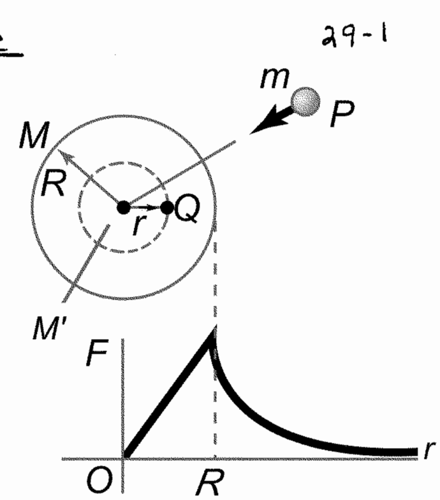
Potential Energy of a Sphere and Particle



The force on a particle when it is outside a uniform solid sphere is given by GMm/r^2 and is directed toward the center. The force on the particle when it is inside such a sphere is proportional to r and goes to zero at the center.

Consider a particle of mass m which is located at a radius r from the center of a homogeneous solid sphere of mass M and radius R.

Case I: $r \geq R$

- Solid sphere is made up of many shells summed to maximum radius R.
- Since force and potential energy due to a shell depend only on mass of shell and distance to particle m from center;

$$F = -\frac{GmM}{r^2}\hat{r} \qquad (r \geq R)$$

$$U = -\frac{GmM}{r} \qquad (r \geq R)$$

Case II: $r \leq R$

- From our result on spherical shells, the force on a particle is due entirely to all the mass between r=0 and r=r. No force due to the mass between r=r and R.

$$\therefore \; F = \frac{Gm\mathcal{M}(r)}{r^2} \qquad (r \leq R)$$

$\mathcal{M}(r) \equiv$ amount of mass in a sphere of radius r.
$M \equiv$ total mass of a sphere of radius R.

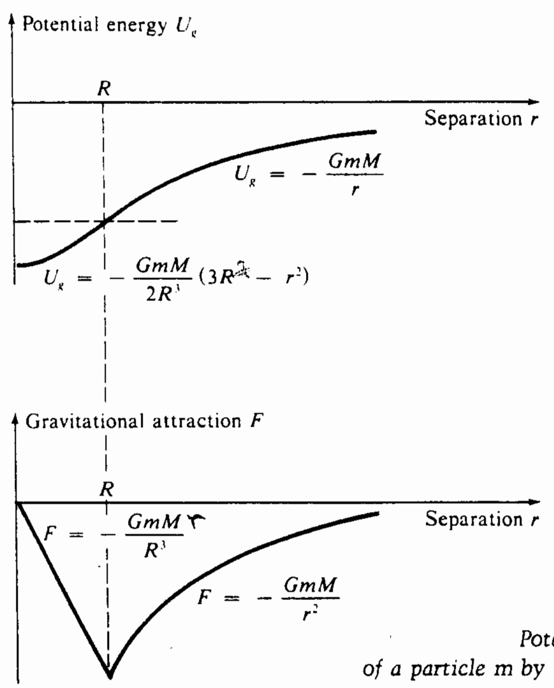
$$\therefore \mathcal{M}(r) = M\left(\frac{r}{R}\right)^3 \qquad \text{(proportional to volume)}$$

$$\therefore F = -\frac{GmMr}{R^3}\,\hat{r} \qquad (r \leq R)$$

The potential energy in the region $r \leq R$ is given by our usual definition:

$$u(r) = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r} = \int_{\infty}^{R} \frac{GmM\,dr'}{r'^2} + \int_{R}^{r} \frac{GmMr'\,dr'}{R^3}$$

$$= -\frac{GmM}{2R^3}\left[3R^2 - r^2\right] \qquad (r \leq R)$$



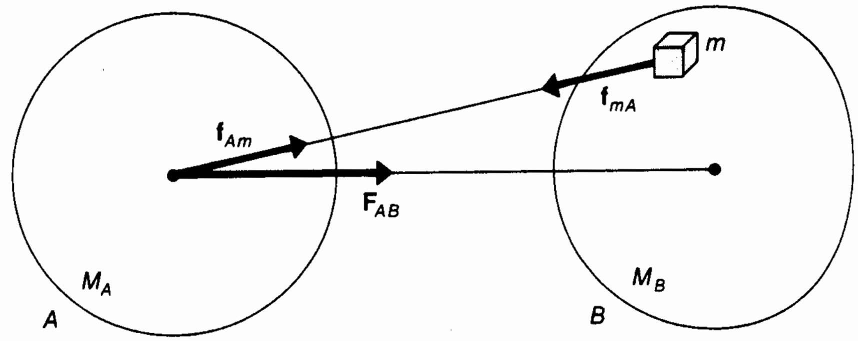
(a) U_g as a function of r

(b) F as a function of r

Potential energy and gravitational attraction of a particle m by a solid sphere of mass M and radius R

