thus Equation (28.6.18) becomes

$$r = -\frac{G m_1 m_2}{2E}\left(1 \pm \varepsilon\right). \tag{28.6.20}$$

A little algebra shows that

$$\begin{aligned}
\frac{r_0}{1-\varepsilon^2} &= \frac{L^2 / \mu G m_1 m_2}{1 - \left(1 + \dfrac{2L^2 E}{\mu (G m_1 m_2)^2}\right)} \\
&= \frac{L^2 / \mu G m_1 m_2}{-2L^2 E / \mu (G m_1 m_2)^2} \\
&= -\frac{G m_1 m_2}{2E}.
\end{aligned} \tag{28.6.21}$$

Substituting the last expression in (28.6.21) into Equation (28.6.20) gives an expression for the points of closest and furthest approach,

$$r = \frac{r_0}{1-\varepsilon^2}(1 \pm \varepsilon)\,. \tag{28.6.22}$$

The minus sign corresponds to the distance of closest approach,

$$r \equiv r_{\text{min}} = \frac{r_0}{1+\varepsilon} \tag{28.6.23}$$

and the plus sign corresponds to the distance of furthest approach,

$$r \equiv r_{\text{max}} = \frac{r_0}{1-\varepsilon}. \tag{28.6.24}$$

Case 3: Parabolic Orbit $E = 0$

The effective potential energy, as given in Equation (28.6.1), approaches zero ($U_{\text{eff}} \to 0$) when the distance r approaches infinity ($r \to \infty$). Since the total energy is zero, when $r \to \infty$ the kinetic energy also approaches zero, $K_{\text{eff}} \to 0$. This corresponds to a parabolic orbit (see Equation (28.5.21)). Recall that in order for a body to escape from a planet, the body must have a total energy $E = 0$ (we set $U_{\text{eff}} = 0$ at infinity). This *escape velocity* condition corresponds to a parabolic orbit.

For a parabolic orbit, the body also has a distance of closest approach. This distance r_{par} can be found from the condition

$$E = U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r} = 0\,. \qquad (28.6.25)$$

Solving Equation (28.6.25) for r yields

$$r_{\text{par}} = \frac{L^2}{2\mu G m_1 m_2} = \frac{1}{2} r_0\,; \qquad (28.6.26)$$

the fact that the minimum distance to the origin (the *focus* of a parabola) is half the semilatus rectum is a well-known property of a parabola.

Case 4: Hyperbolic Orbit $E > 0$

When $E > 0$, in the limit as $r \to \infty$ the kinetic energy is positive, $K_{\text{eff}} > 0$. This corresponds to a hyperbolic orbit (see Equation (28.5.22)). The condition for closest approach is similar to Equation (28.6.15) except that the energy is now positive. This implies that there is only one positive solution to the quadratic Equation (28.6.16), the distance of closest approach for the hyperbolic orbit

$$r_{\text{hyp}} = \frac{r_0}{1+\varepsilon}\,. \qquad (28.6.27)$$

The constant r_0 is independent of the energy and from Equation (28.5.12) as the energy of the reduced body increases, the eccentricity increases, and hence from Equation (28.6.27), the distance of closest approach gets smaller.

28.7 Orbits of the Two Bodies

The orbit of the reduced body can be circular, elliptical, parabolic or hyperbolic, depending on the values of the two constants of the motion, the angular momentum and the energy. Once we have the explicit solution (in this discussion, $r(\theta)$) for the reduced body, we can find the actual orbits of the two bodies.

Choose a coordinate system as we did for the reduction of the two-body problem (Figure 28.5).

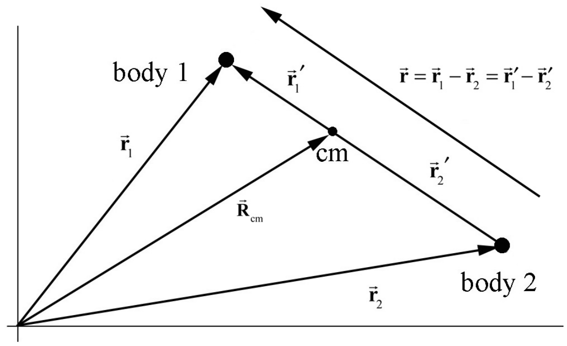


Figure 28.5 Center of mass coordinate system.

The center of mass of the system is given by

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2} . \tag{28.7.1}$$

Let $\vec{\mathbf{r}}_1'$ be the vector from the center of mass to body 1 and $\vec{\mathbf{r}}_2'$ be the vector from the center of mass to body 2. Then, by the geometry in Figure 28.5,

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_1' - \vec{\mathbf{r}}_2' \tag{28.7.2}$$

and hence

$$\vec{\mathbf{r}}_1' = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{\text{cm}} = \vec{\mathbf{r}}_1 - \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_2 \left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)}{m_1 + m_2} = \frac{\mu}{m_1} \vec{\mathbf{r}} . \tag{28.7.3}$$

A similar calculation shows that

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2} \vec{\mathbf{r}} . \tag{28.7.4}$$

Thus each body undergoes a motion about the center of mass in the same manner that the reduced body moves about the central point given by Equation (28.5.13). The only difference is that the distance from either body to the center of mass is shortened by a factor μ / m_i. When the orbit of the reduced body is an ellipse, then the orbits of the two bodies are also ellipses, as shown in Figure 28.6.

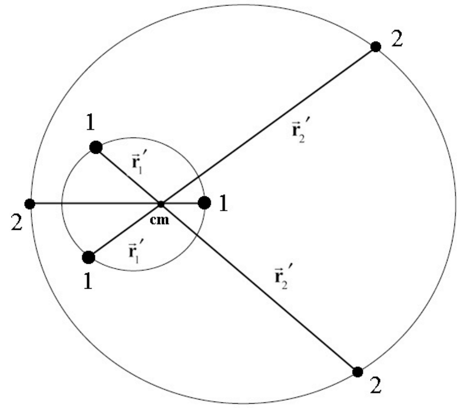


Figure 28.6 The elliptical motion of bodies under mutual gravitation.

When one mass is much smaller than the other, for example $m_1 \ll m_2$, then the reduced mass is approximately the smaller mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \cong \frac{m_1 m_2}{m_2} = m_1 \tag{28.7.5}$$

The center of mass is located approximately at the position of the larger mass, body 2 of mass m_2. Thus body 1 moves according to

$$\vec{\mathbf{r}}_1' = \frac{\mu}{m_1}\vec{\mathbf{r}} \cong \vec{\mathbf{r}} \tag{28.7.6}$$

and body 2 is approximately stationary,

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2}\vec{\mathbf{r}} - \frac{m_1}{m_2}\vec{\mathbf{r}} \cong \vec{\mathbf{0}} \tag{28.7.7}$$

28.8 Kepler's Laws

Elliptic Orbit Law

Each planet moves in an ellipse with the sun at one focus.

When the energy is negative, $E < 0$, and according to Equation (28.5.12),

$$\varepsilon = \left(1 + \frac{2EL^2}{\mu\left(Gm_1m_2\right)^2}\right)^{\frac{1}{2}} \tag{28.8.1}$$

and the eccentricity must fall within the range $0 \le \varepsilon < 1$. These orbits are either circles or ellipses. Note the elliptic orbit law is only valid if we assume that there is only one central force acting. We are ignoring the gravitational interactions due to all the other bodies in the universe, a necessary approximation for our analytic solution.

Equal Area Law

The radius vector from the sun to a planet sweeps out equal areas in equal time.

Using analytic geometry, the sum of the areas of the triangles in Figure 28.7 is given by

$$\Delta A = \frac{1}{2}\left(r\,\Delta\theta\right)r + \frac{\left(r\,\Delta\theta\right)}{2}\Delta r \Delta A = \frac{1}{2}\left(r\,\Delta\theta\right)r + \frac{\left(r\,\Delta\theta\right)}{2}\Delta r \tag{28.8.2}$$

in the limit of small $\Delta\theta$.

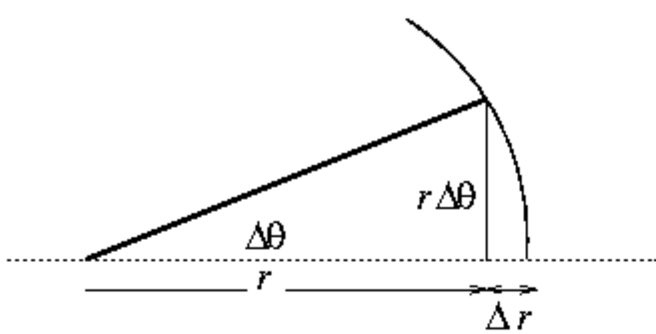


Figure 28.7 Kepler's equal area law.

