Conservation of Angular Momentum

Time Derivative of Angular Momentum for a Point Particle

Time derivative of the angular momentum about S:

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}\right)$$

Product rule

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}\right) = \frac{d\vec{\mathbf{r}}_S}{dt} \times \vec{\mathbf{p}} + \vec{\mathbf{r}}_S \times \frac{d}{dt}\vec{\mathbf{p}}$$

Key Fact:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}_S}{dt} \Rightarrow \frac{d\vec{\mathbf{r}}_S}{dt} \times m\vec{\mathbf{v}} = \vec{\mathbf{v}} \times m\vec{\mathbf{v}} = \vec{\mathbf{0}}$$

Result:

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_S \times \frac{d}{dt}\vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}} = \vec{\boldsymbol{\tau}}_S$$

Torque and the Time Derivative of Angular Momentum: Point Particle

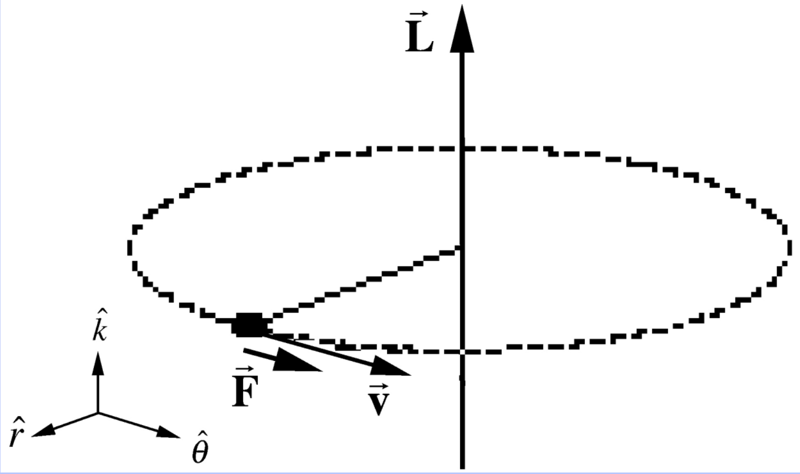
Torque about a point S is equal to the time derivative of the angular momentum about S .

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}$$

Concept Question: Change in Angular Momentum

A person spins a tennis ball on a string in a horizontal circle with velocity $\vec{\mathbf{v}}$ (so that the axis of rotation is vertical). At the point indicated below, the ball is given a sharp blow (force $\vec{\mathbf{F}}$) in the forward direction. This causes a change in angular momentum $\Delta\vec{\mathbf{L}}$ in the

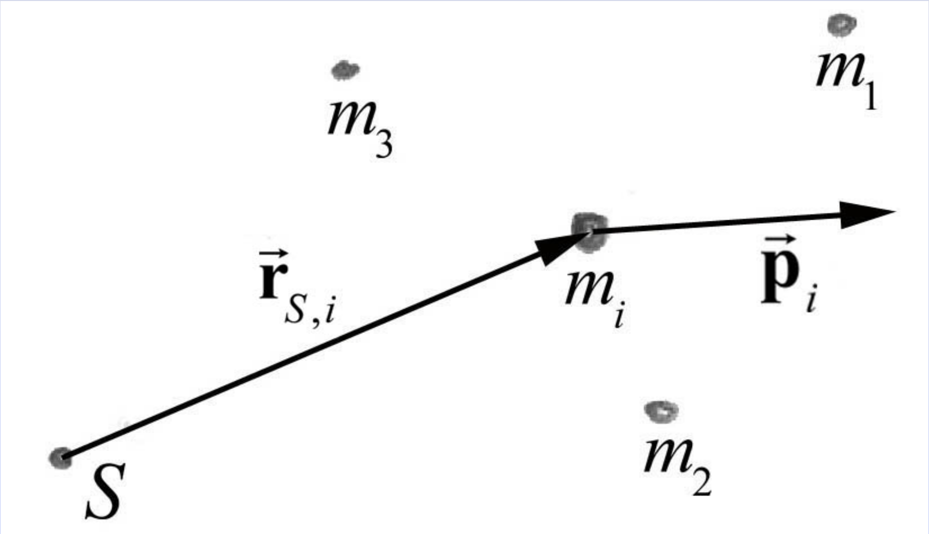
1. $\hat{\mathbf{r}}$ direction
2. $\hat{\boldsymbol{\theta}}$ direction
3. $\hat{\mathbf{k}}$ direction



Angular Momentum for System of Particles

Treat each particle separately

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i$$



Angular momentum for system about S

$$\vec{\mathbf{L}}_S^{\text{sys}} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i$$

Angular Momentum and Torque for a System of Particles

Change in total angular momentum about a point S equals the total torque about the point S