Gravitational Force between Spherically Symmetric Objects

The gravitational force F_{AB} on sphere A due to sphere B acts on the C.M. of A



Two spherically symmetric objects attract each other with a force

$$\vec{F}_{AB} = \frac{G\, m_A m_B}{r^2}$$

r = separation between centers.

Each elementary mass of object B acts on the cm of object A (see point mass + spherical shell). Summing over all elements dm of B the resultant force on A is at cm. By symmetry arguments the force on B is also at its cm. By Newton's 3rd Law they are equal and opposite

If object B does not have spherical symmetry the force on A still acts at the cm of A. The force on B however does not act at the cm of B!!

Gravitational Forces on Extended Objects

The gravitational force due to the earth acting on a particle of mass m_i is

$$\vec{f}_i = m_i \vec{g}$$

$\vec{g} \equiv$ acceleration due to gravity.

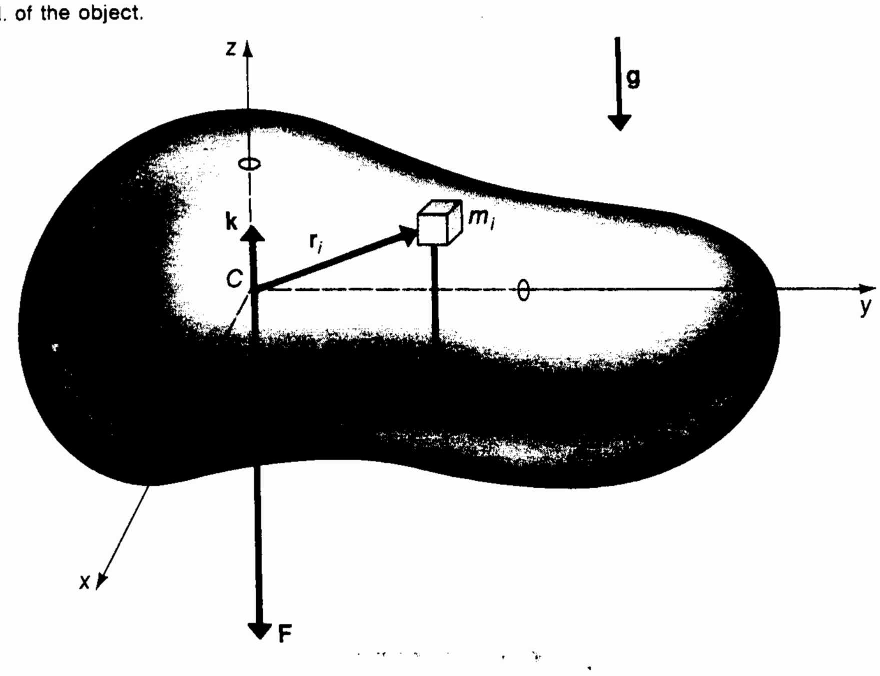
For relatively small objects if $\vec{g}$ can be assumed to be constant over the dimensions of the object;

$$\vec{F} = \Sigma \vec{f}_i = \sum_i m_i \vec{g} = (\Sigma m_i)\vec{g} = M\vec{g}$$

M = total mass of object.

Q: At what point on the object does this single effective force act?

Fig. The individual particles m_i comprising an object are acted upon by the gravitational forces $m_i\mathbf{g}$. The sum of all these individual forces is equivalent to a single force that acts at C, the C.M. of the object.



Calculate the torques about the cm of object resulting from all the forces $\vec{f}_i$.

$$\vec{\tau}_{cm} = \sum \vec{r}_i \times \vec{f}_i = \sum \vec{r}_i \times m_i \vec{g} = \underbrace{\left(\sum m_i \vec{r}_i\right)}_{\vec{R}_{cm} \equiv 0} \times \vec{g}$$

$$\therefore \vec{\tau}_{cm} \equiv 0$$

The single force $\vec{F}$ must also produce no torques about the cm if it reproduces the effects of all the individual forces.