The average rate of the change of area, ΔA, in time, Δt, is given by

$$\Delta A = \frac{1}{2}\frac{\left(r\,\Delta\theta\right)r}{\Delta t} + \frac{\left(r\Delta\theta\right)}{2}\frac{\Delta r}{\Delta t}. \tag{28.8.3}$$

In the limit as $\Delta t \to 0$, $\Delta\theta \to 0$, this becomes

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \tag{28.8.4}$$

Recall that according to Equation (28.A.6) (reproduced below as Equation (28.8.5)), the angular momentum is related to the angular velocity $d\theta / dt$ by

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \tag{28.8.5}$$

and Equation (28.8.4) is then

$$\frac{dA}{dt} = \frac{L}{2\mu}. \tag{28.8.6}$$

Since L and μ are constants, the rate of change of area with respect to time is a constant. This is often familiarly referred to by the expression: *equal areas are swept out in equal times* (see Kepler's Laws at the beginning of this chapter).

Period Law

> *The period of revolution T of a planet about the sun is related to the major axis A of the ellipse by*
>
> $$T^2 = k A^3$$
>
> *where k is the same for all planets.*

When Kepler stated his period law for planetary orbits based on observation, he only noted the dependence on the larger mass of the sun. Since the mass of the sun is much greater than the mass of the planets, his observation is an excellent approximation.

Equation (28.8.6) can be rewritten in the form

$$2\mu \frac{dA}{dt} = L. \tag{28.8.7}$$

Equation (28.8.7) can be integrated as

$$\int_{\text{orbit}} 2\mu \, dA = \int_0^T L \, dt \tag{28.8.8}$$

where T is the period of the orbit. For an ellipse,

$$\text{Area} = \int_{\text{orbit}} dA = \pi\, ab \tag{28.8.9}$$

where a is the semimajor axis and b is the semiminor axis. Thus we have

$$T = \frac{2\mu\pi\, ab}{L}. \tag{28.8.10}$$

Squaring Equation (28.8.10) then yields

$$T^2 = \frac{4\pi^2\mu^2 a^2 b^2}{L^2}. \tag{28.8.11}$$

In Appendix 17.B, the angular momentum is given in terms of the semimajor axis and the eccentricity by Equation (28.B.10). Substitution for the angular momentum into Equation (28.8.11) yields

$$T^2 = \frac{4\pi^2\mu^2 a^2 b^2}{\mu G m_1 m_2\, a(1-\varepsilon^2)}. \tag{28.8.12}$$

In Appendix 17.B, the semi-minor axis is given by Equation (28.B.22) which upon substitution into Equation (28.8.12) yields

$$T^2 = \frac{4\pi^2\mu^2 a^3}{\mu G m_1 m_2}. \tag{28.8.13}$$

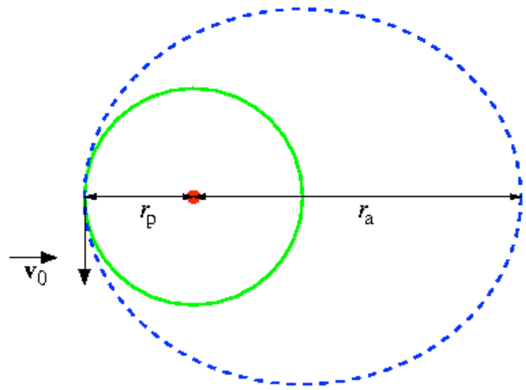
Using Equation (28.3.1) for reduced mass, the square of the period of the orbit is proportional to the semi-major axis cubed,

$$T^2 = \frac{4\pi^2 a^3}{G\left(m_1 + m_2\right)}. \tag{28.8.14}$$

28.9 Worked Examples

28.9.1 Example Satellite Orbits

A satellite of mass m_s is initially in a circular orbit of radius r_0 around the earth. The earth has mass $m_e >> m_s$ and radius R_e. Let G denote the universal gravitational constant. Express all your answers in terms of R_e, m_e, m_s, G and r_0 as needed.



a) Find an expression for the speed v_0 of the satellite when it is in the circular orbit.

b) Find an expression for the mechanical energy E_0 of the satellite when it is in the circular orbit. Take $U(r) \to 0$ as $r \to \infty$.

As a result of an orbital maneuver the satellite trajectory is changed to an elliptical orbit. This is accomplished by firing a rocket for a short time interval thus increasing the tangential speed of the satellite. The apogee (farthest distance from earth) of the elliptical orbit is three times the closest approach (perigee),

$$r_a = 3r_p = 3r_0 .$$

(In the figure, the small red circle represents the earth.)

c) Use conservation of energy and angular momentum for the elliptic orbit to find an expression for the speed of the satellite, v_p, immediately after the rocket has finished firing.

Solution:

a) This preliminary part should be found directly from Newton's Second Law and the Universal Law of Gravitation. The magnitude of the acceleration for the circular orbit is v_0^2 / r_0, and so

$$m_s \frac{v_0^2}{r_0} = G \frac{m_s m_e}{r_0^2}$$
$$v_0 = \sqrt{G m_e / r_0}.$$

b) The total mechanical energy is the sum of the kinetic energy and the gravitational potential energy,

$$E_0 = -G \frac{m_s m_e}{r_0} + \frac{1}{2} m_s v_0^2$$
$$= -\frac{1}{2} G \frac{m_s m_e}{r_0}.$$

c) Since $r_p = r_0$, and $r_a = 3r_0$, the condition that angular momentum is constant $r_p v_p = r_a v_a$ becomes $r_0 v_p = (3r_0) v_a$, so $v_a = v_p / 3$. The condition that the mechanical energy is constant then becomes,

$$\frac{1}{2} m_s v_p^2 - G \frac{m_s m_e}{r_0} = \frac{1}{2} m_s v_a^2 - G \frac{m_s m_e}{r_a}$$
$$= \frac{1}{2} m_s \left(\frac{v_p}{3} \right)^2 - G \frac{m_s m_e}{3r_0}$$
$$\frac{4}{9} v_p^2 = \frac{2}{3} G \frac{m_e}{r_0}$$
$$v_p = \sqrt{(3/2) G m_e / r_0}.$$

As a simple check, note that $v_p > v_0$. As a further check, some minor algebra shows that after the rocket burn, the final mechanical energy is $E_f = -G m_s m_e / (4 r_0) = -G m_s m_e / A$, where $A = r_0 + 3r_0$ is the major axis of the ellipse.

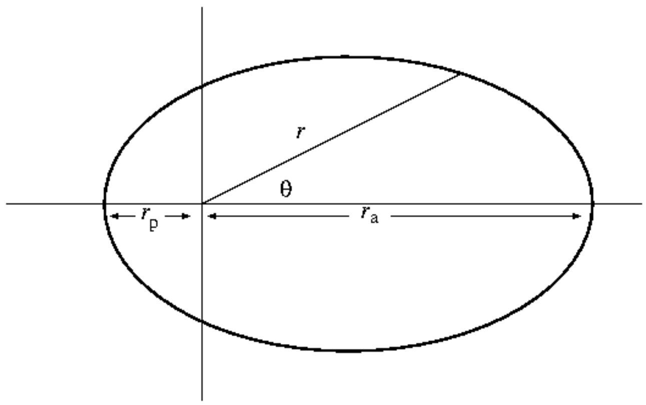
28.9.2 Example Halley's Comet

The equation for any orbit in an inverse square gravitational field is given by

$$r = \frac{r_0}{1 - \varepsilon\cos\theta} \qquad (28.9.1)$$

where

$$r_0 = \frac{L^2}{\mu G m_1 m_2}. \qquad (28.9.2)$$



In Equation (28.9.2), L is the angular momentum, $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass, and ε is the eccentricity of the orbit. When $0 < \varepsilon < 1$, the orbit is an ellipse (in the above figure, $\varepsilon = 3/5$). The period, T, depends only on the length of the major axis, A, of the ellipse, which is given by

$$A = \left(\frac{2T^2 G(m_1 + m_2)}{\pi^2}\right)^{1/3} = \frac{2r_0}{1 - \varepsilon^2} \qquad (28.9.3)$$

.

Halley's Comet is in an elliptic orbit about the sun. The eccentricity of the orbit is $\varepsilon = 0.967$ and the period is $T = 76\,\text{y}$. The mass of the sun is $m_{\text{sun}} = 1.99 \times 10^{30}\,\text{kg}$. The mass of Halley's Comet is negligible compared to the sun.

a) Using this data, determine the distance of Halley's Comet at closest approach r_{p} (perihelion) to the sun, and furthest distance r_{a} (aphelion) from the sun.

b) What is the speed v_{p} of Halley's Comet when it is closest to the sun?

Solution:

Before diving into the numerical calculations, let's do some preliminary math.

