Module 25: Angular Momentum and Torque for a System of Particles

25.1 Angular Momentum of a System of Particles

We now calculate the angular momentum about the point S associated with a system of N point particles. Label each individual particle by the index i, $i = 1,2,\ldots N$. Let the i_{th} particle have mass m_i and velocity $\vec{\mathbf{v}}_i$. The momentum of an individual particle is then $\vec{\mathbf{p}}_i = m_i \vec{\mathbf{v}}_i$. Let $\vec{\mathbf{r}}_{S,i}$ be the vector from the point S to the i_{th} particle, and let θ_i be the angle between the vectors $\vec{\mathbf{r}}_{S,i}$ and $\vec{\mathbf{p}}_i$ (Figure 25.1).

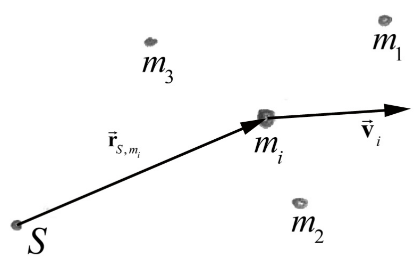


Figure 25.1 A system of particles with a moving center of mass.

The angular momentum $\vec{\mathbf{L}}_{S,i}$ of the i_{th} particle is then

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i . \tag{25.1.1}$$

The total angular momentum for the system of particles is the vector sum of the individual angular momenta,

$$\vec{\mathbf{L}}_S^{\text{total}} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i . \tag{25.1.2}$$

The change in the angular momentum of the system of particles about a point S is given by

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \left(\frac{d\vec{\mathbf{r}}_{S,i}}{dt} \times \vec{\mathbf{p}}_i + \vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right). \tag{25.1.3}$$

Because the velocity of the ith particle is $\vec{\mathbf{v}}_{S,i} = d\vec{\mathbf{r}}_{S,i} / dt$, the first term in the parentheses vanishes (the cross product of a vector with itself is zero because they are parallel to each other)

$$\frac{d\vec{\mathbf{r}}_{S,i}}{dt} \times \vec{\mathbf{p}}_i = \vec{\mathbf{v}}_{S,i} \times m_i \vec{\mathbf{v}}_{S,i} = 0 . \qquad (25.1.4)$$

Recall that $\vec{\mathbf{F}}_i = d\vec{\mathbf{p}}_i / dt$, therefore Eq. **Error! Reference source not found.** becomes

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right) = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i \right) . \qquad (25.1.5)$$

Because

$$\sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i \right) = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i} = \vec{\boldsymbol{\tau}}_S^{\text{total}} \qquad (25.1.6)$$

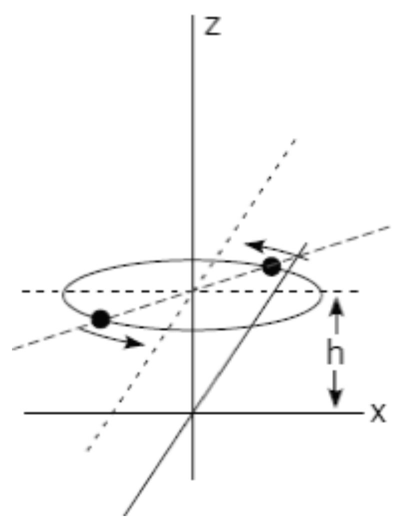
Eq. (25.1.5) becomes

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} . \qquad (25.1.7)$$

Therefore the total torque about the point S is equal to the time derivative of the total angular momentum about the point.

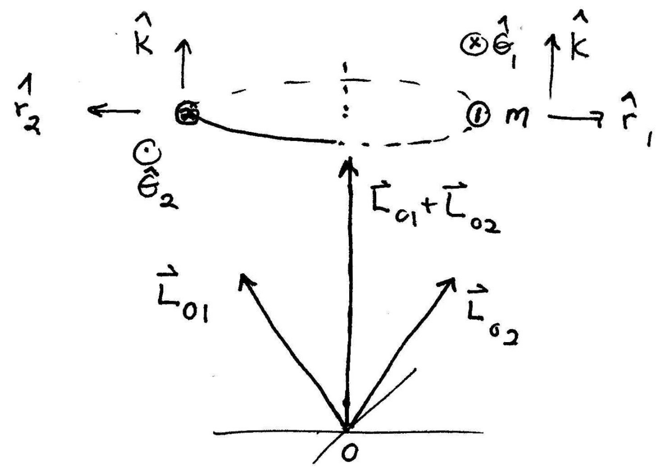
25.1.1 Example

Two identical particles of mass m move in a circle of radius r, 180° out of phase at an angular speed ω about the z axis in a plane parallel to but a distance h above the x-y plane.



Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_0$ relative to the origin.

Solution: The angular momentum about the origin is the sum of the contributions from each object. Since they have the same mass, the angular momentum vectors are shown in the figure below.