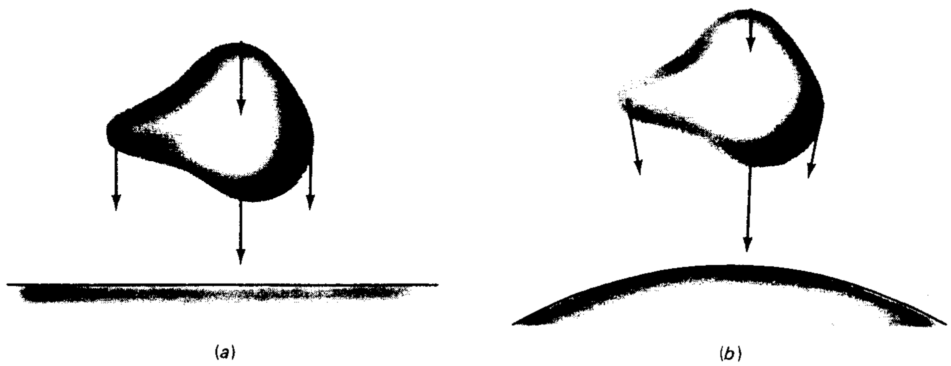
$\therefore \vec{F}$ acts exactly on the cm of the object.

This point is also called the 'center-of gravity' of the object.

If the object is so large that $\vec{g}$ varies over the extent of the object, the center-of-gravity cannot be identified except for spherical objects only !!

Tidal Forces

- when gravitational field is uniform all parts of a freely falling object experience same accel.
- If the field is nonuniform, acceleration varies from place-to-place.
- Result → differential force
  - object may be stretched or compressed.



(a) (b)

Fig.
(a) In a uniform gravitational field, all parts of an object experience the same gravitational acceleration. (b) When the field is not uniform, gravity acts differently on the different parts of the object. The result is a differential force that tends to stretch an object along the field and compress it at right angles to the field.

- Differential forces → Tidal Forces — 'ocean tides'.

Consider two objects of mass m separated a distance 2a located at distance r from mass M.

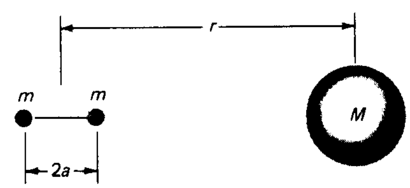


Fig.
The tidal force on an object arises from the difference in gravitational field strength across the object.

'Tidal force' on the object = difference between the gravitational force on either end:

$$F_T = \frac{GMm}{(r-a)^2} - \frac{GMm}{(r+a)^2} = GMm\,\frac{(r+a)^2-(r-a)^2}{(r-a)^2(r+a)^2}$$

$$F_T = \frac{4GMma}{r^3} \qquad \text{(tidal force)}$$

$a \ll r$

- Falls off as $1/r^3$
- Moon more important than massive sun.

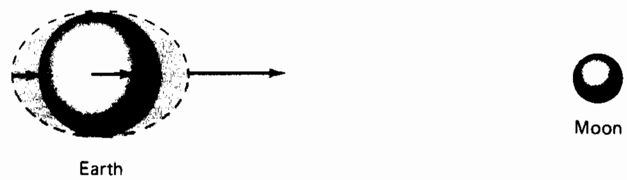


Fig.
Ocean tides result primarily from the differential force of the moon's gravity. The gravitational force is strongest on the moonward ocean, weaker on the solid earth, and weakest on the distant ocean. Thus the water on the moonward side is pulled away from the solid earth, while earth itself is pulled away from the distant water. Two tidal bulges result, so that a given location experiences two high tides each day as earth rotates. (Not shown is the differential force associated with the sun's gravity, which is weaker but also important.)