First, note that when $m_{\text{comet}} \ll m_{\text{sun}}$ (note that m_{comet} is not given in the problem), $m_1 + m_2 \to m_{\text{sun}}$ and

$$\mu = \frac{m_{\text{sun}} m_{\text{comet}}}{m_{\text{sun}} + m_{\text{comet}}} = m_{\text{comet}} \frac{1}{1 + m_{\text{comet}} / m_{\text{sun}}} \to m_{\text{comet}} . \qquad (28.9.4)$$

Next, from Equation (28.9.1), we have

$$r_{\text{p}} = \frac{r_0}{1+\varepsilon}, \quad r_{\text{a}} = \frac{r_0}{1-\varepsilon} \qquad (28.9.5)$$

and combining with Equation (28.9.3) gives

$$r_{\text{p}} = \frac{A}{2} r_0 \left(1-\varepsilon\right), \quad r_{\text{a}} = \frac{A}{2} r_0 \left(1+\varepsilon\right). \qquad (28.9.6)$$

As a quick check, note that $r_{\text{p}} + r_{\text{a}} = A$, the major axis.

Next, anticipating part (b), we expect to use angular momentum considerations. At perihelion, the comet must be moving perpendicular to the vector from the sun to the comet, and so the magnitude of the angular momentum in terms of v_{p}, r_{p} and m_{comet} is

$$L = m_{\text{comet}} v_{\text{p}} r_{\text{p}} . \qquad (28.9.7)$$

Combining with Equation (28.9.2) (with $m_1 = m_{\text{sun}}$, $m_2 = m_{\text{comet}}$, or vice versa) and the simplification for μ as given in Equation (28.9.4) and solving for v_{p} gives

$$v_{\text{p}} = \frac{L}{m_{\text{comet}} r_{\text{p}}} = \frac{\sqrt{G r_0 m_{\text{sun}} m_{\text{comet}}^2}}{m_{\text{comet}} r_{\text{p}}} = \frac{\sqrt{G r_0 m_{\text{sun}}}}{r_{\text{p}}} = \sqrt{\frac{G\left(1+\varepsilon\right) m_{\text{sun}}}{r_{\text{p}}}} . \qquad (28.9.8)$$

The mass of the comet does indeed drop out of this problem if $m_{\text{comet}} \ll m_{\text{sun}}$.

It's time to do the numbers. The calculations presented here were done by computer, keeping almost arbitrary precision in the intermediate calculations, and rounded to three figures (even though the period is given to only two figures). For the Newtonian gravitational constant, $G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$ was used.

a) Since the major axis A is used for determining both r_{p} and r_{a}, find that quantity first. From Equation (28.9.3), with $m_1 + m_2 = m_{\mathrm{sun}}$,

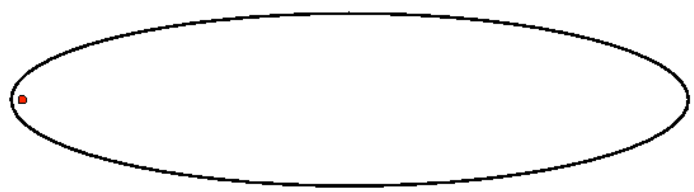
$$\begin{aligned} A &= \left(\frac{2T^2 G m_{\mathrm{sun}}}{\pi^2}\right)^{1/3} \\ &= \left(\frac{2\left(76\,\mathrm{y} \times 3.16\times 10^{7}\,\mathrm{s}\cdot\mathrm{y}^{-1}\right)\left(6.67\times 10^{-11}\,\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}\right)\left(1.99\times 10^{30}\,\mathrm{kg}\right)}{\pi^2}\right)^{1/3} \\ &= 5.37\times 10^{12}\,\mathrm{m} \end{aligned} \tag{28.9.9}$$

from which Equation (28.9.6) gives

$$r_{\mathrm{p}} = 8.86\times 10^{10}\,\mathrm{m}, \quad r_{\mathrm{a}} = 5.28\times 10^{12}\,\mathrm{m}. \tag{28.9.10}$$

These results are roughly half and thirty times the earth-sun distance, respectively; r_{p} is roughly the distance from the sun to the ex-planet Pluto. If fact, the period of Halley's Comet is roughly $1/\sqrt{8}$ the period of Pluto's orbit, consistent with Equation (28.9.3).

A graph of the orbit is shown here:



The tiny dot (red, if viewed in color) represents the sun and is not to scale; a circle representing the sun to scale on this scale is too small to be seen ($r_{\mathrm{p}} > 100\,R_{\mathrm{sun}}$).

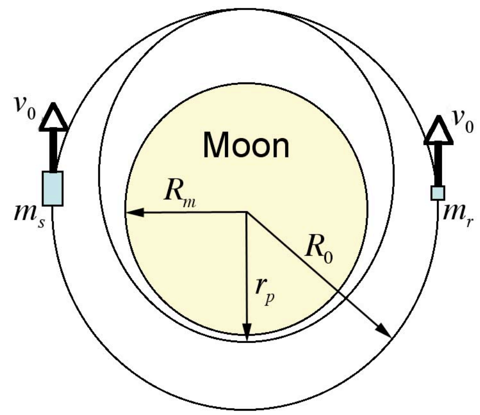
b) Equation (28.9.8) then gives, with the result in Equation (28.9.10),

$$v_{\mathrm{p}} = \sqrt{\frac{G(1+\varepsilon)m_{\mathrm{sun}}}{r_{\mathrm{p}}}} = 5.43\times 10^{4}\,\mathrm{m}\cdot\mathrm{s}^{-1}. \tag{28.9.11}$$

This is essentially (but of course smaller than) the escape velocity from the sun. In fact it's not too hard to show that $v_{\mathrm{p}}^2 / v_{\mathrm{escape}}^2 = \frac{1}{2}(1+\varepsilon)$, which is 0.9835 for the eccentricity of this orbit.

28.9.3 Example Lunar Orbit Collision

A lunar mapping satellite of mass m_s is in a circular orbit around the moon, and the orbit has radius $R_0 = 1.5\, R_m$ where R_m is the radius of the moon. A repair robot of mass $m_r < m_s$ is injected into that orbit, but due to a NASA sign error it orbits in the opposite direction. The two collide and stick together in a useless metal mass. The point of this problem is to find whether they create more junk orbiting the moon or crash into the lunar surface. The mass of the moon is denoted by m_m. The universal gravitational constant is denoted by G.



a. What is the initial orbital velocity of the mapping satellite, v_0? Express your answer in terms of R_0, m_m, and G.

b. What is the speed of the space junk (satellite and robot) immediately after the collision? Write it as $f\, v_0$, where you must determine the number f. Express your answer in terms of m_s and m_r.

c. After the collision, the orbit of the space junk has changed. Use conservation of energy and angular momentum to find an equation for the closest approach r_p of the space junk to the moon.

d. Solve your equation in part c for the number f assuming that the closest approach $r_p = R_m$, the space junk hits the moon.

Solution:

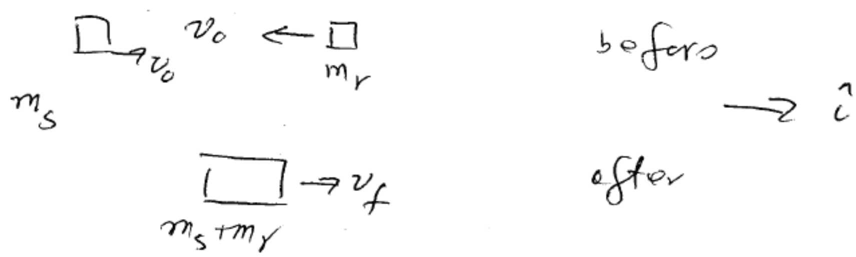
a) The speed of the mapping satellite undergoing uniform circular motion can be found from the force equation,

$$-\frac{Gm_s m_m}{R_0^{\ 2}} = -\frac{m_s v_0^{\ 2}}{R_0} \tag{28.9.12}$$

So the speed is

$$v_0 = \sqrt{Gm_p / R_0} \qquad (28.9.13)$$

b) The momentum flow diagram for the collision is shown below.



