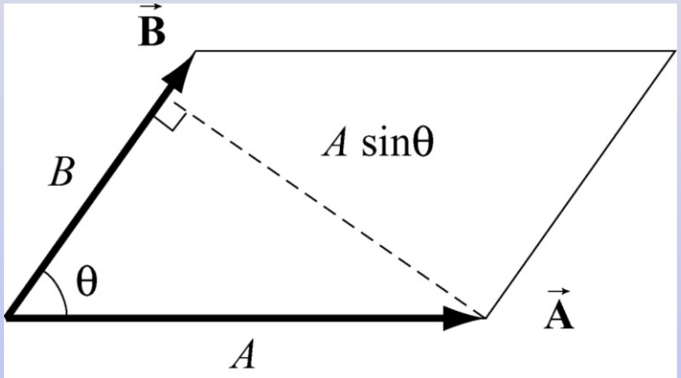
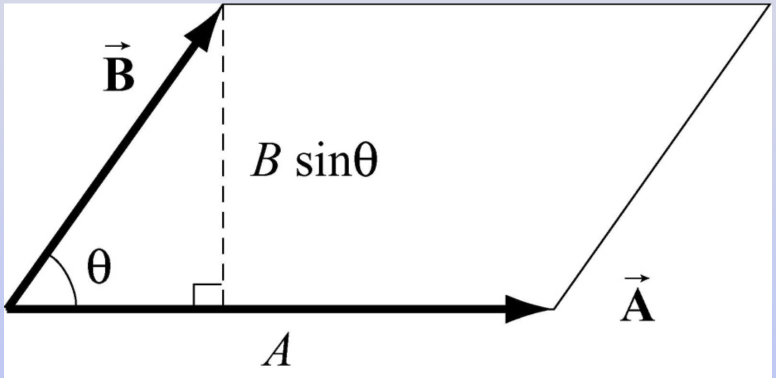
Two-Dimensional Rotational Kinematics: Angular Momentum

Review: Cross Product

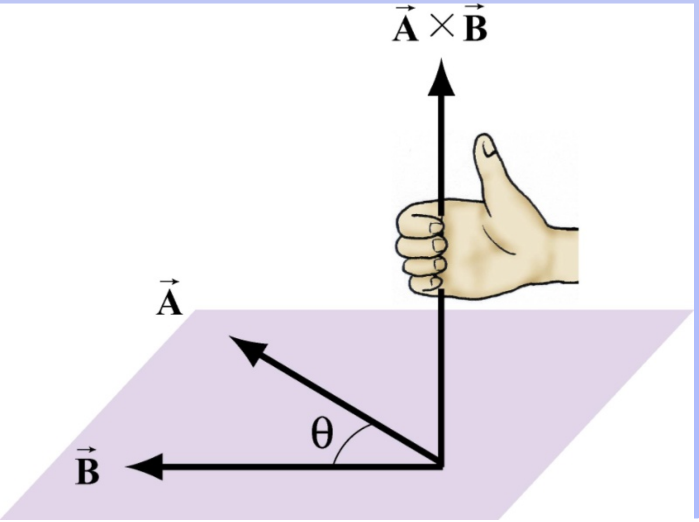
Magnitude: equal to the area of the parallelogram defined by the two vectors

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\theta = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right) = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right| \quad (0 \le \theta \le \pi)$$