Weight and Gravitational Force

Near the surface of the earth the weight of a body of mass m is

$$w = mg$$

We can equate this to the force acting on the freely falling body,

$$mg = \frac{G M_E m}{R_E^2}$$

$$\therefore \quad g = \frac{G M_E}{R_E^2} \qquad \text{[definition of } g\text{]}$$

Suppose object m is at a height h above the earth's surface or a distance r from the center:

$$r = R_E + h$$

If the body is in free fall, its acceleration $= g'$

$$\therefore \quad mg' = \frac{G M_E m}{r^2} = \frac{G M_E m}{(R_E + h)^2}$$

$$\therefore \quad g' = \frac{G M_E}{r^2} = \frac{G M_E}{(R_E + h)^2}$$

g' decreases with increasing altitude. $g' \to 0$ as $h \to \infty$

Consider PE of particle at radius r,

$$U(r) = -\frac{G M_E m}{r}$$

PE of particle at surface of earth, $r = R_E$

$$U(R_E) = -\frac{G M_E m}{R_E}$$

change in PE

$$\Delta U = U(r) - U(R_E) = -G M_E m \left[\frac{1}{r} - \frac{1}{R_E}\right]$$

$$= G M_E m \left(\frac{r - R_E}{r R_E}\right)$$

$$\left.\begin{array}{ll} \text{Let} & r - R_E = z \\ \text{and} & r R_E \sim R_E^2 \end{array}\right\} \text{ for } z \ll R_E$$

$$\Delta U = \frac{G M_E m}{R_E^2} z$$

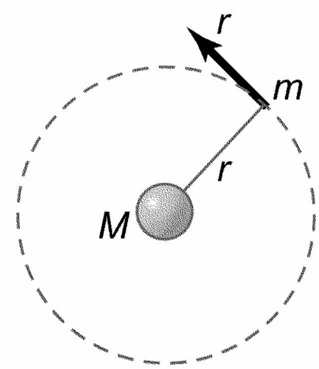
$$\Delta U = m g z$$

change in PE of a particle due to a change in elevation z near earth's surface.

Planetary Motion

Case I : Circular Orbits

- A body of mass m is moving in a circular orbit of radius r_0 about a massive object with mass M (sun).



A body of mass *m* moving in a circular orbit about a body of mass *M*.

Force acting on the planet is

$$F = \frac{GMm}{r_0^2}$$

If the speed of the planet is v, then its centrepetal acceleration is

$$a_c = \frac{v^2}{r_0}$$

$$\therefore \quad \frac{GMm}{r_0^2} = \frac{mv^2}{r_0}$$

or $\quad v^2 = \dfrac{GM}{r_0}$ [radius of orbit fixes speed]

Period for 1-revolution :

$$T = \text{Circumference}/\text{Speed}$$

$$= \frac{2\pi r_0}{v}$$

$$\therefore \quad \frac{GM}{r_0} = \left(\frac{2\pi r_0}{T}\right)^2$$

or $$T^2 = \frac{4\pi^2}{GM} r_0^3 \qquad \text{(Kepler's Law)}$$

- Planetary orbits are nearly circular and this simple analysis is quite good.

- Same analysis applies to the circular motion of the moon or artificial satellite around a planet. In this case the sun's mass is replaced by the mass of the planet.

- Motion dominated by two body (e.g Sun + Planet) interaction. Perturbative effect of other planets is small.

Circular Orbits:

$$K = \frac{1}{2} m v^2 = \frac{1}{2} \frac{GMm}{r_0} \qquad \text{[Kinetic Energy]}$$

$$U = -\frac{GMm}{r_0} \qquad \text{[Potential Energy]}$$

Total Energy:

$$E = K + U$$

$$= \frac{1}{2} \frac{GMm}{r_0} - \frac{GMm}{r_0}$$

$$E = -\frac{1}{2} \frac{GMm}{r_0} \qquad \text{[Total E is negative]}$$

Angular Momentum:

$$K = \frac{L^2}{2I} \qquad I = m r_0^2$$

$$L^2 = 2IK = 2(m r_0^2) \frac{1}{2} \frac{GMm}{r_0}$$

$$L^2 = GMm^2 r_0$$

$$r_0 = \frac{L^2}{GMm^2}$$

Case II : Elliptical Orbits

Kepler's 1'st Law: The orbits of planets are ellipses, with the sun at one focal point.

Sun $\longleftrightarrow$ planets
Earth $\longleftrightarrow$ satellites
M $\qquad$ m
$m \ll M$.