Because there are no external forces, the momentum is constant and so

$$m_s v_0 - m_r v_0 = (m_s + m_r)v_f \,. \qquad (28.9.14)$$

Thus the speed after the collision is

$$v_f = \frac{m_s - m_r}{m_s + m_r} v_0 = f v_0 , \qquad (28.9.15)$$

where the ratio of speed after the collision to the speed before the collision is given by the number

$$f = \frac{v_f}{v_0} = \frac{m_s - m_r}{m_s + m_r} \,. \qquad (28.9.16)$$

c) After the collision, the energy equation is given by

$$\frac{1}{2}(m_s + m_r)v_f^{\,2} - \frac{G(m_s + m_r)m_p}{R_0} = \frac{1}{2}(m_s + m_r)v_p^{\,2} - \frac{G(m_s + m_r)m_p}{r_p} .(28.9.17)$$

Setting $v_f = f v_0$ and simplifying yields

$$\frac{1}{2} f^2 v_0^{\,2} - \frac{Gm_p}{R_0} = \frac{1}{2} v_p^{\,2} - \frac{Gm_p}{r_p} \,. \qquad (28.9.18)$$

The angular momentum equation is

$$(m_s + m_r)R_0 v_f = (m_s + m_r) r_p v_p \,. \tag{28.9.19}$$

Again setting $v_f = f v_0$, Eq. (28.9.19) becomes

$$R_0 f v_0 = r_p v_p \tag{28.9.20}$$

Eq. (28.9.20) implies that $v_p = R_0 f v_0 / r_p$ which we can substitute into Eq. (28.9.18) yielding

$$\frac{1}{2} f^2 v_0^{\,2} - \frac{G m_p}{R_0} = \frac{1}{2}\left(\frac{R_0 f v_0}{r_p}\right)^2 - \frac{G m_p}{r_p} \,. \tag{28.9.21}$$

Collecting terms yields

$$\frac{1}{2} f^2 v_0^{\,2}\left(1 - \frac{R_0^{\,2}}{r_p^{\,2}}\right) = G m_p\left(\frac{1}{R_0} - \frac{1}{r_p}\right). \tag{28.9.22}$$

If we assume that at the closest approach $r_p = R_m$, the space junk hits the moon. Then using the values $r_p = R_m$, $v_0 = \sqrt{G m_p / R_0}$, and $R_0 = (3/2)R_m$, Eq. (28.9.22) becomes

$$f^2 \frac{G m_p}{3R_M}\left(1 - \frac{9R_M^{\,2}}{4R_M^{\,2}}\right) = G m_p\left(\frac{2}{3R_M} - \frac{1}{R_M}\right). \tag{28.9.23}$$

We can solve Eq. (28.9.23) for f:

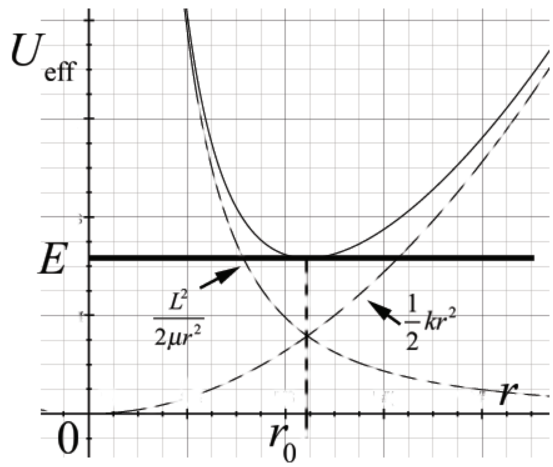
$$f = \sqrt{4/5} \,. \tag{28.9.24}$$

28.9.4 Example Lowest Energy Solution

The effective potential energy for a linear restoring central force $\vec{\mathbf{F}} = -kr\,\hat{\mathbf{r}}$ is given by

$$U_{\text{effective}} = \frac{L^2}{2\mu r^2} + \frac{1}{2}kr^2 .$$

Find the radius and the energy for the motion with the lowest energy. What type of motion does this correspond to? If the energy is slightly greater than the lowest energy, what type of motion would that correspond to?



Solution:

Taking the derivative of the effective potential with respect to the radius r and setting the derivative equal to zero at r_0,

$$-\frac{L^2}{\mu r_0^3} + k\,r_0 = 0$$

$$r_0 = \left(\frac{L^2}{k\mu}\right)^{1/4} .$$

The minimum energy is

$$U_{\text{eff}}(r_0) = \frac{L^2}{2\mu}\left(\frac{k\mu}{L^2}\right)^{1/2} + \frac{k}{2}\left(\frac{L^2}{k\mu}\right)^{1/2}$$

$$= \left(\frac{L^2 k}{\mu}\right)^{1/2} .$$

The orbit at this radius is a circle.

For a slightly larger energy, the orbit will oscillate about the minimum radius. Although it takes a bit more work to show, and is not part of this problem, the orbit will be an ellipse with the origin at the center (not at a focus, as in a Keplerian orbit).

28.9.5 Example ***Effective Potential Energy***

A system of two particles with a reduced mass μ interacts via an attractive central force $\vec{\mathbf{F}} = -ar^3\,\hat{\mathbf{r}}$, where r is the relative distance between the two particles. The magnitude of the angular momentum for the equivalent one particle problem about the origin is L.

a) Find an expression for the effective potential energy and make a graph of the effective potential energy as a function of r. Find an expression for r that minimizes the effective potential energy.

b) Indicate by a horizontal line on the graph sketch the total energy that would correspond to a constant relative distance r between the two particles. Find an expression for this energy. What type of motion does this correspond to for the equivalent one-particle problem?

Solution:

a) The potential energy associated with the given force, denoted U with no subscript, is given by

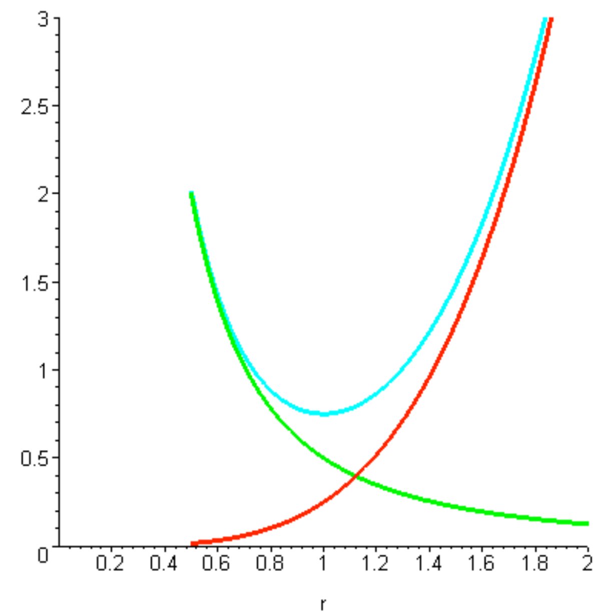
$$U = -\int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{F}}(\vec{\mathbf{r}}')\cdot d\vec{\mathbf{r}}' = \int_{r_0}^{r} a\,r'^3\,dr' = \frac{a}{4}(r^4 - r_0^4) \qquad (28.9.25)$$

where r_0 is the radius at which we choose to set the potential energy equal to zero. While we could choose any (finite) value for r_0, for the purposes of making the graph we'll choose $r_0 = 0$.

The effective potential energy is then

$$U_{\text{eff}} = U + \frac{L^2}{2\mu}\frac{1}{r^2} = \frac{a}{4}r^4 + \frac{L^2}{2\mu}\frac{1}{r^2}. \qquad (28.9.26)$$

From Equation (28.9.26) we see that the effective potential has both an attractive part ($\sim r^4$) and the "repulsive" part ($\sim -1/r^2$). A plot of U_{eff} as a function of r, with, for graphing purposes, $a = L = \mu = 1$ is shown below. Plots of U and the repulsive term are shown as well; U_{eff} is the upper curve, blue if viewed in color.



Differentiating Equation (28.9.26) and setting the derivative equal to zero at $r = r_{\min}$ yields

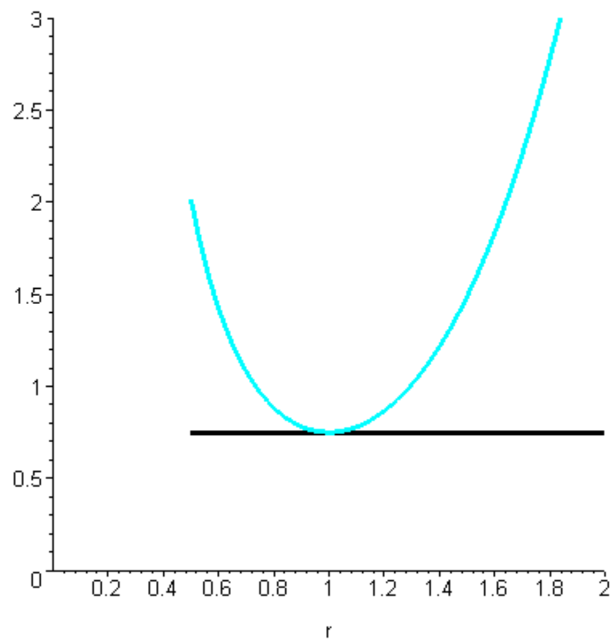
$$a\,r_{\min}^3 - \frac{L^2}{\mu}\frac{1}{r_{\min}^3} = 0$$
$$r_{\min} = \left(\frac{L^2}{\mu\,a}\right)^{1/6}. \tag{28.9.27}$$

In the above plot, with $a = L = \mu = 1$, $r_{\min} = 1$, consistent with the graph.

b) Using the value of $r_{\min}$ from (28.9.27) in Equation (28.9.26) yields

$$\begin{aligned} U_{\mathrm{eff}}(r_{\min}) &= \frac{a}{4}\left(\frac{L^2}{\mu\,a}\right)^{2/3} + \frac{L^2}{2\mu}\left(\frac{L^2}{\mu\,a}\right)^{1/3} \\ &= \frac{a^{1/3}L^{4/3}}{\mu^{2/3}}\left(\frac{1}{4}+\frac{1}{2}\right) \\ &= \frac{3}{4}\frac{a^{1/3}L^{4/3}}{\mu^{2/3}}. \end{aligned} \tag{28.9.28}$$

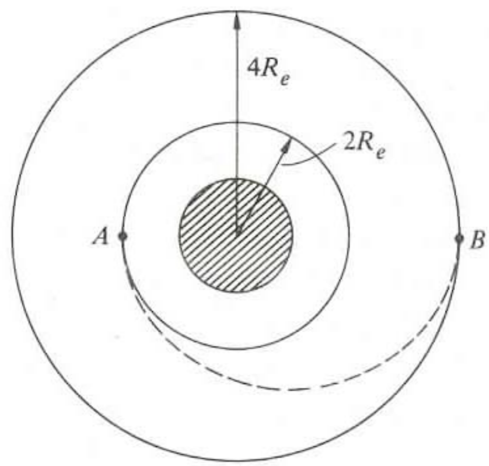
This value is shown as the black horizontal line in the plot below; with $a = L = \mu = 1$, $r_{\min} = 1$, $U_{\mathrm{eff}}(r_{\min}) = 3/4$.



