Dynamics of a Rigid Body

Dynamics of a particle

$$\vec{F} = m\vec{a} \quad \text{(Newton's 2'ND Law)}$$

Dynamics of a rigid body

$$\vec{F} \longleftrightarrow \alpha \quad \text{(Angular acceleration)}$$

$\downarrow$

Torques

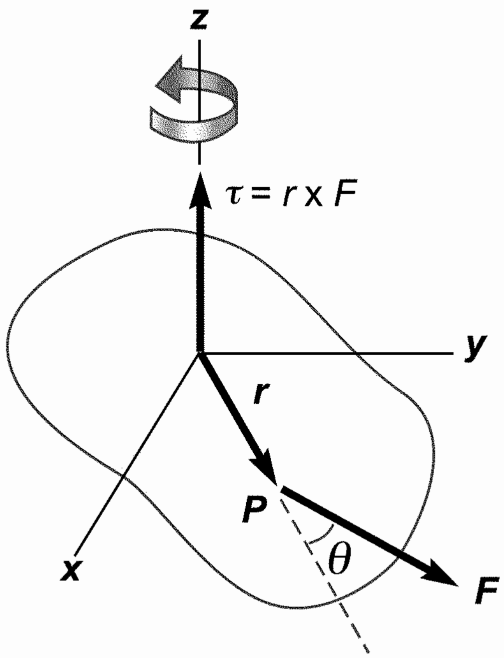
- When a force is properly applied to a rigid body pivoted about an axis, the body will rotate
- The tendency of a body to rotate due to a force is measured by a quantity called a torque.

Consider a force acting on a particle having a position vector $\vec{r}$. The torque with respect to the origin as a reference axis is:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

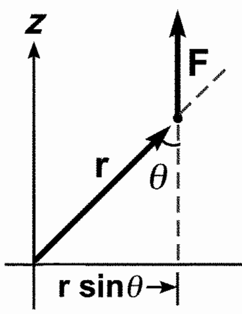
$$\tau = rF\sin\theta$$

$\vec{\tau}$ lies in a direction which is $\perp$ to the plane defined by $\vec{r}$ and $\vec{F}$.



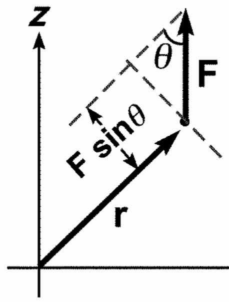
The torque vector τ lies in a direction perpendicular to the plane formed by the position vector r and the applied force F.

(a)



(a) Here r sinθ is the perpendicular distance between the line of action of the force and the origin of coordinates.

(b)



(b) Here *F* sinθ is the component of F perpendicular to r.

a) $d = r\sin\theta$

Moment arm (or lever arm) of force. It is the perpindicular distance from the rotation axis to the line-of-action of $\vec{F}$.

Torque (or Moment) = Moment arm x Force.

$= |r_\perp||F|$

b) Represent $\vec{F}$ in terms of components:

$F_t = F_\perp$, component perpindicular to $\vec{r}$

$F_t = F\sin\theta$

$F_{//} = F\cos\theta$, component parallel to $\vec{r}$

Torque = Radial distance x Transverse Force

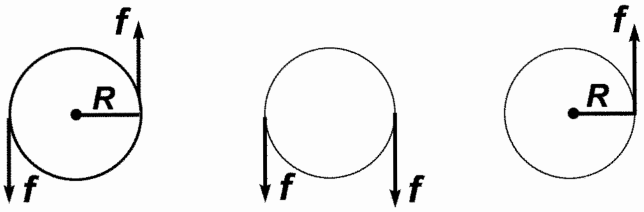
$= |r||F_t|$

$F_{//}$ has no contribution to torque or rotations.

$r_{//}$ has no contribution to torque or rotations.