Consider the planet at one of the turning points: r_p or r_a

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$ [Total Energy = constant]

$$L = mvr$$ [Total Angular Momentum = const]

$$\therefore \frac{L^2}{2mr^2} - \frac{GMm}{r} = E$$

$$r^2 + \frac{GMm}{E}r - \frac{L^2}{2mE} = 0$$

$$r = \frac{-\frac{GMm}{E} \pm \sqrt{\left(\frac{GMm}{E}\right)^2 + \frac{4L^2}{2mE}}}{2}$$

$$r_a = r_{max} = (\quad) + \sqrt{\quad}$$
$$r_p = r_{min} = (\quad) - \sqrt{\quad}$$

$$A = r_a + r_p = -\frac{GMm}{E}$$

Length of major axis is independent of L. Orbits with same major axis have the same total energy.

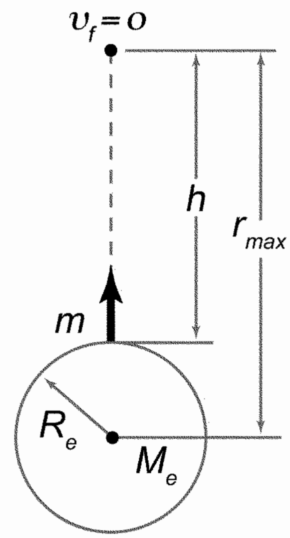
can show that,

$$\epsilon = \sqrt{1 + \frac{2EL^2}{m(GMm)^2}}$$

Example

An object is projected upward with an initial speed v_i. How high does the object go?

(Analysis assumes stat. earth)



An object of mass m projected upward from the earth's surface with an initial speed v_f reaches a maximum altitude h (where $M_e > m$).

Energy is conserved:

$$K_i + \mathcal{U}_i = K_f + \mathcal{U}_f$$

$$\frac{1}{2} m v_i^2 - \frac{G M_E m}{r_i} = \frac{1}{2} m v_f^2 - \frac{G M_E m}{r_f}$$

At $r_f = r_{max}$, $v_f = 0$. $r_i = R_E$ (earth radius)

$$\therefore \; v_i^2 = 2 G M_E \left(\frac{1}{R_E} - \frac{1}{r_{max}} \right)$$

$$h = r_{max} - R_E .$$

What is the minimum speed for object to escape earth completely?

This corresponds reaching $r_{max} = \infty$ with $v_f = 0$.

$$\therefore \; v_{esc} = \sqrt{\frac{2 G M_E}{R_E}}$$

$$= \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{6.37 \times 10^{6}}}$$

$$= 1.12 \times 10^{4} \text{ m/s} \quad \text{or } 25{,}000 \text{ mi/h}$$

Escape Velocity

Object of mass m in the gravitational field of a large body of mass M.

Total Energy:

$$E = K + U$$
$$= \frac{1}{2}mv^2 - \frac{GMm}{r} \qquad (M \gg m)$$

Assume object is launched from surface of M at radius R with a velocity v such that its KE is zero at $r = \infty$.

$$E_i = E_f$$

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 + 0$$

$$\therefore v = \sqrt{\frac{2GM}{R}} \qquad \text{(escape velocity)}$$

$$\left.\begin{array}{l} M = M_{\odot} \\ R = R_{\odot} \end{array}\right\} \quad v_{\odot} = 618 \text{ km/s}$$

$$\left.\begin{array}{l} M = M_E \\ R = R_E \end{array}\right\} \quad v_E = 11.2 \text{ km/s}$$

General Planetary Motion

Two-Particle Systems

- Consider two interacting particles only.

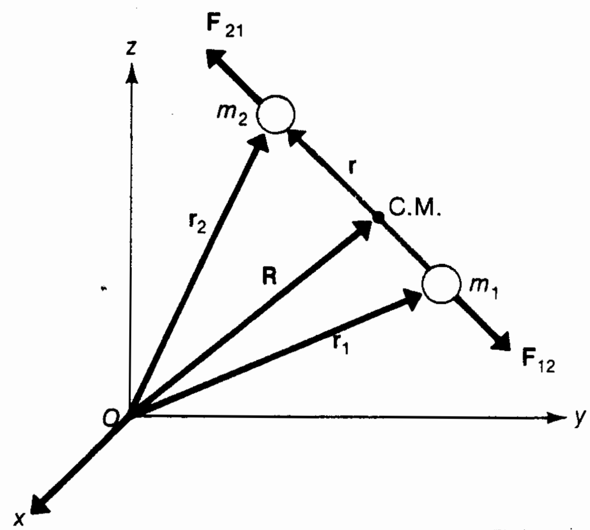


Fig. A pair of interacting particles with masses m_1 and m_2. Two equivalent coordinate representations are shown—that consisting of the individual coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ in the inertial frame O, and that consisting of the relative coordinate $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and the center-of-mass coordinate $\mathbf{R}$. The central forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, are shown as repulsive forces; they could equally well be attractive forces.