Direction: determined by the Right-Hand-Rule



Angular Momentum of a Point Particle

- Point particle of mass *m* moving with a velocity $\vec{\mathbf{v}}$

- Momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$

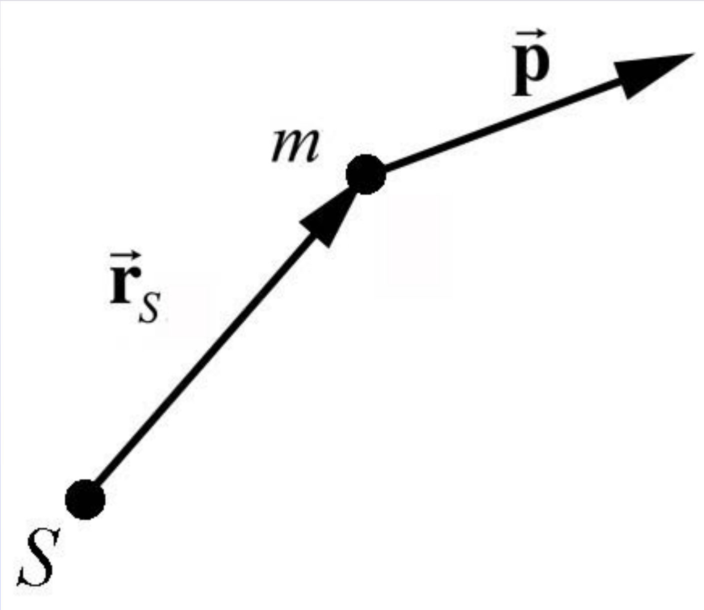
- Fix a point S

- Vector $\vec{\mathbf{r}}_S$ from the point S to the location of the object

- Angular momentum about the point S

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}$$

- SI Unit $[\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}]$



Cross Product: Angular Momentum of a Point Particle

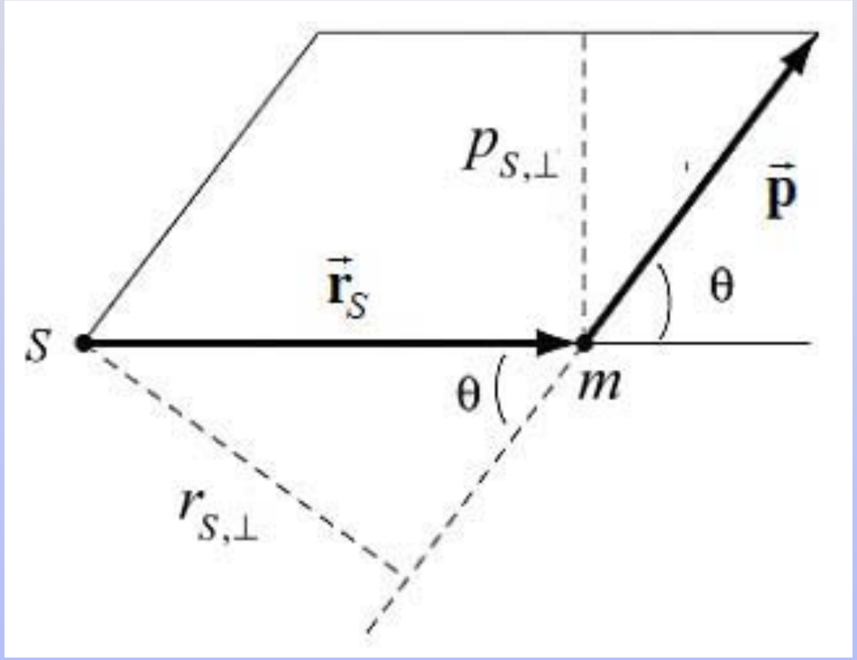
$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}$$

Magnitude:

$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right|\left|\vec{\mathbf{p}}\right|\sin\theta$$