The motion is a circle of radius $r_{\min} = 1$.

28.9.6 Example *Transfer Orbit*

A space vehicle is in a circular orbit about the earth. The mass of the vehicle is m_s and the radius of the orbit is $2R_e$. It is desired to transfer the vehicle to a circular orbit of radius $4R_e$. The mass of the earth is M_e.



a) What is the difference in energy between the outer and inner circular orbits?

b) An efficient way to accomplish the transfer is to use an elliptical orbit from point A at the inner circular orbit at to point B at the outer circular orbit (known as a Hohmann transfer orbit). This is accomplished by firing a rocket for a short time interval during each change of orbit thus increasing the tangential speed of the satellite. What velocity changes are required at the points of intersection, A and B?

c) Assume that the rocket burns fuel at a steady rate and the exhaust speed relative to the rocket is u. Using our early results for the rocket equation, how much fuel is burned at each of the rocket firings at the points of intersection, A and B? Assume that before the firing at A, the mass of the rocket is m_0.

Solution:

a) Because $m_s << m_p$, the reduced mass $\mu \cong m_s$. The total mechanical energy is the sum of the kinetic and potential energies,

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2} m_{\mathrm{s}} v^2 - G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{r}. \end{aligned} \tag{28.9.29}$$

For a circular orbit, the orbital speed and orbital radius must be related by the radial component of Newton's Second Law

$$-G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{r^2} = -m_{\mathrm{s}}\frac{v^2}{r}. \tag{28.9.30}$$

We can rewrite Eq. (28.9.30) as

$$\frac{1}{2} m_{\mathrm{s}} v^2 = \frac{1}{2} G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{r}. \tag{28.9.31}$$

Substituting Eq. (28.9.31) into Equation (28.9.29) gives

$$E = \frac{1}{2} G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{r} - G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{r} = -\frac{1}{2} G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{r} = \frac{1}{2} U(r). \tag{28.9.32}$$

In moving from a circular orbit of radius $2R_{\mathrm{e}}$ to an orbit of radius $4R_{\mathrm{e}}$, the total energy increases, as the energy becomes less negative. The change in energy is

$$\begin{aligned} \Delta E &= E\left(r = 4R_{\mathrm{e}}\right) - E\left(r = 2R_{\mathrm{e}}\right) \\ &= -\frac{1}{2} G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{4R_{\mathrm{e}}} - \left(-\frac{1}{2} G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{2R_{\mathrm{e}}}\right) \\ &= \frac{1}{8} G\frac{m_{\mathrm{s}} M_{\mathrm{e}}}{R_{\mathrm{e}}}. \end{aligned} \tag{28.9.33}$$

b) The satellite must increase its speed at point A in order to move to the larger orbit radius and increase its speed again at point B to stay in a circular orbit. Denote the satellite speed at point A while in the circular orbit as $v_{A,i}$ and after the speed increase (a "rocket burn") as $v_{A,f}$. Similarly, denote the satellite's speed when it first reaches point B as $v_{B,i}$ and the speed of the satellite in the circular orbit at point B as $v_{B,f}$. The speeds $v_{A,i}$ and $v_{B,f}$ are given by Equation (28.9.31).

While the satellite is moving from point A to point B (that is, during the transfer, after the first burn and before the second), both mechanical energy and angular momentum are conserved. Conservation of energy relates the speeds and radii by

$$\frac{1}{2}m_s(v_{A,f})^2 - G\frac{m_s M_e}{2R_e} = \frac{1}{2}m_s(v_{B,i})^2 - G\frac{m_s M_e}{4R_e}. \qquad (28.9.34)$$

Conservation of angular momentum relates the speeds and radii by

$$m_s\, v_{A,f}\,(2R_e) = m_s\, v_{B,i}\,(4R_e) \;\Rightarrow\; v_{A,f} = 2\, v_{B,i}. \qquad (28.9.35)$$

Substitution of Equation (28.9.35) into Equation (28.9.34) yields, after minor algebra,

$$v_{A,f} = \sqrt{\frac{2}{3}\frac{G M_e}{R_e}}, \quad v_{B,i} = \sqrt{\frac{1}{6}\frac{GM_e}{R_e}}. \qquad (28.9.36)$$

Equation (28.9.31) gives

$$v_{A,i} = \sqrt{\frac{1}{2}\frac{G M_e}{R_e}}, \quad v_{B,f} = \sqrt{\frac{1}{4}\frac{G M_e}{R_e}}. \qquad (28.9.37)$$

Thus the change in speeds at the respective points is given by

$$\Delta v_A = v_{A,f} - v_{A,i} = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right)\sqrt{\frac{G M_e}{R_e}} \qquad (28.9.38)$$

and

$$\Delta v_B = v_{B,f} - v_{B,i} = \left(\sqrt{\frac{1}{4}} - \sqrt{\frac{1}{6}}\right)\sqrt{\frac{G M_e}{R_e}}. \qquad (28.9.39)$$

Note that at both points, the speed must increase.

c) Recall from our rocket equation (add correct link W08D2 Worked Example Rocket Problem) that the change in the rocket speed is given by

$$\Delta v_r \equiv v_{r,f} - v_{r,0} = u\ln\left(\frac{m_0}{m_0 - \Delta m_f}\right) \qquad (28.9.40)$$

where the mass of the rocket after the burn is $m_{r,f} = m_0 - \Delta m_f$ with $\Delta m_f > 0$ the amount of fuel that is necessary to burn in order to increase the speed by Δv_r. We can write Eq. (28.9.40) as

$$-\frac{\Delta v_r}{u} = \ln\left(\frac{m_0 - \Delta m_f}{m_0}\right). \qquad (28.9.41)$$

Exponentiating both sides yields

$$e^{-\Delta v_r / u} = \frac{m_0 - \Delta m_f}{m_0}. \qquad (28.9.42)$$

We can now solve for Δm_f :

$$\Delta m_f = m_0(1 - e^{-\Delta v_r / u}). \qquad (28.9.43)$$

So for the first firing at point A we have that

$$\Delta m_{f,A} = m_0(1 - e^{-\Delta v_A / u}) \qquad (28.9.44)$$

where Δv_A is given by Eq. (28.9.38). The amount of fuel after the firing is then

$$m_{0,B} = m_0 e^{-\Delta v_A / u} \qquad (28.9.45)$$

The amount of fuel burned during the second firing at point B is then

$$\Delta m_{f,B} = m_{0,B}(1 - e^{-\Delta v_B / u}) = m_0 e^{-\Delta v_A / u}(1 - e^{-\Delta v_B / u}) \qquad (28.9.46)$$

where Δv_B is given by Eq. (28.9.39).