$[\tau] = N \cdot m \qquad = \cancel{J}$ (not used)

Torque and Force



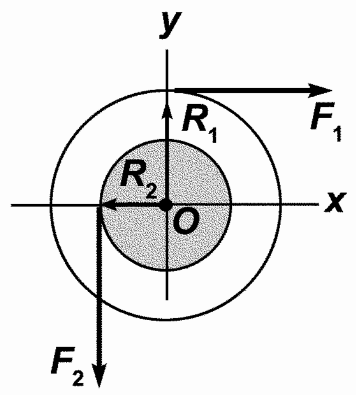
$\tau = 2Rf$	$\tau = 0$	$\tau = Rf$
$F = 0$	$F = 2f$	$F = f$

$$\vec{F} = m\vec{a}$$

$\vec{a} \equiv 0$	$a \neq 0$	$a \neq 0$
$\alpha \neq 0$	No Rotation	$\alpha \neq 0$
	$\alpha = 0$	

Example : Net Torque on a Cylinder

Forces $\vec{F_1}$ at radius $\vec{R_1}$ and $\vec{F_2}$ at radius $\vec{R_2}$ exert torques on the cylinder by means of a rope wrapped around the cylinder.



A solid cylinder is pivoted about the *z* axis through *O*. The moment arm of F_1 is R_1, and the moment arm of F_2 is R_2

Force $\vec{F_1}$ has a moment arm R_1 and exerts a clockwise torque or moment. Call this negative

$$\tau_1 = -F_1 R_1$$

Torque due to F_2 is counterclockwise (positive).

$$\tau_2 = +F_2 R_2$$

Net torque

$$\tau_{Net} = \tau_1 + \tau_2 = -F_1 R_1 + F_2 R_2 .$$

If $F_1 = 5N$ $R_1 = 1.0m$
$F_2 = 6N$ $R_2 = 0.5m$.

$$\tau_{Net} = -(5)\times(1) + (6)\times(0.5)$$

$$= -2\,N\cdot m.$$

Torque is negative, cylinder will rotate clockwise

Angular Momentum and Torque

For a single particle,

$$\vec{L} = \vec{r} \times \vec{p}$$

For a system of particles,

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

How does $\vec{L}$ change with time?

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

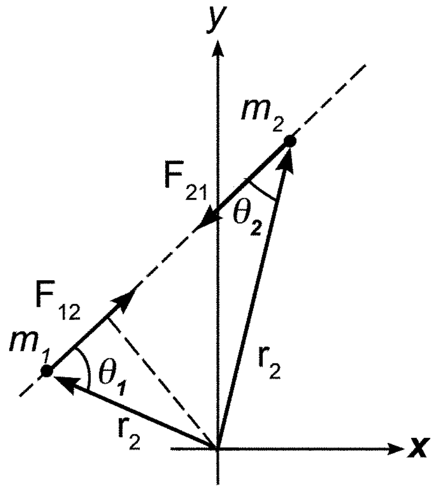
$$= \underbrace{\sum_i \vec{v}_i \times m\vec{v}_i}_{\equiv 0} + \sum \vec{r}_i \times \vec{F}_i$$

$$\therefore \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times F_i = \sum \vec{\tau}_i$$

· Time rate of change of $\vec{L}$ is equal to the applied torque.

Internal Forces - Isolated System

- 3'rd Law: action-reaction pairs.
- Equal and opposite
- Assume forces lie along line joining particles



The forces that two particles exert on one another are of equal magnitude and of opposite direction. Furthermore, we assume that the forces act along the line joining the particles.

$\vec{r}_1 \times \vec{F}_{12} = F_{12} r_1 \sin\theta_1$ (mag)

$\vec{r}_2 \times \vec{F}_{21} = F_{21} r_2 \sin\theta_2$ (mag)

$r_1 \sin\theta_1 = r_2 \sin\theta_2$
[⊥ dist. from origin to line-of-action of forces]

∴ Torques cancel internally!!

Summary:

$\frac{d\vec{L}}{dt} = \sum \vec{r}_i \times \vec{F}_{i,ext} = \sum \tilde{\tau}_{i,ext}$

$\frac{d\vec{L}}{dt} = \vec{\tilde{\tau}}_{ext}$

Use common origin either in an inertial frame <u>or</u> at the CM.

If $\sum \tilde{\tau}_{ext} \equiv 0$

$\vec{L} \equiv$ Constant

Law of Conservation of Angular Momentum.