The components that lie in the x-y plane cancel leaving only a non-zero z-component so

$$\vec{\mathbf{L}}_0 = \vec{\mathbf{L}}_{0,1} + \vec{\mathbf{L}}_{0,2} = 2mr^2\omega\hat{\mathbf{k}} . \tag{25.1.8}$$

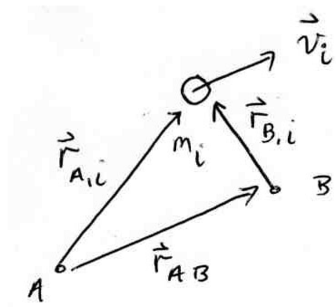
If you explicitly calculate the cross product in polar coordinates you must be careful because the units vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ at the position of objects 1 and 2 are different. If we set $\vec{\mathbf{r}}_{0,1} = r\hat{\mathbf{r}}_1 + h\hat{\mathbf{k}}$ and $\vec{\mathbf{v}}_1 = r\omega\,\hat{\boldsymbol{\theta}}_1$ such that $\hat{\mathbf{r}}_1 \times \hat{\boldsymbol{\theta}}_1 = \hat{\mathbf{k}}$ and similarly set $\vec{\mathbf{r}}_{0,2} = r\hat{\mathbf{r}}_2 + h\hat{\mathbf{k}}$ and $\vec{\mathbf{v}}_2 = r\omega\,\hat{\boldsymbol{\theta}}_2$ such that $\hat{\mathbf{r}}_2 \times \hat{\boldsymbol{\theta}}_2 = \hat{\mathbf{k}}$ then $\hat{\mathbf{r}}_1 = -\hat{\mathbf{r}}_2$ and $\hat{\boldsymbol{\theta}}_1 = -\hat{\boldsymbol{\theta}}_2$. With this in mind we can compute

$$\begin{aligned}\vec{\mathbf{L}}_0 &= \vec{\mathbf{L}}_{0,1} + \vec{\mathbf{L}}_{0,2} = \vec{\mathbf{r}}_{0,1} \times m\vec{\mathbf{v}}_1 + \vec{\mathbf{r}}_{0,2} \times m\vec{\mathbf{v}}_2 \\ &= (r\hat{\mathbf{r}}_1 + h\hat{\mathbf{k}}) \times mr\omega\,\hat{\boldsymbol{\theta}}_1 + (r\hat{\mathbf{r}}_2 + h\hat{\mathbf{k}}) \times mr\omega\,\hat{\boldsymbol{\theta}}_2 \\ &= 2mr^2\omega\hat{\mathbf{k}} + hmr\omega(-\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2) = 2mr^2\omega\hat{\mathbf{k}} + hmr\omega(-\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_1) = 2mr^2\omega\hat{\mathbf{k}}\end{aligned} \quad .(25.1.9)$$

The important point about this example is that the two objects are symmetrically distributed with respect to the z-axis (opposite sides of the circular orbit). Therefore the angular momentum about any point S along the z-axis has the same value $\vec{\mathbf{L}}_S = 2mr^2\omega\hat{\mathbf{k}}$ that is constant in magnitude and points in the +z-direction for the motion shown in the figure above.

Example 25.1.2: Angular Momentum of a System of Particles about Different Points

Consider a system of N particles, and two points A and B.



The angular momentum of the ith particle about the point A is given by

$$\vec{\mathbf{L}}_{A,i} = \vec{\mathbf{r}}_{A,i} \times m_i \vec{\mathbf{v}}_i \,. \tag{1.10}$$

So the angular momentum of the system of particles about the point A is given by the sum

$$\vec{\mathbf{L}}_{A} = \sum_{i=1}^{N} \vec{\mathbf{L}}_{A,i} = \sum_{i=1}^{N} \vec{\mathbf{r}}_{A,i} \times m_i \vec{\mathbf{v}}_i \tag{1.11}$$

The angular momentum about the point B can be calculated in a similar way and is given by

$$\vec{\mathbf{L}}_{B} = \sum_{i=1}^{N} \vec{\mathbf{L}}_{B,i} = \sum_{i=1}^{N} \vec{\mathbf{r}}_{B,i} \times m_i \vec{\mathbf{v}}_i \,. \tag{1.12}$$

From the above figure, the vectors

$$\vec{\mathbf{r}}_{A,i} = \vec{\mathbf{r}}_{B,i} + \vec{\mathbf{r}}_{A,B} \,. \tag{1.13}$$

We can substitute Eq. (1.13) into Eq. (1.11) yielding

$$\vec{\mathbf{L}}_{A} = \sum_{i=1}^{N} (\vec{\mathbf{r}}_{B,i} + \vec{\mathbf{r}}_{A,B}) \times m_i \vec{\mathbf{v}}_i = \sum_{i=1}^{N} \vec{\mathbf{r}}_{B,i} \times m_i \vec{\mathbf{v}}_i + \sum_{i=1}^{N} \vec{\mathbf{r}}_{A,B} \times m_i \vec{\mathbf{v}}_i \,. \tag{1.14}$$