a) moment arm

$$r_{S,\perp} = \left|\vec{\mathbf{r}}_S\right|\sin\theta$$

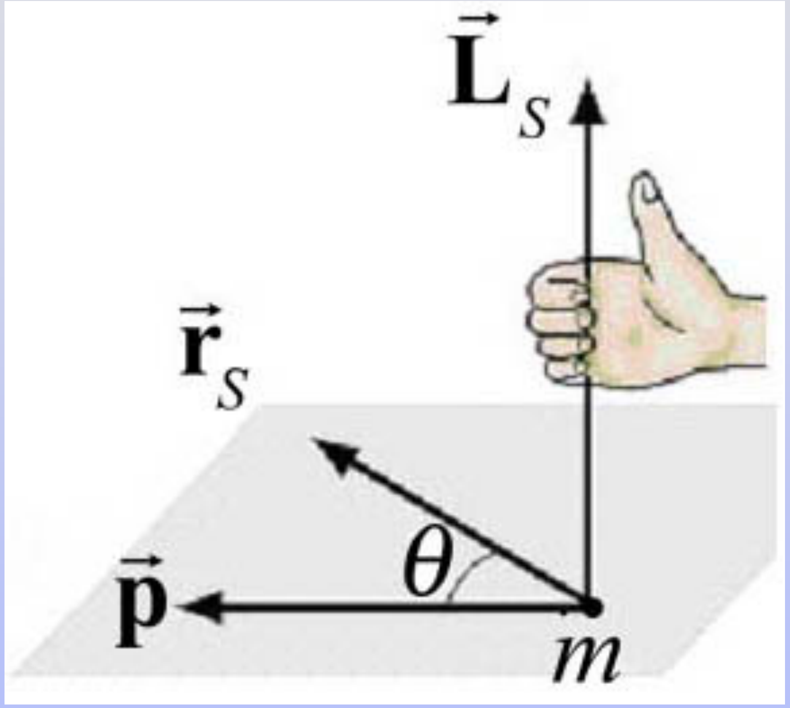


b) Perpendicular momentum

$$\left|\vec{\mathbf{L}}_S\right| = r_{S,\perp}\left|\vec{\mathbf{p}}\right|$$

$$p_{S,\perp} = \left|\vec{\mathbf{p}}\right|\sin\theta \qquad \left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right| p_\perp$$

Angular Momentum of a Point Particle: Direction



Direction: Right Hand Rule

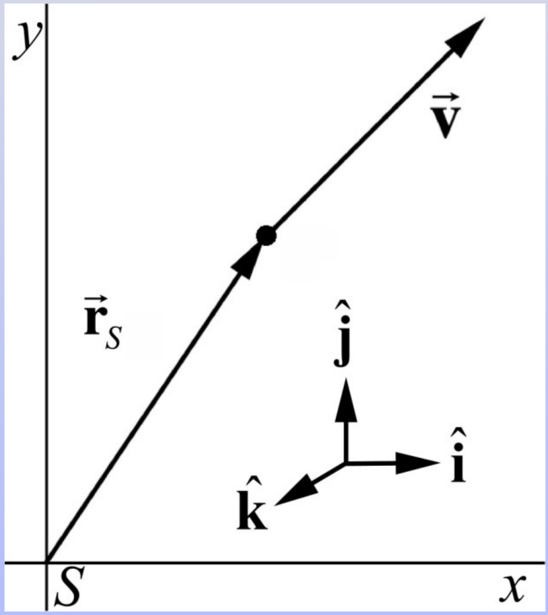
Worked Example: Angular Momentum and Cross Product

A particle of mass $m = 2\ \mathrm{kg}$ moves with a uniform velocity

$$\vec{\mathbf{v}} = 3.0\ \mathrm{m} \cdot \mathrm{s}^{-1}\ \hat{\mathbf{i}} + 3.0\ \mathrm{m} \cdot \mathrm{s}^{-1}\ \hat{\mathbf{j}}$$

At time t, the position vector of the particle with respect ot the point S is

$$\vec{\mathbf{r}}_S = 2.0\ \mathrm{m}\ \hat{\mathbf{i}} + 3.0\ \mathrm{m}\ \hat{\mathbf{j}}$$

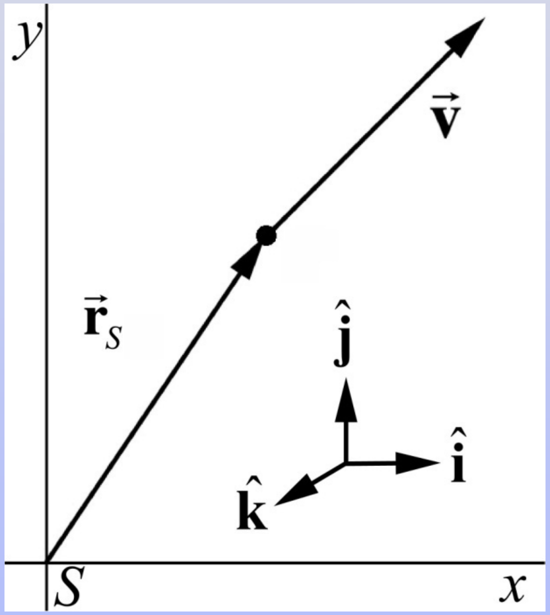


Find the direction and the magnitude of the angular momentum about the origin, (the point S) at time t.