Torque on a Conical Pendulum

Origin at A:

Forces on bob:

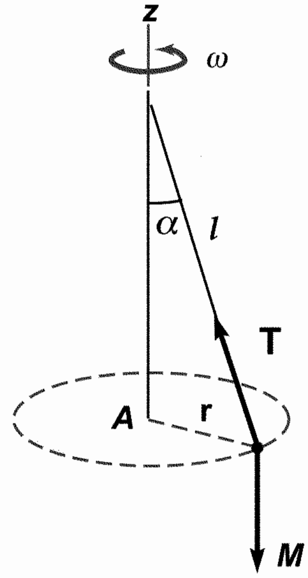
$T\cos\alpha - Mg = 0$

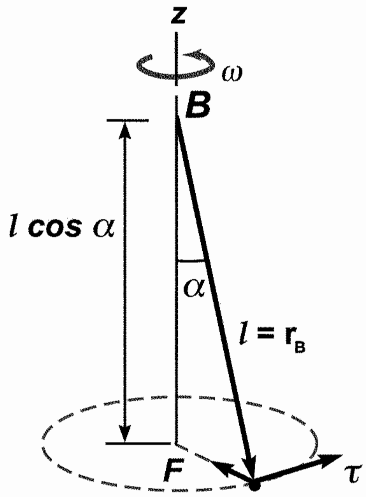
$\vec{F} = -T\sin\alpha\,\hat{r}$

$\vec{\tau}_A = \vec{r}_A \times \vec{F} \equiv 0$

since $\vec{r}_A \parallel \vec{F}$

$\therefore \dfrac{d\vec{L}_A}{dt} = 0 \qquad \vec{L}_A = \text{constant}$





Origin at B:

$\vec{\tau}_B = \vec{r}_B \times \vec{F}$

$|\tau_B| = L\cos\alpha\, F = L\cos\alpha\, T\sin\alpha$

$= MgL\sin\alpha$

$\hookrightarrow T\cos\alpha = Mg.$

$\vec{\tau}_B$ is tangent to line of motion of M

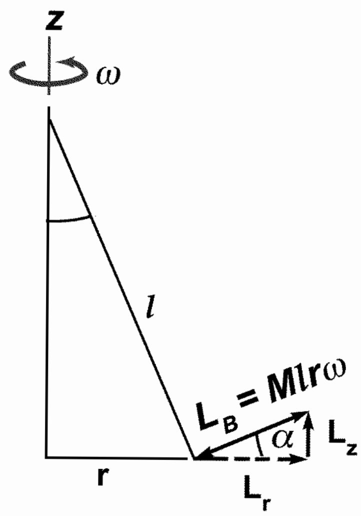
$\vec{\tau}_B = MgL\sin\alpha\,\hat{\theta}$ $\qquad \hat{\theta}$: tangential unit vector

Want to show that,

$\vec{\tau}_B = \dfrac{d\vec{L}_B}{dt}$

$$|L_B| = MLr\omega$$

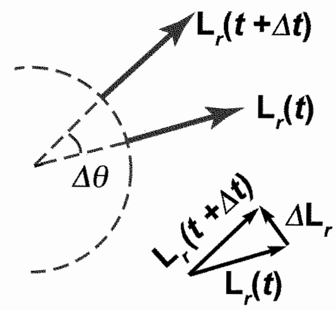
$$\vec{L_B} = L_r\hat{r} + L_z\hat{k} = \vec{L_r} + \vec{L_z}$$



$$L_z = MLr\omega \sin\alpha$$
$$L_r = MLr\omega \cos\alpha$$

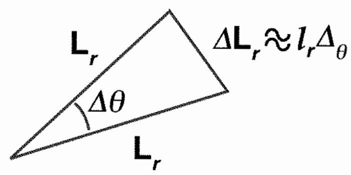
L_r at times t and $t + \Delta t$ swings through angle

$$\Delta\theta = \omega\,\Delta t$$



$$|\Delta L_r| = L_r(t+\Delta t) - L_r(t)$$

$$|\Delta L_r| = L_r\,\Delta\theta$$



As $\Delta t \to 0$

$$\frac{dL_r}{dt} = L_r\frac{d\theta}{dt} = L_r\omega$$

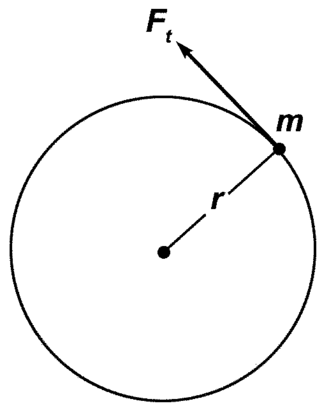
$$\therefore \frac{dL_r}{dt} = MLr\omega^2\cos\alpha$$

$$T\sin\alpha = Mr\omega^2$$
$$T\cos\alpha = Mg.$$

$$\therefore \frac{dL_r}{dt} = MgL\sin\alpha$$

Rotational Dynamics: Torque and Angular Acceleration

Consider a particle moving in a circle of radius r under the action of a tangential force $\vec{F}_t$. This force produces a tangential acceleration a_t, and