The first term in Eq. (1.14) is the angular momentum about the point B. The vector $\vec{\mathbf{r}}_{A,B}$ is a constant and so can be pulled out of the sum in the second term, and Eq. (1.14) becomes

$$\vec{\mathbf{L}}_{A} = \vec{\mathbf{L}}_{B} + \vec{\mathbf{r}}_{A,B} \times \sum_{i=1}^{N} m_i \vec{\mathbf{v}}_i \tag{1.15}$$

The sum in the second term is the momentum of the system

$$\vec{\mathbf{p}}_{sys} = \sum_{i=1}^{N} m_i \vec{\mathbf{v}}_i \,. \tag{1.16}$$

Therefore the angular momentum about the points A and B are related by

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B + \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{p}}_{sys} \tag{1.17}$$

Thus if the momentum of the system is zero, the angular momentum is the same about any point. In particular, the momentum of a system of particles is zero by definition in the center of mass reference frame. Hence the angular momentum is the same about any point in the center of mass reference frame.

25.2 Angular Momentum and Torque for Fixed Axis of Rotation

We have shown that, for fixed axis rotation, the component of torque that causes the angular velocity to change is the rotational analog of Newton's Second Law,

$$\tau_S^{\text{total}} = I_S \,\alpha = I_S \frac{d\omega}{dt} \,. \tag{25.2.1}$$

We shall now see that this is a special case of the more general result

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d}{dt} \vec{\mathbf{L}}_S^{\text{total}} \,. \tag{25.2.2}$$

Consider a rigid body rotating about a fixed axis passing through the point S .and take the fixed axis of rotation to be the z-axis.

Recall that all the points in the rigid body rotate about the z-axis with the same angular velocity $\vec{\boldsymbol{\omega}} \equiv (d\theta / dt)\hat{\mathbf{k}} \equiv \omega \hat{\mathbf{k}}$. In a similar fashion, all points in the rigid body have the same angular acceleration, $\vec{\boldsymbol{\alpha}} \equiv (d^2\theta / dt^2)\,\hat{\mathbf{k}} \equiv \alpha \hat{\mathbf{k}}$ $\alpha = d^2\theta / dt^2$. The velocity vector $\vec{\mathbf{v}}_i$ for any point in the rigid body is then given by

$$\vec{\mathbf{v}}_i = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_i \tag{25.2.3}$$

The angular momentum is a vector, and will have a component along the direction of the fixed z-axis. Let the point S lie somewhere along the z-axis. As before, the body is divided into individual elements; we calculate the contribution of each element to the angular momentum, and then sum over all the elements. The summation will become an integral for a continuous body.

Each individual element has a mass Δm_i and is moving in a circle of radius $r_{S,\perp,i}$ about the axis of rotation. Let $\vec{\mathbf{r}}_{S,i}$ be the vector from the point S to the element. The velocity of the element, $\vec{\mathbf{v}}_i$, is tangent to this circle (Figure 25.2).

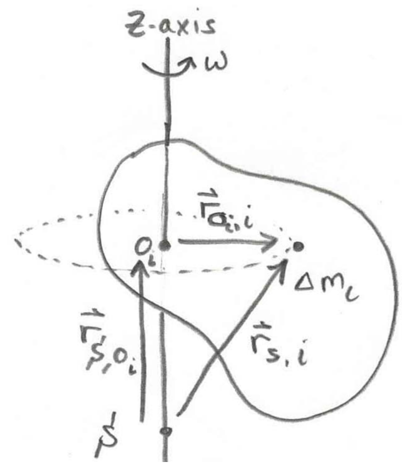


Figure 25.2 Geometry of instantaneous rotation.

The angular momentum of the i_{th} element about the point S is given by

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i = \vec{\mathbf{r}}_{S,i} \times \Delta m_i \, \vec{\mathbf{v}}_i \,. \tag{25.2.4}$$

Let the point O_i denote the center of the circular orbit of the element. Define a vector $\vec{\mathbf{r}}_{O_i,i} = \vec{\mathbf{r}}_{S,\perp,i}$ from the point O_i to the element. Let $\vec{\mathbf{r}}_{S,O_i} = \vec{\mathbf{r}}_{S,\|,i}$ denote the vector from the point S to the point O_i. Note that $\vec{\mathbf{r}}_{S,O_i} = \vec{\mathbf{r}}_{S,\|,i}$ is directed along the *z*-axis, as in Figure 25.2. The three vectors are related by

$$\vec{\mathbf{r}}_{S,i} = \vec{\mathbf{r}}_{S,O_i} + \vec{\mathbf{r}}_{O_i,i} = \vec{\mathbf{r}}_{S,\|,i} + \vec{\mathbf{r}}_{S,\perp,i} \,. \tag{25.2.5}$$