Solution: Angular Momentum and Cross Product

The angular momentum vector of the particle about the point S is given by:

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times m\vec{\mathbf{v}}$$
$$= (2.0\,\mathrm{m}\,\hat{\mathbf{i}} + 3.0\,\mathrm{m}\,\hat{\mathbf{j}}) \times (2\,\mathrm{kg})(3.0\,\mathrm{m}\cdot\mathrm{s}^{-1}\hat{\mathbf{i}} + 3.0\,\mathrm{m}\cdot\mathrm{s}^{-1}\hat{\mathbf{j}})$$
$$= 12\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}\,\hat{\mathbf{k}} + 18\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}(-\hat{\mathbf{k}})$$
$$= -6.0\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}\,\hat{\mathbf{k}}.$$



The direction is in the negative $\hat{\mathbf{k}}$ direction, and the magnitude is

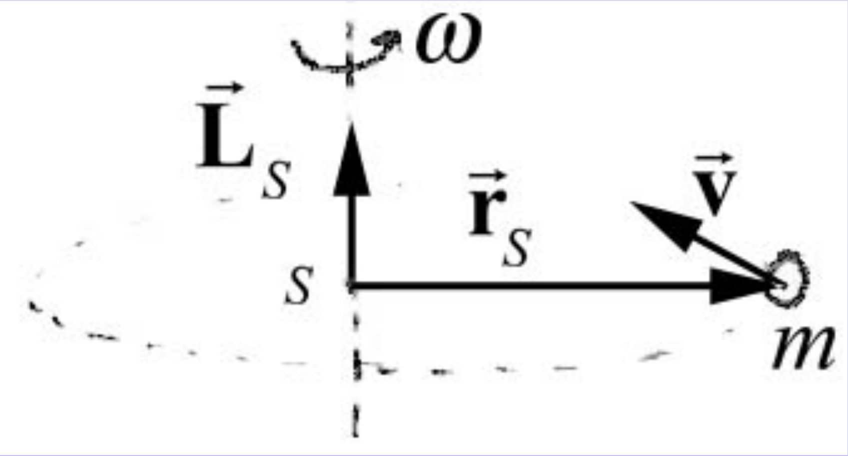
$$\left|\vec{\mathbf{L}}_S\right| = 6.0\ \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}.$$

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}},$$
$$\vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}},$$
$$\vec{\mathbf{i}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \times \vec{\mathbf{j}} = \vec{\mathbf{0}}$$

Angular Momentum and Circular Motion of a Point Particle:

Fixed axis of rotation: z-axis

Angular velocity $\vec{\boldsymbol{\omega}} \equiv \omega\,\hat{\mathbf{k}}$



Velocity

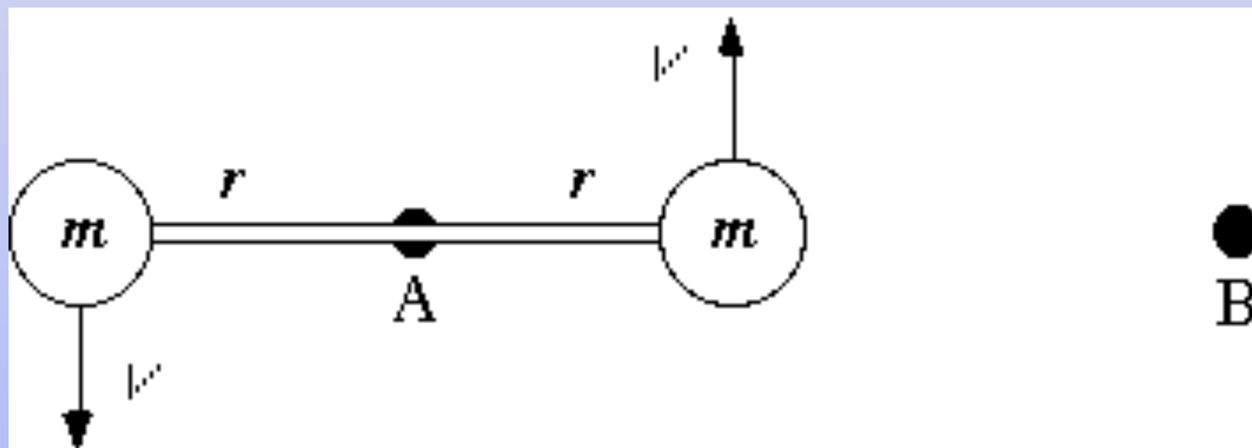
$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = \omega\,\hat{\mathbf{k}} \times R\,\hat{\mathbf{r}} = R\omega\,\hat{\boldsymbol{\theta}}$$

Angular momentum about the point S

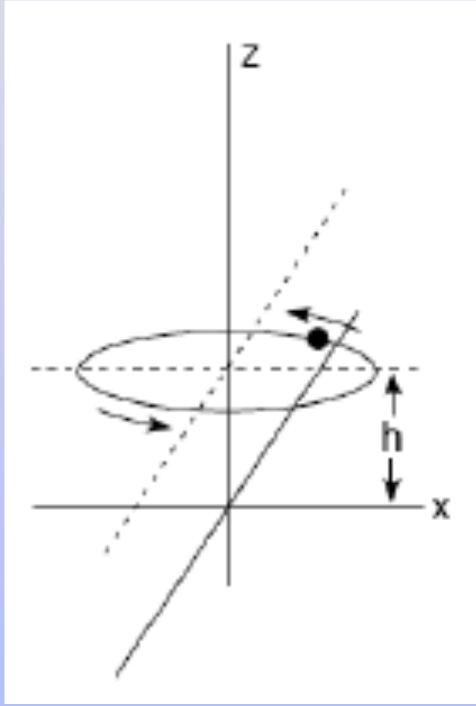
$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times m\vec{\mathbf{v}} = Rmv\,\hat{\mathbf{k}} = RmR\omega\,\hat{\mathbf{k}} = mR^2\omega\,\hat{\mathbf{k}}$$

Checkpoint Problem: angular momentum of dumbbell

A dumbbell is rotating at a constant angular speed about its center (point *A*). How does the angular momentum about the point *B* compared to the angular momentum about point *A*, (as shown in the figure)?



Checkpoint Problem: angular momentum of a single particle

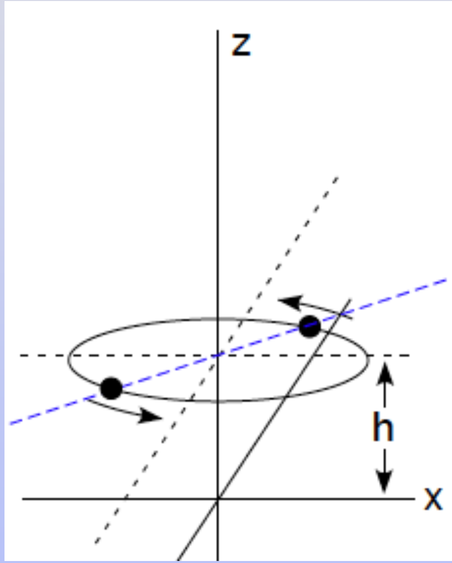


A particle of mass m moves in a circle of radius R at an angular speed ω about the z axis in a plane parallel to but a distance h above the x-y plane.

a) Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_0$ relative to the origin.

b) Is this angular momentum relative to the origin constant? If yes, why? If no, why is it not constant?