Chapter 28 Appendices

Appendix 28.A: Derivation of the Orbit Equation Using Energy Methods

Consider the reduced body with reduced mass given by Equation (28.3.1), orbiting about a central point under the influence of a radially attractive force given by Equation (28.3.3). Since the force is conservative, the potential energy with choice of zero reference point $U(\infty) = 0$ is given by

$$U(r) = -\frac{G m_1 m_2}{r}. \tag{28.A.1}$$

The total energy E is constant, and the sum of the kinetic energy and the potential energy is

$$E = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}. \tag{28.A.2}$$

The kinetic energy term, $\mu v^2 / 2$, has the reduced mass and the relative speed v of the two bodies. The velocity is cylindrical coordinates is given by (add link)

$$\begin{gathered}\vec{\mathbf{v}} = v_{\text{rad}}\,\hat{\mathbf{r}} + v_{\tan}\,\hat{\boldsymbol{\theta}}, \\ v^2 = v_{\text{rad}}{}^2 + v_{\tan}{}^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\end{gathered} \tag{28.A.3}$$

where $v_{\text{rad}} = dr/dt$ and $v_{\tan} = r(d\theta/dt)$. Equation (28.A.2) then becomes

$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{G\,m_1\,m_2}{r}. \qquad (28.A.4)$$

The magnitude of the angular momentum with respect to the center of mass is

$$L = \mu\, r\, v_{\tan} = \mu\, r^2 \frac{d\theta}{dt}. \qquad (28.A.5)$$

We shall explicitly eliminate the θ dependence from Equation (28.A.4) by using our expression in Equation (28.A.5),

$$\frac{d\theta}{dt} = \frac{L}{\mu\, r^2}. \qquad (28.A.6)$$

The mechanical energy as expressed in Equation (28.A.4) then becomes

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{\mu\, r^2} - \frac{G\,m_1\,m_2}{r}. \qquad (28.A.7)$$

Equation (28.A.7) is a separable differential equation involving the variable r as a function of time t and can be solved for the first derivative dr/dt,

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}}\left(E - \frac{1}{2}\frac{L^2}{\mu\, r^2} + \frac{G\,m_1\,m_2}{r}\right)^{\frac{1}{2}}. \qquad (28.A.8)$$

Instead of solving for the position of the reduced body as a function of time, we shall find a geometric description of the orbit by finding $r(\theta)$. We first divide Equation (28.A.6) by Equation (28.A.8) to obtain

$$\frac{d\theta}{dr} = \frac{\dfrac{d\theta}{dt}}{\dfrac{dr}{dt}} = \frac{\dfrac{L}{\mu\, r^2}}{\left(\dfrac{2}{\mu}\right)^{\frac{1}{2}}\left(E - \dfrac{1}{2}\dfrac{L^2}{\mu\, r^2} + \dfrac{G\,m_1\,m_2}{r}\right)^{\frac{1}{2}}}. \qquad (28.A.9)$$

The variables r and θ are separable;

$$d\theta = \frac{L}{\sqrt{2\mu}}\frac{(1/r^2)}{\left(E - \dfrac{L^2}{2\mu\, r^2} + \dfrac{G\,m_1\,m_2}{r}\right)^{1/2}}dr. \qquad (28.A.10)$$

What follows involves a good deal of hindsight, allowing selection of convenient substitutions in the math in order to get a clean result. First, note the many factors of the reciprocal of r. Make the substitution $u = 1/r$, $du = -(1/r^2)\,dr$, with the result

$$d\theta = -\frac{L}{\sqrt{2\mu}} \frac{du}{\left(E - \dfrac{L^2}{2\mu}u^2 + G\,m_1\,m_2\,u\right)^{1/2}}. \qquad (28.A.11)$$

Experience in evaluating integrals suggests that we make the absolute value of the factor multiplying u^2 inside the square root equal to unity. That is, multiplying numerator and denominator by $\sqrt{2\mu}/L$,

$$d\theta = -\frac{du}{(2\mu E/L^2 - u^2 + 2(\mu G\,m_1\,m_2/L^2)u)^{1/2}}. \qquad (28.A.12)$$

As both a check and a motivation for the next steps, note that the left side $(d\theta)$ of Equation (28.A.12) is dimensionless, and so the right side must be. This means that the factor of $\mu G\,m_1\,m_2/L^2$ in the square root must have the same dimensions as u, or length^{-1}; so, define

$$r_0 \equiv L^2/\mu G\,m_1\,m_2. \qquad (28.A.13)$$

a quantity we previously encountered in Eq. (28.5.10) called the semilatus rectum. The differential equation then becomes

$$\begin{aligned} d\theta &= -\frac{du}{(2\mu E/L^2 - u^2 + 2u/r_0)^{1/2}} \\ &= -\frac{du}{(2\mu E/L^2 + 1/r_0^{\,2} - u^2 + 2u/r_0 - 1/r_0^{\,2})^{1/2}} \\ &= -\frac{du}{(2\mu E/L^2 + 1/r_0^{\,2} - (u - 1/r_0)^2)^{1/2}} \\ &= -\frac{r_0\,du}{(2\mu E r_0^{\,2}/L^2 + 1 - (r_0 u - 1)^2)^{1/2}}. \end{aligned} \qquad (28.A.14)$$

Next, we note that the combination of terms $2\mu E r_0^{\,2}/L^2 + 1$ is dimensionless, so define

$$\varepsilon = \sqrt{1 + 2\mu E r_0^{\,2}/L^2} \qquad (28.A.15)$$

a quantity we also previously encountered (Eq. (28.5.12)) called the eccentricity. The last expression in is then

$$d\theta = -\frac{r_0\, du}{(\varepsilon^2 - (r_0 u - 1)^2)^{1/2}} . \tag{28.A.16}$$

From here, we'll combine a few calculus steps, going immediately to the substitution $r_0\, u - 1 = \varepsilon \cos\alpha$, $r_0\, du = -\varepsilon \sin\alpha\, d\alpha$, yielding

$$d\theta = -\frac{-\varepsilon \sin\alpha\, d\alpha}{(\varepsilon^2 - \varepsilon^2 \cos^2\alpha)^{1/2}} = d\alpha . \tag{28.A.17}$$

We can now integrate and find that

$$\theta = \int d\theta = \int d\alpha = \alpha + \text{constant} \tag{28.A.18}$$

We have a choice in selecting the constant, and if we pick $\theta = \alpha - \pi$, $\alpha = \theta + \pi$, $\cos\alpha = -\cos\theta$, the result is

$$r = \frac{1}{u} = \frac{r_0}{1 - \varepsilon \cos\theta} \tag{28.A.19}$$

agreeing with our result in Equation (28.5.13).

Note that if we chose the constant of integration to be zero, the result would be

$$r = \frac{1}{u} = \frac{r_0}{1 + \varepsilon \cos\theta} \tag{28.A.20}$$

which is the same trajectory reflected about the "vertical" axis in Figure 28.3, indeed the same as rotating by π .

Appendix 28.B: Properties of an Elliptical Orbit

We now consider the special case of an elliptical orbit.

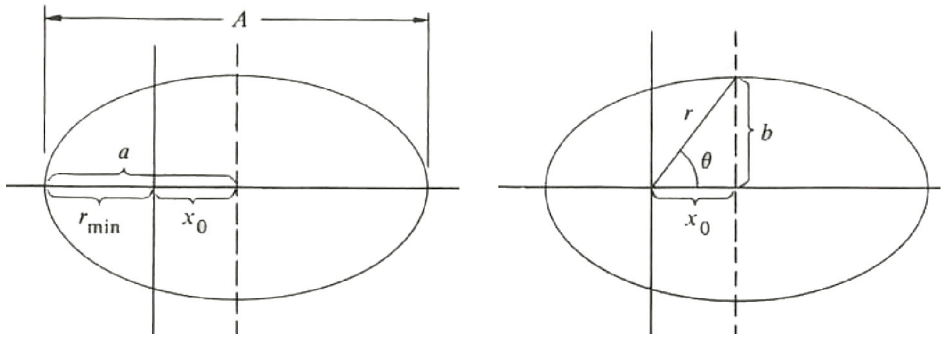


Figure 28.B.1 Ellipse.

In Figure 28.B.1, let a denote the semimajor axis, b denote the semiminor axis and x_0 denote the distance from the center of the ellipse to the origin of our coordinate system (r,θ). We shall now express the parameters a, b and x_0 in terms of the constants of the motion L, E, μ, m_1 and m_2.

The semimajor axis:

See Equation (28.A.19) above. The major axis $A = 2a$ is given by

$$A = 2a = r_{max} + r_{min} \tag{28.B.1}$$

where the distance of furthest approach occurs when $\theta = 0$, hence

$$r_{max} = r(\theta = 0) = \frac{r_0}{1-\varepsilon} \tag{28.B.2}$$

and the distance of nearest approach occurs when $\theta = \pi$, hence

$$r_{min} = r(\theta = \pi) = \frac{r_0}{1+\varepsilon}. \tag{28.B.3}$$

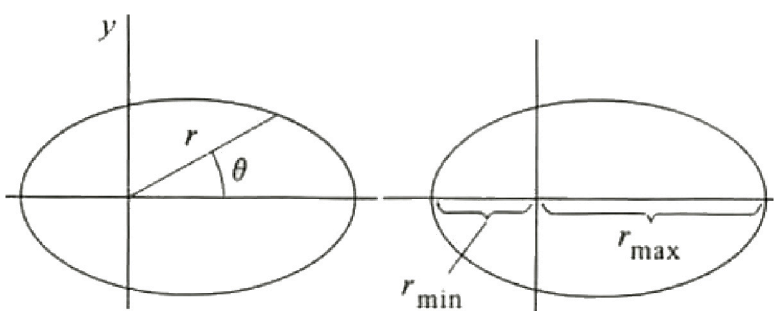


Figure 28.B.2: nearest and furthest approach

Thus

$$a = \frac{1}{2}\left(\frac{r_0}{1-\varepsilon} + \frac{r_0}{1+\varepsilon}\right) = \frac{r_0}{1-\varepsilon^2}. \tag{28.B.4}$$