The angular momentum about S is then expressed as

$$\begin{aligned} \vec{\mathbf{L}}_{S,i} &= \vec{\mathbf{r}}_{S,i} \times \Delta m_i \, \vec{\mathbf{v}}_i = (\vec{\mathbf{r}}_{S,\|,i} + \vec{\mathbf{r}}_{S,\perp,i}) \times \Delta m_i \vec{\mathbf{v}}_i \\ &= (\vec{\mathbf{r}}_{S,\|,i} \times \Delta m_i \vec{\mathbf{v}}_i) + (\vec{\mathbf{r}}_{S,\perp,i} \times \Delta m_i \, \vec{\mathbf{v}}_i). \end{aligned} \tag{25.2.6}$$

In the last expression in Equation (25.2.6), the first term has a direction that is perpendicular to the z-axis; the direction of a cross product of two vectors is always perpendicular to the direction of either vector. Since $\vec{\mathbf{r}}_{S,\|,i}$ is in the z-direction, the first in the last expression in Equation (25.2.6) has no component along the z-axis. Therefore the vector component along the z-axis of the angular momentum about the point S, $(\vec{\mathbf{L}}_{S,i})_z \equiv (L_{S,i})_z$, arises entirely from the second term,

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,\perp,i} \times \Delta m_i \, \vec{\mathbf{v}}_i \,. \tag{25.2.7}$$

The vectors $\vec{\mathbf{r}}_{S,\perp,i} = \vec{\mathbf{r}}_{S,O_i}$ and $\vec{\mathbf{v}}_i$ are perpendicular, as shown in Figure 25.3.

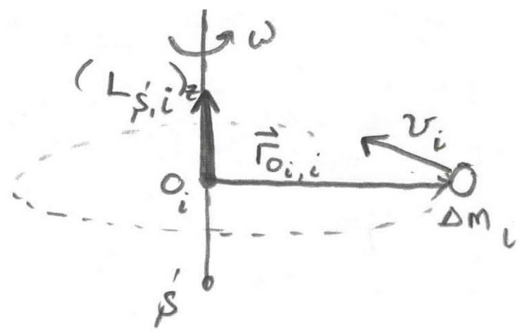


Figure 25.3 The z-component of angular momentum.

Therefore the z-component of the angular momentum about S is just the product of the radius of the circle, $r_{S,\perp,i}$, and the component $\Delta m_i \, v_i$ of the momentum,

$$(L_{S,i})_z = r_{S,\perp,i} \, \Delta m_i \, v_i \,. \tag{25.2.8}$$

For a rigid body, all elements have the same angular velocity, $\omega \equiv d\theta/dt$, and the tangential velocity is

$$v_i = r_{S,\perp,i} \, \omega \,. \tag{25.2.9}$$

The expression in Equation (25.2.8) for the z-component of the momentum about S is

$$(L_{S,i})_z = r_{S,\perp,i} \, \Delta m_i \, v_i = \Delta m_i (r_{S,\perp,i})^2 \omega \,. \tag{25.2.10}$$

The z-component of the total angular momentum about S is the summation over all the elements,

$$(L_S^{\text{total}})_z = \sum_i (L_{S,i})_z = \sum_i \Delta m_i (r_{S,\perp,i})^2 \omega \,. \tag{25.2.11}$$

For a continuous mass distribution the summation becomes an integral over the body,

$$(L_S^{\text{total}})_z = \int_{\text{body}} dm \, (r_\perp)^2 \omega \,, \tag{25.2.12}$$

where $r_\perp$ is the distance form the fixed z-axis to the infinitesimal element of mass dm.

The moment of inertia of a rigid body about a fixed z-axis passing through a point S is given by an integral over the body

$$I_S = \int_{\text{body}} dm\,(r_\perp)^2 . \tag{25.2.13}$$

Thus the z-component of the total angular momentum about S for a fixed axis that passes through S in the z-direction is proportional to the angular velocity, ω,

$$(L_S^{\text{total}})_z = I_S\,\omega . \tag{25.2.14}$$

For fixed-axis rotation, our result that torque about a point is equal to the time derivative of the angular momentum about that point,

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d}{dt}\vec{\mathbf{L}}_S^{\text{total}} , \tag{25.2.15}$$

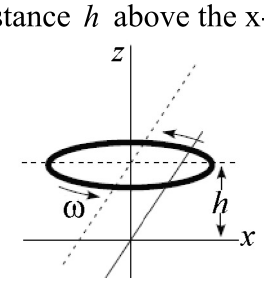
can now be resolved in the z-direction,

$$(\tau_S^{\text{total}})_z = \frac{d}{dt}(L_S^{\text{total}})_z = \frac{d}{dt}(I_S\,\omega) = I_S\frac{d\omega}{dt} = I_S\frac{d^2\theta}{dt^2} = I_S\,\alpha , \tag{25.2.16}$$

in agreement with our earlier result that the z-component of torque about the point S is equal to the product of moment of inertia about I_S, and the magnitude α of the angular acceleration.