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \left(\frac{d\vec{\mathbf{r}}_{S,i}}{dt} \times \vec{\mathbf{p}}_i + \vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right)$$

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right) = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i \right) = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i} = \vec{\boldsymbol{\tau}}_S^{\text{total}}$$

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \vec{\boldsymbol{\tau}}_S^{\text{total}}$$

Internal and External Torques

The total external torque is the sum of the torques due to the net external force acting on each element

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i}^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i^{\text{ext}}$$

The total internal torque arise from the torques due to the internal forces acting between pairs of elements

$$\vec{\boldsymbol{\tau}}_S^{\text{int}} = \sum_{j=1}^{N} \vec{\boldsymbol{\tau}}_{S,j}^{\text{int}} = \sum_{j=1}^{i=N} \sum_{\substack{i=1 \\ j \neq i}}^{i=N} \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{i,j}$$

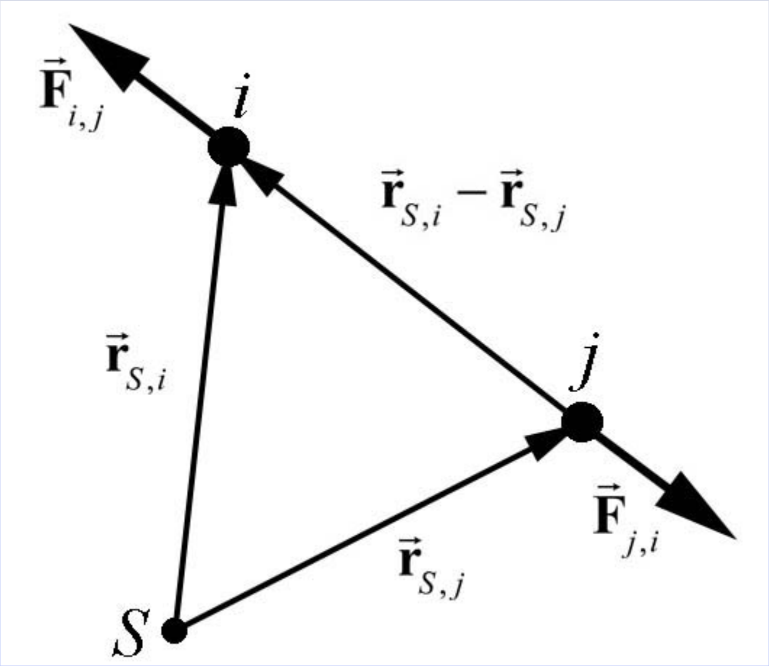
The total torque about S is the sum of the external torques and the internal torques

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \vec{\boldsymbol{\tau}}_S^{\text{ext}} + \vec{\boldsymbol{\tau}}_S^{\text{int}}$$

Internal Torques

We know by Newton's Third Law that the internal forces cancel in pairs and hence the sum of the internal forces is zero

$$\vec{\mathbf{F}}_{i,j} = -\vec{\mathbf{F}}_{j,i} \qquad \vec{\mathbf{0}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{i=N} \vec{\mathbf{F}}_{i,j}$$



Does the same statement hold about pairs of internal torques?

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{i,j} + \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_{j,i}$$

By the Third Law this sum becomes

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{i,j}$$

The vector $\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}$ points from the j^{th} element to the i^{th} element.

Central Forces: Internal Torques Cancel in Pairs

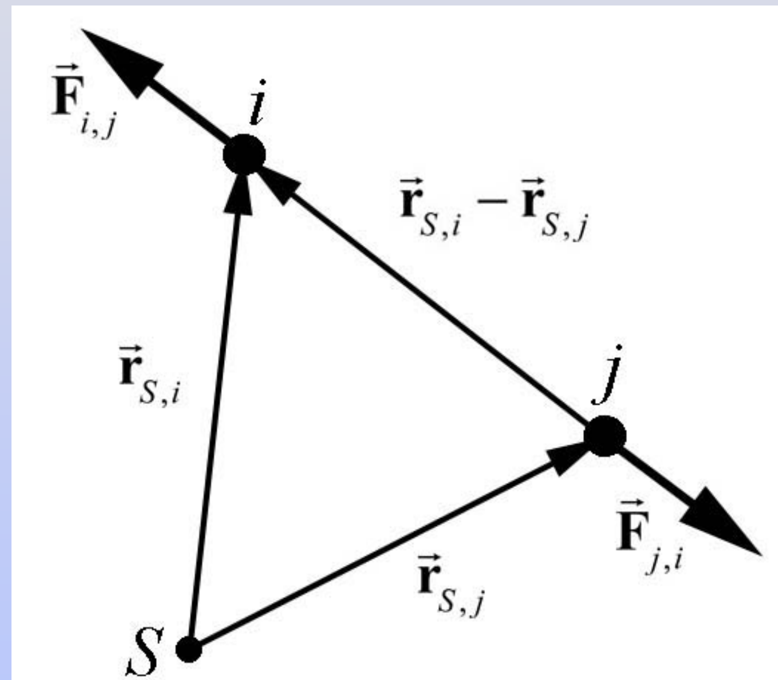
If the internal forces between a pair of particles are directed along the line joining the two particles then the torque due to the internal forces cancel in pairs.