The *semilatus rectum* r_0 can be re-expressed in terms of the semimajor axis and the eccentricity,

$$r_0 = a(1-\varepsilon^2). \tag{28.B.5}$$

We can now express the distance of nearest approach, Equation (28.B.3), in terms of the semimajor axis and the eccentricity,

$$r_{\text{min}} = \frac{r_0}{1+\varepsilon} = \frac{a(1-\varepsilon^2)}{1+\varepsilon} = a(1-\varepsilon). \tag{28.B.6}$$

In a similar fashion the distance of furthest approach is

$$r_{\text{max}} = \frac{r_0}{1-\varepsilon} = \frac{a(1-\varepsilon^2)}{1-\varepsilon} = a\left(1+\varepsilon\right). \tag{28.B.7}$$

Figure 17.B.2 shows the distances of nearest and furthest approach. Using our results for r_0 and ε from Equations (17.3.12) and (17.3.13), we have for the semimajor axis

$$\begin{aligned} a &= \frac{L^2}{\mu G m_1 m_2} \frac{1}{(1-(1+2EL^2/\mu\left(Gm_1m_2\right)^2))} \\ &= -\frac{Gm_1m_2}{2E} \end{aligned}. \tag{28.B.8}$$

and so the energy is determined by the semimajor axis,

$$E = -\frac{Gm_1m_2}{2a}. \tag{28.B.9}$$

The angular momentum is related to the *semilatus rectum* r_0 by Equation (17.3.12). Using Equation (28.B.5) for r_0, we can express the angular momentum (28.B.3) in terms of the semimajor axis and the eccentricity,

$$L = \sqrt{\mu G m_1 m_2 r_0} = \sqrt{\mu G m_1 m_2 a(1-\varepsilon^2)}. \tag{28.B.10}$$

Note that

$$\sqrt{(1-\varepsilon^2)} = \frac{L}{\sqrt{\mu G m_1 m_2 a}} . \qquad (28.B.11)$$

Location x_0 of the center of the ellipse:

From Figure 28.B.1, the distance from a focal point to the center of the ellipse is

$$x_0 = r_{\max} - a . \qquad (28.B.12)$$

Using Equation (28.B.7) for $r_{\max}$, we have that

$$x_0 = a(1+\varepsilon) - \varepsilon = \varepsilon\, a. \qquad (28.B.13)$$

Thus, from Equations (17.3.12), (28.B.13) and (28.B.8),

$$x_0 = \varepsilon\, a = -\frac{G m_1 m_2}{2E}\sqrt{(1+2E L^2 / \mu(G m_1 m_2)^2)} . \qquad (28.B.14)$$

The semi-minor axis:

From Figure 28.B.1,

$$b = \sqrt{(r_b{}^2 - x_0{}^2)} \qquad (28.B.15)$$

where

$$r_b = \frac{r_0}{1-\varepsilon\cos\theta_b} , \qquad (28.B.16)$$

which can be rewritten as

$$r_b - r_b\, \varepsilon \cos\theta_b = r_0 . \qquad (28.B.17)$$

Note that from Figure 28.B.1,

$$x_0 = r_b \cos\theta_b , \qquad (28.B.18)$$

so that

$$r_b = r_0 + \varepsilon\, x_0 . \qquad (28.B.19)$$

Substituting Equation (28.B.13) for x_0 and Equation (28.B.5) for r_0 into Equation (28.B.19) yields

$$r_b = a(1-\varepsilon^2) + a\varepsilon^2 = a\,. \qquad (28.B.20)$$

The fact that $r_b = a$ is a well-known property of an ellipse reflected in the geometric construction, that the sum of the distances from the two foci to any point on the ellipse is a constant. Thus the semi-minor axis b becomes

$$b = \sqrt{r_b^{\,2} - x_0^{\,2}} = \sqrt{a^2 - \varepsilon^2 a^2} = a\sqrt{1-\varepsilon^2}\,. \qquad (28.B.21)$$

Using Equation (28.B.11) for $\sqrt{1-\varepsilon^2}$, we have for the semi-minor axis

$$b = \sqrt{aL^2 / \mu\, G m_1 m_2}\,. \qquad (28.B.22)$$

We can now use Equation (28.B.8) for a in the above expression, yielding

$$b = \sqrt{aL^2 / \mu\, G m_1 m_2} = L\sqrt{-\frac{G m_1 m_2}{2E} / \mu\, G m_1 m_2} = L\sqrt{-\frac{1}{2\mu E}} \qquad (28.B.23)$$

Speeds at nearest and furthest distances:

At nearest approach the velocity vector is tangent to the orbit, so the angular momentum is

$$L = \mu r_{\min} v_p \qquad (28.B.24)$$

and the speed at nearest approach is

$$v_p = L / \mu r_{\min}\,. \qquad (28.B.25)$$

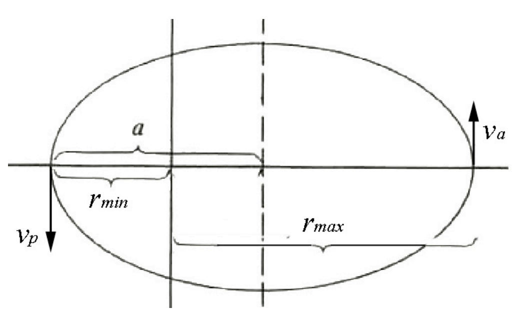


Figure 28.B.3 Speeds at nearest and furthest approach

Using Equation (28.B.10) for the angular momentum and Equation (28.B.6) for $r_{\min}$, Equation (28.B.25) becomes

$$v_p = \frac{L}{\mu r_{\min}} = \frac{\sqrt{\mu G m_1 m_2 (1-\varepsilon^2)}}{\mu a(1-\varepsilon)} = \sqrt{\frac{G m_1 m_2 (1-\varepsilon^2)}{\mu a(1-\varepsilon)^2}} = \sqrt{\frac{G m_1 m_2 (1+\varepsilon)}{\mu a(1-\varepsilon)}} . \quad (28.B.26)$$

A similar calculation show that the speed v_a at furthest approach,

$$v_a = \frac{L}{\mu r_{\max}} = \frac{\sqrt{\mu G m_1 m_2 (1-\varepsilon^2)}}{\mu a(1+\varepsilon)} = \sqrt{\frac{G m_1 m_2 (1-\varepsilon^2)}{\mu a(1+\varepsilon)^2}} = \sqrt{\frac{G m_1 m_2 (1-\varepsilon)}{\mu a(1+\varepsilon)}} . \quad (28.B.27)$$

Appendix 28.C: Analytic Geometric Properties of Ellipses

Consider Equation (28.5.18), and for now take $\varepsilon < 1$, so that the equation is that of an ellipse. It takes some, but not a great deal, of algebra to put this into the more familiar form

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (28.C.1)$$

where the ellipse has axes parallel to the x and y coordinate axes, center at $(x_0, 0)$, semimajor axis a and semiminor axis b.

We rewrite Equation (28.5.18) as

$$x^2 - \frac{2\varepsilon r_0}{1-\varepsilon^2} x + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{1-\varepsilon^2} . \quad (28.C.2)$$

We next complete the square,

$$\begin{gathered} x^2 - \frac{2\varepsilon r_0}{1-\varepsilon^2} x + \frac{\varepsilon^2 r_0^2}{(1-\varepsilon^2)^2} + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{1-\varepsilon^2} + \frac{\varepsilon^2 r_0^2}{(1-\varepsilon^2)^2} \\ \left(x - \frac{\varepsilon r_0}{1-\varepsilon^2} \right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{(1-\varepsilon^2)^2} \end{gathered} . \quad (28.C.3)$$

We now multiply each side by the factor $(1-\varepsilon^2)^2 / r_0^2$ yielding

$$\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2} \right)^2 \frac{(1-\varepsilon^2)^2}{r_0^2} + \frac{y^2 (1-\varepsilon^2)^2}{r_0^2 (1-\varepsilon^2)} = 1 \quad (28.C.4)$$

Eq. (28.C.4) simplifies to

$$\frac{\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2}{\left(\frac{r_0}{(1-\varepsilon^2)}\right)^2} + \frac{y^2}{\left(\frac{r_0}{\sqrt{1-\varepsilon^2}}\right)^2} = 1 \qquad (28.C.5)$$

The last expression in (28.C.3) is the equation of an ellipse

$$\frac{(x-\varepsilon a)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (28.C.6)$$

with semimajor axis

$$a = \frac{r_0}{1-\varepsilon^2}, \qquad (28.C.7)$$

semiminor axis

$$b = \frac{r_0}{\sqrt{1-\varepsilon^2}} = a\sqrt{1-\varepsilon^2} \qquad (28.C.8)$$

and center at $(\varepsilon r_0 / (1-\varepsilon^2), 0) = (\varepsilon a, 0)$, as found in Equation (28.B.13).