25.2.1 Example

A circular ring of radius r, and mass m out of phase at an angular speed ω about the z axis in a plane parallel to but a distance h above the x-y plane.



Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_0$ relative to the origin.

Solution: Use the same symmetry argument as we did in Example 25.1.1 . The ring can be thought of as made up of pairs of point like objects on opposite sides of the ring each of mass Δm. Each pair has a non-zero z-component of the angular momentum taken about any point along the z-axis, $\vec{\mathbf{L}}_{0,pair} = \vec{\mathbf{L}}_{0,1} + \vec{\mathbf{L}}_{0,2} = 2\Delta m r^2 \omega \hat{\mathbf{k}}$. So summing up over all the pairs gives

$$\vec{\mathbf{L}}_S = m r^2 \omega \hat{\mathbf{k}} . \tag{25.2.17}$$

Recall that the moment of inertia of a ring is given by

$$I_S = \int_{\text{body}} dm\,(r_\perp)^2 = m r^2 . \tag{25.2.18}$$

So for the symmetric ring the angular momentum points in the direction of the angular velocity and is equal to

$$\vec{\mathbf{L}}_S = I_S \omega \hat{\mathbf{k}} \tag{25.2.19}$$

25.3 Conservation of Angular Momentum about a Point

Consider a system of particles. We begin with our result that we derived in Section 25.1 that the total torque about a point S is equal to the time derivative of the angular momentum about that point S:

$$\vec{\boldsymbol{\tau}}_S^{\text{ total}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} . \tag{25.3.1}$$

The total torque about the point S is the sum of the external torques and the internal torques

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \vec{\boldsymbol{\tau}}_S^{\text{ext}} + \vec{\boldsymbol{\tau}}_S^{\text{int}} . \tag{25.3.2}$$

The total external torque about the point S is the sum of the torques due to the net external force acting on each element

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i}^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i^{\text{ext}} . \tag{25.3.3}$$

The total internal torque arise from the torques due to the internal forces acting between pairs of elements

$$\vec{\boldsymbol{\tau}}_S^{\text{int}} = \sum_{j=1}^{N} \vec{\boldsymbol{\tau}}_{S,j}^{\text{int}} = \sum_{j=1}^{i=N} \sum_{\substack{i=1 \\ j \neq i}}^{i=N} \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{i,j} . \tag{25.3.4}$$

We know by Newton's Third Law that the internal forces cancel in pairs, $\vec{\mathbf{F}}_{i,j} = -\vec{\mathbf{F}}_{j,i}$, and hence the sum of the internal forces is zero

$$\vec{\mathbf{0}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{i=N} \vec{\mathbf{F}}_{i,j} . \tag{25.3.5}$$

Does the same statement hold about pairs of internal torques? Consider the sum of internal torques arising from the interaction between the ith and jth particles

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{i,j} + \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_{j,i} . \tag{25.3.6}$$

By the Third Law this sum becomes

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{i,j} . \tag{25.3.7}$$

In the Figure 25.4, the vector $\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}$ points from the jth element to the ith element.

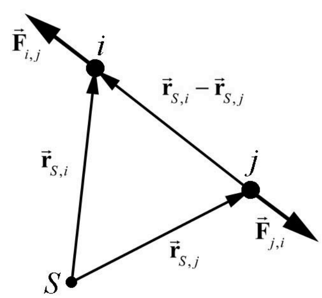


Figure 25.4 Special case when the internal force is along the line connecting to the ith and jth particles

If the internal forces between a pair of particles are directed along the line joining the two particles then the torque due to the internal forces cancel in pairs.

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{i,j} = \vec{\mathbf{0}} . \tag{25.3.8}$$

This is a stronger version of Newton's Third Law than we have so far since we have added the additional requirement regarding the direction of all the internal forces between pairs of particles. With this assumption, the total torque is just due to the external forces

$$\vec{\boldsymbol{\tau}}_{S}^{\text{ext}} = \frac{d\vec{\mathbf{L}}_{S}^{\text{sys}}}{dt} . \tag{25.3.9}$$

However, so far no isolated system has been encountered such that the angular momentum is not constant.

Principle of Conservation of Angular Momentum

If the external torque acting on a system is zero, then the angular momentum of the system is constant. So for any change of state of the system the change in angular momentum is zero

$$\Delta \vec{\mathbf{L}}_S^{\text{sys}} \equiv (\vec{\mathbf{L}}_S^{\text{sys}})_f - (\vec{\mathbf{L}}_S^{\text{sys}})_0 = \vec{\mathbf{0}}. \quad (25.3.10)$$

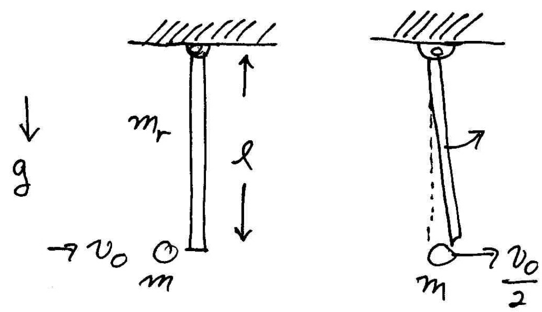
Equivalently the angular momentum is constant

$$(\vec{\mathbf{L}}_S^{\text{sys}})_f = (\vec{\mathbf{L}}_S^{\text{sys}})_0. \quad (25.3.11)$$

So far no isolated system has been encountered such that the angular momentum is not constant.