Checkpoint Problem: angular momentum of two particles

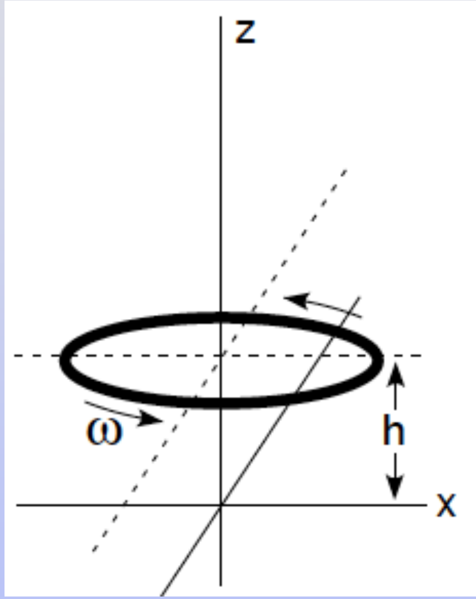


Two identical particles of mass m move in a circle of radius R, 180º out of phase at an angular speed ω about the z axis in a plane parallel to but a distance h above the x-y plane.

a) Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_0$ relative to the origin.

b) Is this angular momentum relative to the origin constant? If yes, why? If no, why is it not constant?

Checkpoint Problem: angular momentum of a ring



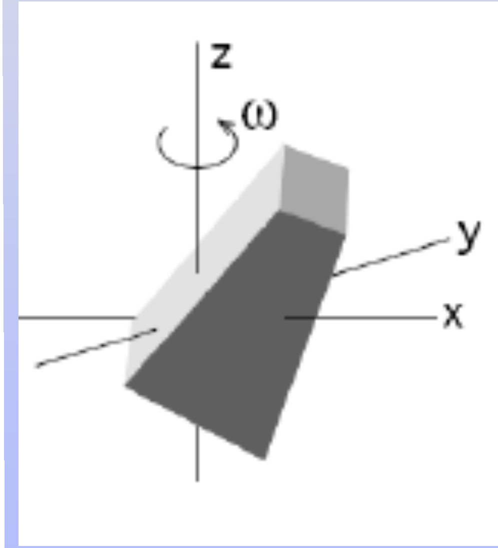
A circular ring of radius R and mass dm rotates at an angular speed ω about the z-axis in a plane parallel to but a distance h above the x-y plane.

a) Find the magnitude and the direction of the angular momentum $\vec{L}_0$ relative to the origin.

b) Is this angular momentum relative to the origin constant? If yes, why? If no, why is it not constant?

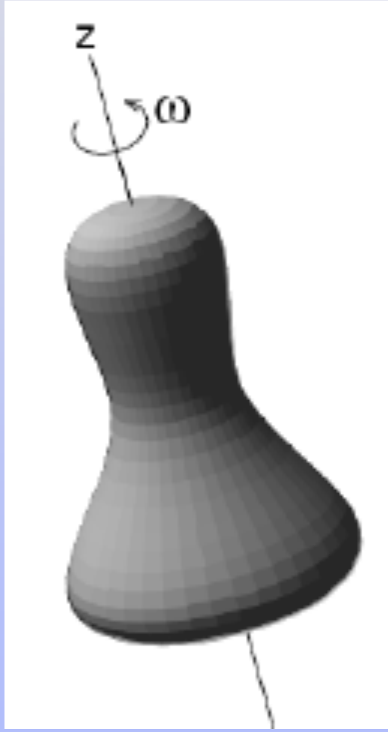
Checkpoint Problem: Angular momentum of non-symmetric body

A non-symmetric body rotates with constant angular speed ω about the z axis. Relative to the origin



1. $\vec{\mathbf{L}}_0$ is constant.
2. $\left|\vec{\mathbf{L}}_0\right|$ is constant but $\vec{\mathbf{L}}_0 / \left|\vec{\mathbf{L}}_0\right|$ is not.
3. $\vec{\mathbf{L}}_0 / \left|\vec{\mathbf{L}}_0\right|$ is constant but $\left|\vec{\mathbf{L}}_0\right|$ is not.
4. $\vec{\mathbf{L}}_0$ has no z-component.