A particle rotating in a circle under the influence of a tangential force F_t. A centripetal force F_r (not shown) must also be present to maintain the circular motion.

$$F_t = m a_t$$

The torque of F about the origin, which is axis of rotation

$$\tau = F_t r = (m a_t) r$$

The tangential acceleration is related to the angular acceleration α, by

$$a_t = r\alpha$$

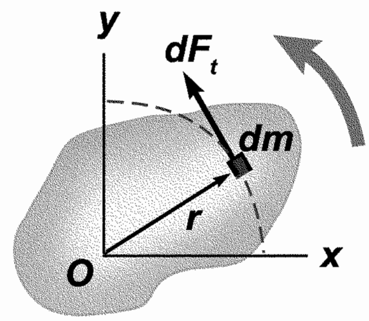
$$\therefore \quad \tau = (mr^2)\alpha$$

$$\boxed{\tau = I\alpha}$$

The torque acting on the particle is proportional to the angular acceleration.
Rotational analogue of Newton's 2nd Law.

Rigid Body Dynamics

- Rotation about a fixed axis.
- Body consists of infinite number of mass elements dm.
- Each mass element rotates in a circle about the origin and has a tangential acceleration a_t produced by a tangential force F_t.
- Newton's 2ND Law for mass element,



A rigid body pivoted about an axis through O. Each mass element *dm* rotates about O with the same angular acceleration α, and the net torque on the body is proportional to α.

$$dF_t = (dm)\, a_t$$

Associated torque is given by

$$d\tau = r\, dF_t = (r\, dm)\, a_t$$

We have $a_t = r\alpha$

$$d\tau = (r\,dm)\, r\alpha = (r^2 dm)\, \alpha$$

Different points have different a_t, but same α.

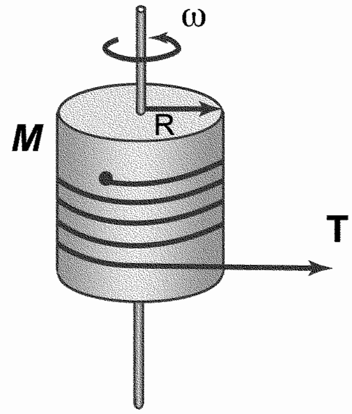
Integrating:

$$\tau_{net} = \int (r^2 dm)\, \alpha = \alpha \int r^2 dm$$

$$\vec{\tau}_{net} = I \vec{\alpha}$$

Example

A string is wrapped around a cylinder of mass M and radius R. The cylinder is free to rotate about its axis.
The string is pulled tangentially by a force maintaining a constant tension T.



$M = 15\,kg$

$R = 6\,cm$

$T = 2N$

a) What is the angular acceleration of the cylinder?

$$\vec{\tau} = I\vec{\alpha}$$

$$\tau = RT = (0.06)(2) = 0.12\ N \cdot m.$$

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(15kg)(0.06)^2 = 2.70 \times 10^{-2}\ kg/m^2$$

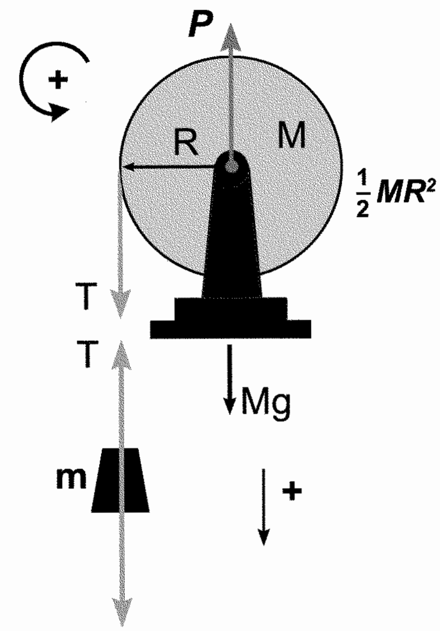
$$\alpha = \frac{\tau}{I} = \frac{RT}{\frac{1}{2}MR^2} = \frac{2T}{MR} = \frac{2 \times 2}{15 \times .06} = \frac{4}{0.9} = 4.44\ r/s^2.$$

b) What is the angular speed after t = 2s?