25.3.1 Example Collision

An object of mass m and speed v_0 strikes a rigid uniform rod of length l and mass m_r that is hanging by a frictionless pivot from the ceiling. Immediately after striking the rod, the object continues forward but its speed decreases to $v_0/2$. The moment of inertia of the rod about its center of mass is $I_{cm} = (1/12)m_r l^2$. Gravity acts with acceleration g downward.



a) For what value of v_0 will the rod just touch the ceiling on its first swing? You may express your answer in terms of g, m_r, m, and l.

b) For what ratio m_r / m will the collision be elastic?

Solution:

We begin by identifying our system which consists of the object and the uniform rod. We identify as well three states:

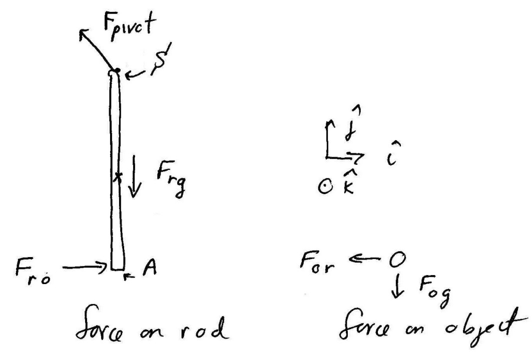
State 1: Immediately before the collision
State 2: Immediately after the collision.
State 3: The instant the rod touches the ceiling when the final angular speed is zero.

and two state changes:

State 1 $\rightarrow$ State 2
State 2 $\rightarrow$ State 3

We would like to know if any of our fundamental quantities, momentum, energy, angular momentum are constant during these state changes.

We start with State 1 $\rightarrow$ State 2. The pivot force holding the rod to the ceiling is an external force acting at the pivot point S. There is also the gravitational force acting on at the center of mass of the rod and on the object. There are also internal forces due to the collision of the rod and the object at point A.



The external force means that momentum is not constant. However the external force is fixed and so does no work. However, we do not know whether the collision is elastic or not so we cannot assume that mechanical energy is constant. If we choose the pivot point S as the point in which to calculate torque, then the torque about the pivot is

$$\vec{\boldsymbol{\tau}}_S^{\text{sys}} = \vec{\mathbf{r}}_{S,S} \times \vec{\mathbf{F}}_{pivot} + \vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{o,r} + \vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{r,o} + \vec{\mathbf{r}}_{S,cm} \times \vec{\mathbf{F}}_{r,g}$$

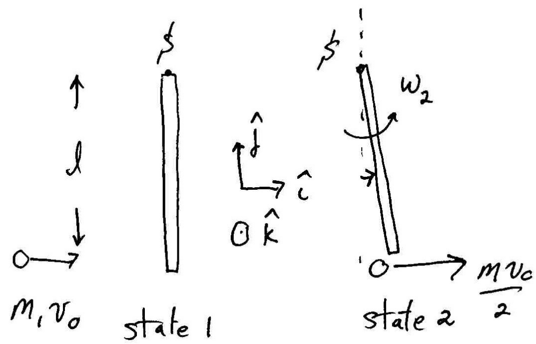
The external pivot force does not contribute any torque because $\vec{\mathbf{r}}_{S,S} = \vec{\mathbf{0}}$. The internal forces between the rod and the object are equal in magnitude and opposite in direction, $\vec{\mathbf{F}}_{o,r} = -\vec{\mathbf{F}}_{r,o}$ (Newton's Third Law), and so their contributions to the torque add to zero, $\vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{o,r} + \vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{r,o} = \vec{\mathbf{0}}$. If the collision is instantaneous then the gravitational force is parallel to $\vec{\mathbf{r}}_{S,cm}$ and so $\vec{\mathbf{r}}_{S,cm} \times \vec{\mathbf{F}}_{r,g} = \vec{\mathbf{0}}$. Therefore the torque on the system about the pivot point is zero,

$$\vec{\boldsymbol{\tau}}_S^{\text{sys}} = \vec{\mathbf{0}}.$$

Thus the angular momentum about the pivot point is constant,

$$(\vec{\mathbf{L}}_S^{\text{sys}})_1 = (\vec{\mathbf{L}}_S^{\text{sys}})_2.$$

In order to calculate the angular momentum we draw a diagram showing the momentum of the object and the angular speed of the rod in the figure below.