Checkpoint Problem: Angular momentum of symmetric body



A rigid body with rotational symmetry body rotates at a constant angular speed ω about it symmetry (z-axis). In this case

1. $\vec{\mathbf{L}}_0$ is constant.

2. $\left|\vec{\mathbf{L}}_0\right|$ is constant but $\vec{\mathbf{L}}_0 / \left|\vec{\mathbf{L}}_0\right|$ is not.

3. $\vec{\mathbf{L}}_0 / \left|\vec{\mathbf{L}}_0\right|$ s constant but $\left|\vec{\mathbf{L}}_0\right|$ is not.

4. $\vec{\mathbf{L}}_0$ has no z-component.

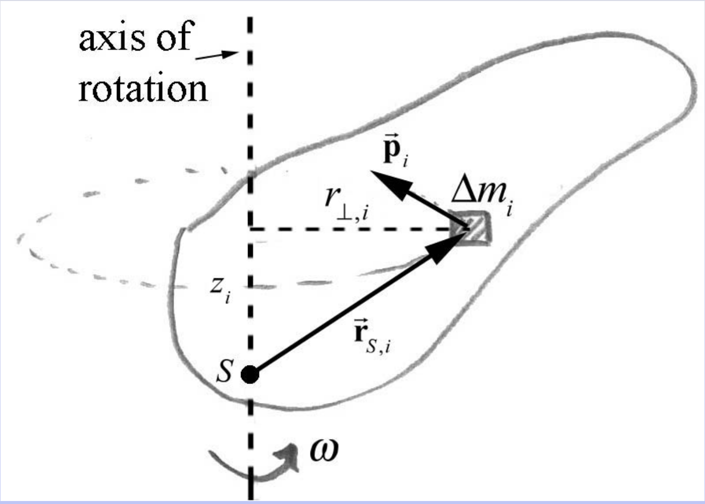
5. Two of the above are true.

Angular Momentum for Fixed Axis Rotation

Angular Momentum about the point S

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i = (r_{\perp,i}\ \hat{\mathbf{r}} + z_i\ \hat{\mathbf{k}}) \times p_{\tan,i}\ \hat{\boldsymbol{\theta}}$$

$$\vec{\mathbf{L}}_{S,i} = r_{\perp,i}\ p_{\tan,i}\ \hat{\mathbf{k}} - z_i\ p_{\tan,i}\hat{\mathbf{r}}$$



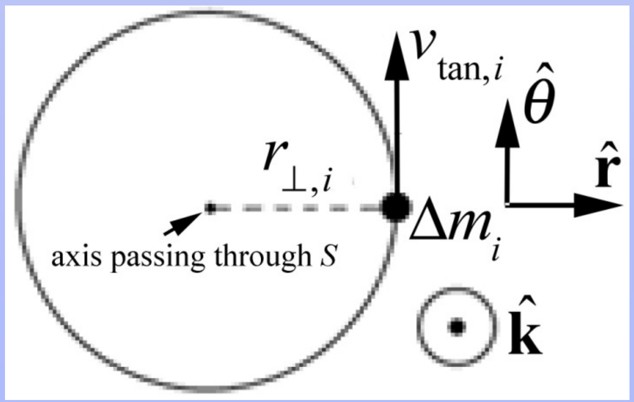
Tangential component of momentum

$$p_{\tan,i} = \Delta m_i v_{\tan,i} = \Delta m_i r_{\perp,i}\omega$$

z-component of angular momentum about S:

$$L_{S,z,i} = r_{\perp,i} p_{\tan,i} = r_{\perp,i}\Delta m_i r_{\perp,i}\omega = \Delta m_i r_{\perp,i}{}^2\omega$$

$$L_{S,z} = \sum_{i=1}^{i=N} L_{S,z,i} = \sum_{i=1}^{i=N} \Delta m_i r_{\perp,i}{}^2\omega = I_S\omega$$



Checkpoint Problem: angular momentum of disk about point on the rim

A disk with mass M and radius R is spinning with angular velocity ω about an axis that passes through the rim of the disk perpendicular to its plane. Find the angular momentum about the point where the rotation axis intersects the disk.