$$\omega = \alpha t = 4.44 \times 2 = 8.89\ rad/s.$$

Example

A rope is wrapped around a cylinder of mass M and radius R. The rope is pulled by a mass m. What are the accelerations of the two masses?



① $mg - T = ma$ $\qquad (\vec{F} = m\vec{a})$

② $RT = I\alpha = \frac{1}{2}MR^2\alpha$ $\qquad (\vec{\tau} = I\vec{\alpha})$

③ $a = R\alpha$ $\qquad$ (Geometry)

sub. ③ into ② $\quad T = \frac{1}{2}Ma$ $\quad$ ④

sub for T into ① $\quad mg - \frac{Ma}{2} = ma$

$$a = \frac{g}{1 + M/2m} \qquad \alpha = \frac{a}{R} = \frac{g/R}{1 + M/2m}$$

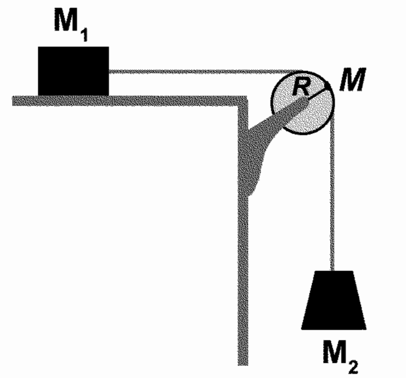
$$T = mg - ma = \frac{mg}{1 + 2m/M}$$

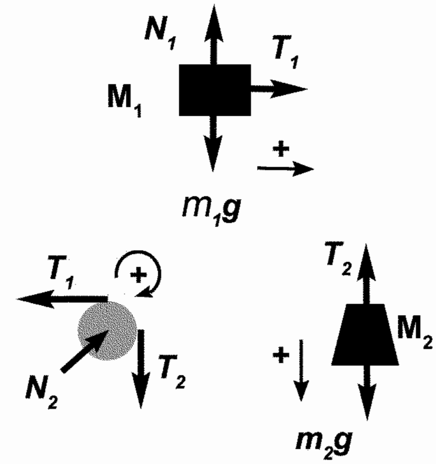
If $M = 0$ $\quad T = 0$

$a = g.$

Example:

m_1 slides without friction on a horizontal surface. The pulley is a thin cylinder of mass M and radius R. String attached to mass m_2 pulls m_1 without slipping on pulley.





$$T_1 = m_1 a_1$$

$$m_2 g - T_2 = m_2 a_2$$

$$+T_2 R - T_1 R = I\alpha = (MR^2)\alpha$$

$$a_1 = a_2 = R\alpha$$ [string does not stretch nor slip]

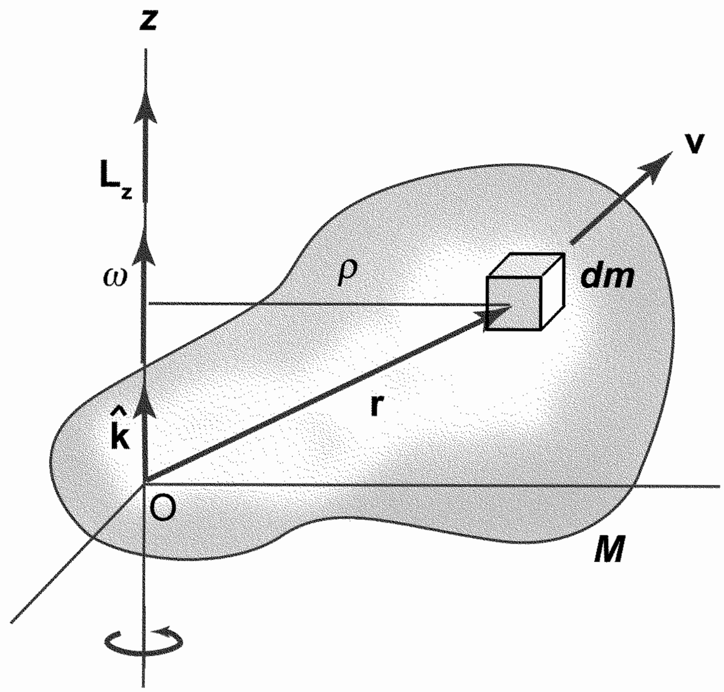
$$\left.\begin{aligned} T_1 &= m_1 a_1 \\ m_2 g - T_2 &= m_2 a_1 \\ T_2 - T_1 &= M a_1 \end{aligned}\right\}$$ Add to eliminate T_1 and T_2 and solve for a_1.