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{i,j} = \vec{\mathbf{0}}$$

This is a stronger version of Newton's Third Law than we have so far used requiring that internal forces are *central forces*. With this assumption, the total torque is just due to the external forces

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$$

However, so far no isolated system has been encountered such that the angular momentum is not constant.



$$\vec{\mathbf{F}}_{i,j} \quad (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j})$$

Angular Impulse and Change in Angular Momentum

Angular impulse

$$\vec{\mathbf{J}} = (\vec{\boldsymbol{\tau}}_S{}^{\text{ext}})_{\text{ave}} \Delta t_{\text{int}} = \Delta \vec{\mathbf{L}}_S^{\text{sys}}$$

$$\vec{\mathbf{J}} = \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S{}^{\text{ext}} \, dt$$

Change in angular momentum

$$\Delta \vec{\mathbf{L}}_S^{\text{sys}} \equiv \left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_f - \left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_i$$

Rotational dynamics

$$\int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S{}^{\text{ext}} \, dt = \left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_f - \left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_i$$

Conservation of Angular Momentum

Rotational dynamics $\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \dfrac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$

No external torques $\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S^{\text{ext}} = \dfrac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$

Change in Angular momentum is zero

$$\Delta\vec{\mathbf{L}}_S^{\text{sys}} \equiv \left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_f - \left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_0 = \vec{\mathbf{0}}$$

Angular Momentum is conserved

$$\left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_f = \left(\vec{\mathbf{L}}_S^{\text{sys}}\right)_0$$

So far no isolated system has been encountered such that the angular momentum is not constant.

Constants of the Motion

When are the quantities, angular momentum about a point S, energy, and momentum constant for a system?

- No external torques about point S : angular momentum about S is constant

$$\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S^{\,\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$$

- No external work: mechanical energy constant

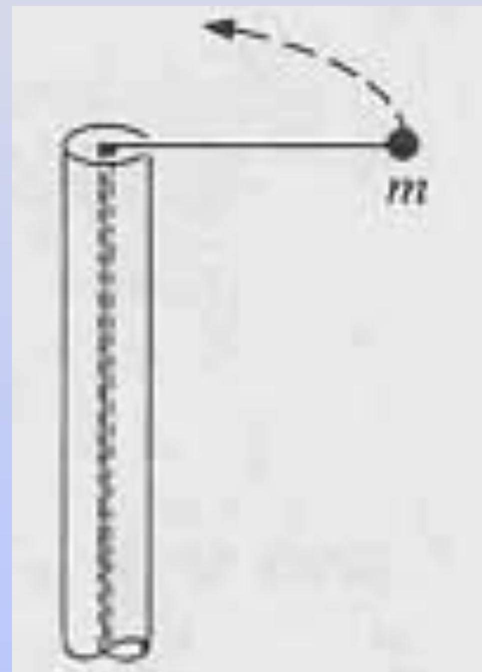
$$0 = W_{\text{ext}} = \Delta E_{\text{mechanical}}$$

- No external forces: momentum constant

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt}$$

Checkpoint Problem: Conservation Laws

A tetherball of mass m is attached to a post of radius by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0 . The string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. Ignore gravity. Until the ball hits the post,



1. The energy and angular momentum about the center of the post are constant.

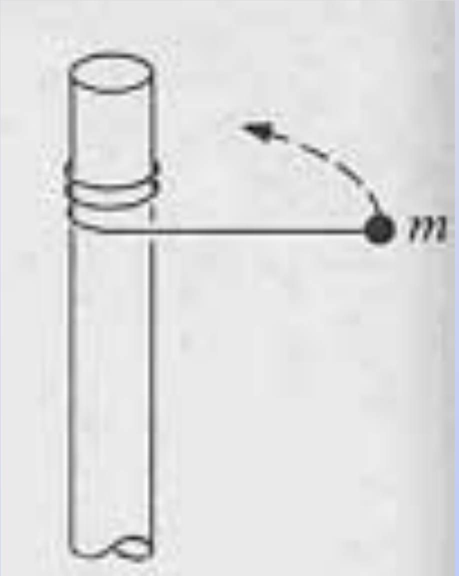
2. The energy of the ball is constant but the angular momentum about the center of the post changes.