Two particles of mass m_1 and m_2 have position vectors $\vec{r}_1$ and $\vec{r}_2$ in an inertial frame O. A central force acts between the particles. $\vec{F}_{12}$ is the force on m_1 due to m_2.

$$\left.\begin{aligned} &(1) \quad \vec{F}_{12} = m_1 \frac{d^2\vec{r}_1}{dt^2} \\ &(2) \quad \vec{F}_{21} = m_2 \frac{d^2\vec{r}_2}{dt^2} \end{aligned}\right\} \text{describes motions of } m_1 \text{ and } m_2.$$

We want to look at the motion using cm coordinates for the particles.

$\vec{R} \equiv$ Position vector of cm system relative to O.
$\vec{r} \equiv$ relative coordinate of the two particles.

(3) $$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1+m_2} \qquad M\vec{R} = m_1\vec{r_1} + m_2\vec{r_2}$$

(4) $$\vec{r} = \vec{r_2} - \vec{r_1}$$

solving $$\vec{r_1} = \vec{R} - \frac{m_2}{M}\vec{r} \qquad (5)$$

$$\vec{r_2} = \vec{R} + \frac{m_1}{M}\vec{r} \qquad (6)$$

where $$M = m_1 + m_2$$

$$\vec{F_{12}} = -\vec{F_{21}} \qquad \text{Newton's 3rd Law}$$

(1) + (2) $$M\frac{d^2\vec{R}}{dt^2} = 0 \implies \frac{d\vec{R}}{dt} = \vec{V}_{cm} = \text{constant !!}$$

cm motion described by $\vec{R}(t)$ takes place at constant velocity and is independent of the interaction between particles.

$m_2 \times$ Eq (1) $- m_1 \times$ Eq (2) and using Eq. (5) and (6)

$$m_1 m_2 \frac{d^2\vec{r}}{dt^2} = M\vec{F_{21}}$$

Let $$\mu = \frac{m_1 m_2}{m_1+m_2}$$ Reduced mass of 2-particle system

$$\therefore \mu\frac{d^2\vec{r}}{dt^2} = \vec{F_{21}} \qquad (7)$$

The relative coordinate $\vec{r}(t)$ of the two particle system has the same behavior as the coordinate of a single particle of mass μ moving in a force field $\vec{F}_{21}(\vec{r})$, which is the mutual or internal force between the actual pair of particles.

Kinetic Energy and Momentum

The total kinetic energy in the inertial frame O can be expressed in terms of $\vec{V} = d\vec{R}/dt$, the velocity of the cm and $\vec{v} = d\vec{r}/dt$ the relative velocity of the two particles.

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_1^2 = \frac{d\vec{r}_1}{dt} \cdot \frac{d\vec{r}_1}{dt} \qquad v_2^2 = \frac{d\vec{r}_2}{dt} \cdot \frac{d\vec{r}_2}{dt}$$

Using Eq's 5 and 6 for $\vec{r}_1$ and $\vec{r}_2$

$$K = \frac{1}{2} M \left(\frac{d\vec{R}}{dt}\right) \cdot \left(\frac{d\vec{R}}{dt}\right) + \frac{1}{2} \mu \left(\frac{d\vec{r}}{dt}\right) \cdot \left(\frac{d\vec{r}}{dt}\right)$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2$$

relative velocity

cm velocity.

Total KE in frame O is the sum of the KE of a hypothetical particle of mass $M = m_1 + m_2$, moving with velocity V of cm plus the KE of a particle of mass μ moving with the relative velocity v.

Total Linear Momentum in frame O

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = M\vec{V}$$

'Relative' linear momenta of the pair of particles is

$$\mu \vec{v} = \frac{m_1 m_2}{M} (\vec{v}_2 - \vec{v}_1)$$

$$= \frac{1}{M} (m_1 \vec{p}_2 - m_2 \vec{p}_1)$$

If the system of particles is isolated (no external forces)

$$\vec{P} = \text{constant}.$$

Total $\vec{P}$ in cm frame $= 0$.