Appendix 28.D: Even More on Kepler Orbits

We've seen so far that for a Kepler Orbit, we have two constants of the motion: the angular momentum and the total energy. Since the angular momentum is a vector with three components, these constitute a total of four scalar constants of the motion. The Kepler Problem has six "degrees of freedom" (three position, three velocity), and so we expect to be able to find two more scalar constants of the motion.

We might expect to be able to further identify any orbit by a vector in the plane of the orbit, perpendicular to the angular momentum, and this is indeed the case. Symmetry suggests that this vector would be along the major axis, and we'll see that this is the case as well. What follows uses a good deal of vector algebra, but minimal calculus, and leads to the orbit equation in a surprisingly simple form.

We'll need two results from vector algebra that we haven't had to use yet. Specifically, for vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, we have

$$\begin{aligned} \vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) &= \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \\ \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) &= \vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{a}}) = \vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}). \end{aligned} \qquad (28.D.1)$$

These relations are not hard to derive in Cartesian coordinates; the derivations will not be reproduced here. As a check, however, note that the vector on the right side of the first relation is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}\times\vec{\mathbf{c}}$. If $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are non-coplanar, the common magnitude of the scalars in the second relation is the expression for the volume of a parallelepiped with the three vectors forming the sides.

Let's start with the known constant angular momentum,

$$\vec{\mathbf{L}} = \vec{\mathbf{r}}\times\vec{\mathbf{p}} = \mu\,\vec{\mathbf{r}}\times\vec{\mathbf{v}} \tag{28.D.2}$$

and re-express this quantity in a way that will allow us to use Newton's Laws. Specifically, consider the velocity in terms of polar coordinates,

$$\vec{\mathbf{v}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \tag{28.D.3}$$

so that the angular momentum can be expressed as

$$\vec{\mathbf{L}} = \mu(r\,\hat{\mathbf{r}})\times\left(r\frac{d\hat{\mathbf{r}}}{dt}\right) = \mu r^2\left(\hat{\mathbf{r}}\times\frac{d\hat{\mathbf{r}}}{dt}\right) \tag{28.D.4}$$

from which

$$\hat{\mathbf{r}}\times\frac{d\hat{\mathbf{r}}}{dt} = \frac{1}{r^2}\frac{\vec{\mathbf{L}}}{\mu}. \tag{28.D.5}$$

The advantage to this operation is that we now have an explicit scalar factor of $1/r^2$, which can and should be related to the same factor that appears in Newton's Law of Gravitation. However, in order to use that law, we need a vector relation involving $\hat{\mathbf{r}}$, and so we'll cross $\hat{\mathbf{r}}$ into both sides of Equation (28.D.5), yielding

$$\begin{aligned}\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\frac{d\hat{\mathbf{r}}}{dt}\right) &= \frac{\hat{\mathbf{r}}}{r^2}\times\frac{\vec{\mathbf{L}}}{\mu}\\ -\frac{d\hat{\mathbf{r}}}{dt} &= -\frac{1}{Gm_1m_2}\left(\frac{d^2}{dt^2}\vec{\mathbf{r}}\right)\times\vec{\mathbf{L}}.\end{aligned} \tag{28.D.6}$$

In the above, the first relation in Equation (28.D.1) was used to simplify the left side, and Newton's Law of Gravitation, in the form $\mu(d^2\vec{\mathbf{r}}/dt^2) = -(Gm_1m_2/r^2)\hat{\mathbf{r}}$ was used on the right side. Note the cancellation of the factor of the reduced mass μ.

Equation (28.D.6) may now be integrated to obtain

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{v}} \times \vec{\mathbf{L}}}{Gm_1 m_2} + \vec{\mathbf{A}}$$
$$\vec{\mathbf{A}} = \hat{\mathbf{r}} - \frac{\vec{\mathbf{v}} \times \vec{\mathbf{L}}}{Gm_1 m_2}, \qquad (28.D.7)$$

where $\vec{\mathbf{A}}$ is a constant vector. Since $\hat{\mathbf{r}}$ is in the plane of the orbit, and $\vec{\mathbf{L}}$ is perpendicular to the plane of the orbit, $\vec{\mathbf{A}}$ must lie in the plane of the orbit, as indicated above. Further, by considering extreme points of the orbit, where $\vec{\mathbf{v}} \perp \hat{\mathbf{r}}$, and hence $\vec{\mathbf{v}} \times \vec{\mathbf{L}} \parallel \hat{\mathbf{r}}$, we see that at these points $\vec{\mathbf{A}}$ is in the direction parallel to the major axis. Since $\vec{\mathbf{A}}$ is a constant vector, $\vec{\mathbf{A}}$ must always be in this direction. By considering the vector $\vec{\mathbf{A}}$ at perihelion (or at any point on the orbit of a circular orbit), we can see that the direction of $\vec{\mathbf{A}}$ is that from the perihelion point to the focus; we'll need this result below, when we find the orbit equation.

The magnitude of $\vec{\mathbf{A}}$ is readily found by calculating

$$A^2 = \vec{\mathbf{A}} \cdot \vec{\mathbf{A}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} - \frac{2}{Gm_1 m_2} \hat{\mathbf{r}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{L}}) + \frac{1}{\left(Gm_1 m_2\right)^2} (\vec{\mathbf{v}} \times \vec{\mathbf{L}}) \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{L}}) . \qquad (28.D.8)$$

The first dot product is manifestly 1. The middle term, the "cross term," is found using the second relation in Equation (28.D.1),

$$\hat{\mathbf{r}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{L}}) = \vec{\mathbf{L}} \cdot (\hat{\mathbf{r}} \times \vec{\mathbf{v}}) = \frac{1}{\mu r} \vec{\mathbf{L}} \cdot (\vec{\mathbf{r}} \times \mu \vec{\mathbf{v}}) = \frac{1}{\mu r} L^2 . \qquad (28.D.9)$$

The third term is most easily evaluated by recalling that $\vec{\mathbf{v}} \perp \vec{\mathbf{L}}$, so that $\left|\vec{\mathbf{v}} \times \vec{\mathbf{L}}\right| = vL$ and $(\vec{\mathbf{v}} \times \vec{\mathbf{L}}) \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{L}}) = v^2 L^2$. Combining, we see that

$$\begin{aligned} A^2 &= 1 - \frac{2}{Gm_1 m_2} \frac{1}{\mu r} L^2 + \frac{1}{(Gm_1 m_2)^2} v^2 L^2 \\ &= 1 + \frac{2L^2}{(Gm_1 m_2)^2 \mu} \left(\frac{1}{2} \mu v^2 - \frac{Gm_1 m_2}{r} \right) \\ &= 1 + \frac{2L^2 E}{(Gm_1 m_2)^2 \mu} = \varepsilon^2. \end{aligned} \qquad (28.D.10)$$

Thus, the constant vector $\vec{\mathbf{A}}$ is directed along the major axis and has magnitude equal to the eccentricity.

The orbit equation is now found algebraically by taking the dot product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{r}}$;

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{r}} = Ar\cos\theta = \hat{\mathbf{r}} \cdot \vec{\mathbf{r}} - \frac{1}{Gm_1 m_2} \vec{\mathbf{r}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{L}}) . \qquad (28.D.11)$$

The first term is merely the magnitude r of the position vector. The second term, repeating the calculation of Equation (28.D.9) with $\vec{\mathbf{r}}$ instead of $\hat{\mathbf{r}}$, is

$$\frac{1}{Gm_1 m_2 \mu} L^2 = r_0 , \qquad (28.D.12)$$

with the result

$$Ar\cos\theta = \varepsilon r\cos\theta = r - r_0 . \qquad (28.D.13)$$

Solving for r gives the orbit equation in the form

$$r = \frac{r_0}{1 - \varepsilon\cos\theta} . \qquad (28.D.14)$$

It should be noted that what we call the vector $\vec{\mathbf{A}}$ is a negative scalar multiple of the "Laplace-Runge-Lenz" vector (yes, it took three people to come up with this). Specifically, the L-R -L vector is in many sources given as

$$\vec{\mathbf{A}}_{\mathrm{LRL}} = \vec{\mathbf{p}} \times \vec{\mathbf{L}} - \mu\, Gm_1 m_2 \frac{\vec{\mathbf{r}}}{r} = -\mu\, Gm_1 m_2\, \vec{\mathbf{A}} . \qquad (28.D.15)$$

Our choice of the form for $\vec{\mathbf{A}}$ allows $\left|\vec{\mathbf{A}}\right| = \varepsilon$ and the direction of $\vec{\mathbf{A}}$ to lead to Equation (28.D.11) without introduction of extra minus signs.