The angular momentum about S immediately before the collision is

$$(\vec{\mathbf{L}}_S^{\text{sys}})_1 = \vec{\mathbf{r}}_{S,0} \times m_1\vec{\mathbf{v}}_0 = l(-\hat{\mathbf{j}}) \times m_1 v_0 \hat{\mathbf{i}} = l m_1 v_0 \hat{\mathbf{k}} .$$

The angular momentum about S immediately after the collision is

$$(\vec{\mathbf{L}}_S^{\text{sys}})_2 = \vec{\mathbf{r}}_{S,0} \times m_1\vec{\mathbf{v}}_0 / 2 + I_s\vec{\boldsymbol{\omega}}_2 = l(-\hat{\mathbf{j}}) \times m_1(v_0 / 2)\hat{\mathbf{i}} = \frac{l m_1 v_0}{2}\hat{\mathbf{k}} + I_s\omega_2\hat{\mathbf{k}} .$$

Therefore the condition that the angular momentum about S is constant during the collision becomes

$$l m_1 v_0 \hat{\mathbf{k}} = \frac{l m_1 v_0}{2}\hat{\mathbf{k}} + I_s \omega_2 \hat{\mathbf{k}} .$$

We can solve for the angular speed immediately after the collision

$$\omega_2 = \frac{l m_1 v_0}{2 I_s} .$$

By the parallel axis theorem the moment of inertial of a uniform rod about the pivot point is

$$I_S = m(l/2)^2 + I_{cm} = (1/4)m_r l^2 + (1/12)m_r l^2 = (1/3)m_r l^2 .$$

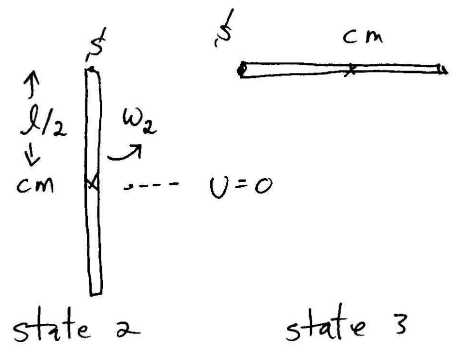
Therefore the angular speed immediately after the collision is

$$\omega_2 = \frac{3 m_1 v_0}{2 m_r l} .$$

For the transition State 2 $\rightarrow$ State 3, we know that the gravitational force is conservative and the pivot force does no work so mechanical energy is constant.

$$E_{mech,2} = E_{mech,3}$$

We draw an energy diagram with a choice of zero for the potential energy at the center of mass. We only show the rod because the object undergoes no energy transformation during the transition State 2 $\rightarrow$ State 3.



The mechanical energy immediately after the collision is

$$E_{mech,2} = \frac{1}{2} I_S \omega_2^2 + \frac{1}{2} m_1 (v_0 / 2)^2 .$$

Using our results for the moment of inertia I_S and ω_2, we have that

$$E_{mech,2} = \frac{1}{2}(1/3) m_r l^2 \left(\frac{3m_1 v_0}{2m_r l} \right)^2 + \frac{1}{2} m_1 (v_0 / 2)^2 = \frac{3{m_1}^2{v_0}^2}{8m_r} + \frac{1}{2} m_1 (v_0 / 2)^2 .$$

The mechanical energy when the rod just reaches the ceiling when the final angular speed is zero is then

$$E_{mech,3} = m_r g(l/2) + \frac{1}{2} m_1 (v_0 / 2)^2 .$$

Then the condition that the mechanical energy is constant becomes

$$\frac{3{m_1}^2{v_0}^2}{8m_r} + \frac{1}{2} m_1 (v_0 / 2)^2 = m_r g(l/2) + \frac{1}{2} m_1 (v_0 / 2)^2 .$$

We can now solve this equation for the initial speed of the object

$$v_0 = \frac{m_r}{m_1}\sqrt{\frac{4gl}{3}} \, .$$

We now return to the transition State 1 $\rightarrow$ State 2 and determine the constraint on the mass ratio in order for the collision to be elastic. The mechanical energy before the collision is

$$E_{mech,1} = \frac{1}{2} m_1 v_0^{\,2} \, .$$

If we impose the condition that the collision is elastic then

$$E_{mech,1} = E_{mech,2} \, .$$

Using our result above we have that

$$\frac{1}{2} m_1 v_0^{\,2} = \frac{3 m_1^{\,2} v_0^{\,2}}{8 m_r} + \frac{1}{2} m_1 (v_0 / 2)^2 \, .$$

This simplifies to

$$\frac{3}{8} m_1 v_0^{\,2} = \frac{3 m_1^{\,2} v_0^{\,2}}{8 m_r}$$

Hence we can solve for the mass ratio necessary to insure that the collision is elastic if the final speed of the object is half it's initial speed

$$\frac{m_r}{m_1} = 1 \, .$$

Notice that the result in independent of the initial speed of the object.