3. Both the energy and the angular momentum about the center of the post, change.

4. The energy of the ball changes but the angular momentum about the center of the post is constant.

Checkpoint Problem: Conservation laws

A tetherball of mass m is attached to a post of radius R by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0. The string wraps around the outside of the post. Ignore gravity. Until the ball hits the post,



1. The energy and angular momentum about the center of the post are constant.

2. The energy of the ball is constant but the angular momentum about the center of the post changes.

3. Both the energy of the ball and the angular momentum about the center of the post, change.

4. The energy of the ball changes but the angular momentum about the center of the post is constant.

Home Experiment: Rotating on a Chair

A person holding dumbbells in his/her arms spins in a rotating stool. When he/she pulls the dumbbells inward, the moment of inertia changes and he/she spins faster.

Checkpoint Problem: Figure Skater

A figure skater stands on one spot on the ice (assumed frictionless) and spins around with her arms extended. When she pulls in her arms, she reduces her rotational moment of inertia and her angular speed increases. Assume that her angular momentum is constant. How does her initial rotational kinetic energy compare in to her rotational kinetic energy after she has pulled in her arms?

Checkpoint Problem: Impact Parameter

A meteor of mass m is approaching earth as shown on the sketch. The distance h on the sketch below is called the impact parameter. The radius of the earth is R_e. The mass of the earth is m_e. Suppose the meteor has an initial speed of v_0. Assume that the meteor started very far away from the earth. Suppose the meteor just grazes the earth. You may ignore all other gravitational forces except the earth. Find the impact parameter h and the cross section πh^2 .

Strategy: Impact Parameter

1. Draw a force diagram for the forces acting on the meteor.

2. Find a point about which the gravitational torque of the earth's force on the meteor is zero for the entire orbit of the meteor.

3. What is the initial angular momentum and final angular momentum (when it just grazes the earth) of the meteor about that point?

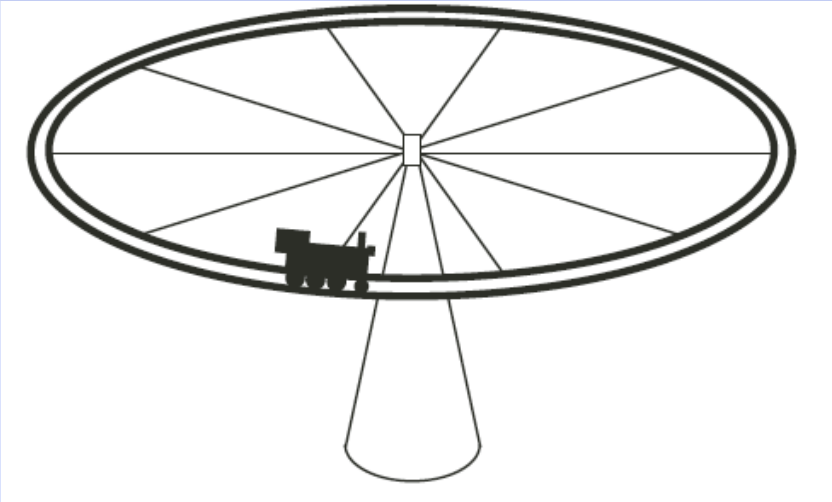
4. Apply conservation of angular momentum to find a relationship between the meteor's final velocity and the impact parameter.

5. Apply conservation of energy to find a relationship between the final velocity of the meteor and the initial velocity of the meteor.

6. Use your above results to calculate the impact parameter and the effective scattering cross section.

Checkpoint Problem: Train on Track

A toy locomotive of mass M_L runs on a horizontal circular track of radius R and total mass M_T . The track forms the rim of an otherwise massless wheel which is free to rotate without friction about a vertical axis. The locomotive is started from rest and accelerated without slipping to a final speed of v relative to the track. What is the locomotive's final speed, v_f , relative to the floor?



Checkpoint Problem: Collisions and Angular Momentum

A steel washer, is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. Assume that the frictional torque on the axle remains the same throughout the slowing down. The washer is set into motion. When it reaches an initial angular velocity ω_0, at t = 0, the power to the motor is shut off, and the washer slows down during the time interval $\Delta t_1 = t_a$ until it reaches an angular velocity of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{col} = t_b - t_a$. Assume the frictional torque on the axle remains the same. The two washers continue to slow down during the time interval $\Delta t_2 = t_f - t_b$ until they stop at $t = t_f$. Express your answers in terms of I_0, I_w, ω_0, ω_a, Δt_1, Δt_{col}, and Δt_2.

a) What is the angular deceleration α_1 while the washer and motor are slowing down during the interval $\Delta t_1 = t_a$?

b) What is the angular impulse during the collision?

c) What is the angular velocity of the two washers immediately after the collision is finished?

d) What is the angular deceleration α_2 after the collision?