25.4 External Impulse and Change in Angular Momentum

Define the *external impulse* about a point S applied as the integral of the external torque about S

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}} \, dt \, . \qquad (25.4.1)$$

Then the external impulse about S is equal to the change in angular momentum

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}} dt = \int_{t_0}^{t_f} \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} dt = (\vec{\mathbf{L}}_S^{\text{sys}})_f - (\vec{\mathbf{L}}_S^{\text{sys}})_0 . \tag{25.4.2}$$

Notice that this is the rotation analog to our statement about impulse and momentum,

$$\vec{\mathbf{I}}_S^{\text{ext}} \equiv \int_{t_0}^{t_f} \vec{\mathbf{F}}^{\text{ext}} dt = \int_{t_0}^{t_f} \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt} dt = (\vec{\mathbf{p}}^{\text{sys}})_f - (\vec{\mathbf{p}}^{\text{sys}})_0 . \tag{25.4.3}$$

25.4.1 Example: A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular speed ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down until it reaches an angular speed of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\text{int}} = t_b - t_a$. Assume the frictional torque on the axle is independent of speed, and remains the same when the second washer is dropped. The two washers continue to slow down during the time interval $\Delta t_2 = t_f - t_b$ until they stop at time $t = t_f$.

a) What is the angular acceleration while the washer and motor are slowing down during the interval $\Delta t_1 = t_a$?

b) Suppose the collision is nearly instantaneous, $\Delta t_{\text{int}} = (t_b - t_a) \simeq 0$. What is the angular speed ω_b of the two washers immediately after the collision is finished (when the washers rotate together)?

Now suppose the collision is not instantaneous but that the frictional torque is independent of the speed of the rotor.

c) What is the angular impulse during the collision?

d) What is the angular velocity ω_b of the two washers immediately after the collision is finished (when the washers rotate together)?

e) What is the angular deceleration α_2 after the collision?

Solution:

a) The angular acceleration of the motor and washer from the instant when the power is shut off until the second washer was dropped is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{\Delta t_1} < 0. \qquad (25.4.4)$$

b) If the collision is nearly instantaneous, then there is no angular impulse and therefore the z-component of the angular momentum about the rotation axis of the motor remains constant

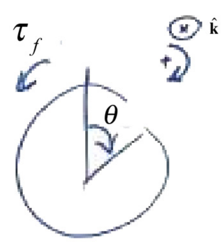
$$0 = \Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_{\mathrm{w}})\omega_b - I_0\omega_a. \qquad (25.4.5)$$

We can solve Eq. (25.4.5) for the angular speed ω_b of the two washers immediately after the collision is finished

$$\omega_b = \frac{I_0}{I_0 + I_{\mathrm{w}}}\omega_a. \qquad (25.4.6)$$

c) The angular acceleration found in part a) is due to the frictional torque in the motor. Let $\vec{\boldsymbol{\tau}}_f = -\tau_f \hat{\mathbf{k}}$ where τ_f is the magnitude of the frictional torque then

$$-\tau_f = I_0\alpha_1 = \frac{I_0(\omega_a - \omega_0)}{\Delta t_1}. \qquad (25.4.7)$$



During the collision with the second washer, the frictional torque exerts an angular impulse (pointing along the z-axis in the figure),

$$J_z = -\int_{t_a}^{t_b} \tau_f \, dt = -\tau_f \, \Delta t_{\mathrm{int}} = I_0(\omega_a - \omega_0)\frac{\Delta t_{\mathrm{int}}}{\Delta t_1}. \qquad (25.4.8)$$

c) The z-component of the angular momentum about the rotation axis of the motor changes during the collision,

$$\Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_\mathrm{w})\omega_b - I_0\omega_a \tag{25.4.9}$$

The change in the z-component of the angular momentum is equal to the z-component of the angular impulse

$$J_z = \Delta L_z \,. \tag{25.4.10}$$

Thus, equating the expressions in Equations (25.4.8) and (25.4.9),

$$I_0(\omega_a - \omega_0)\left(\frac{\Delta t_\mathrm{int}}{\Delta t_1}\right) = (I_0 + I_\mathrm{w})\omega_b - (I_0)\omega_a \,. \tag{25.4.11}$$

Solving Equation (25.4.11) for the angular velocity immediately after the collision,

$$\omega_b = \frac{I_0}{(I_0 + I_\mathrm{w})}\left((\omega_a - \omega_0)\left(\frac{\Delta t_\mathrm{int}}{\Delta t_1}\right) + \omega_a\right). \tag{25.4.12}$$

If there were no frictional torque, then the first term in the brackets would vanish, and the second term of Equation would be the only contribution to the final angular speed.

d) The final angular acceleration α_2 is given by

$$\alpha_2 = \frac{0 - \omega_b}{\Delta t_2} = -\frac{I_0}{(I_0 + I_\mathrm{w})\Delta t_2}\left((\omega_a - \omega_0)\left(\frac{\Delta t_\mathrm{int}}{\Delta t_1}\right) + \omega_a\right). \tag{25.4.13}$$