$$a_1 = \frac{m_2 g}{m_1 + m_2 + M}$$

$$T_1 = \frac{m_1 m_2 g}{m_1 + m_2 + M} \qquad T_2 = \frac{(m_1 + M) m_2 g}{m_1 + m_2 + M}$$

If $M \equiv 0$ $\quad a_1 = \dfrac{m_2 g}{m_1 + m_2}$ and $T_1 = T_2$ [as before]

Rigid Body - L_z



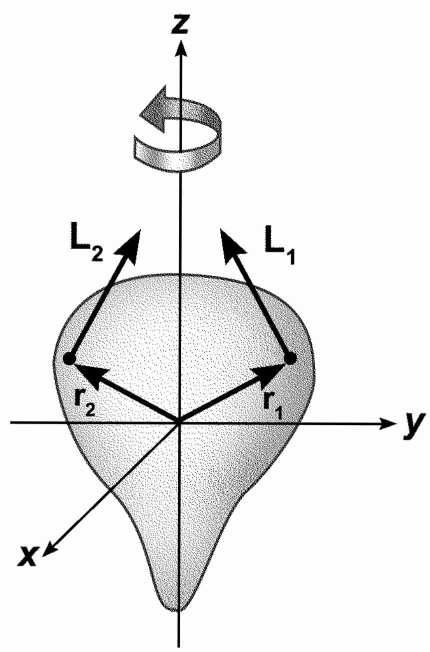
Consider a rigid body of mass M rotating about the fixed z-axis.

A differential mass element, dm:

- radial component of ang. momentum.
- z-component of ang. mom.

- z-component of angular momentum influences the rotational motion of the object about fixed axis.
- ρ-component is associated with forces that the axis exerts on the support bearings.

- For a symmetric rigid body the radial component of $\vec{L}$ vanishes. z-components add up.
- Forces on bearings vanish.



A rigid body rotating about its axis of symmetry.

For mass element dm

$$d\ell_z = \rho^2 \omega \, dm$$

The total z-component, L_z, of the angular momentum of the object is obtained by integrating over all the mass elements.

$$L_z = \int d\ell_z = \omega \int \rho^2 dm$$

$$L_z = I_z \omega$$

$$\text{or} \quad \omega = \left(\frac{L_z}{I_z}\right)$$

Total Kinetic Energy

$$K = \frac{1}{2} I_z \omega^2 = \frac{1}{2} I_z \left(\frac{L_z}{I_z}\right)^2$$

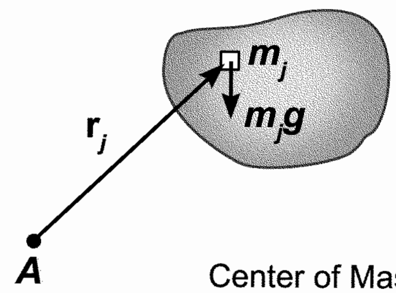
$$= \frac{1}{2} \frac{L_z^2}{I}$$

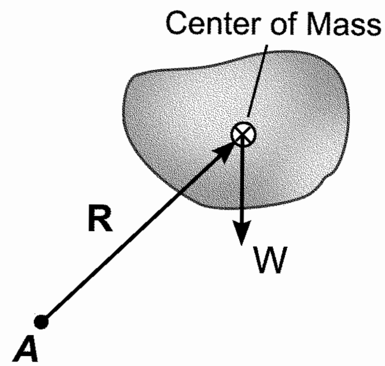
[Drop subscript - z]

Torque due to Gravity

- Body of mass M
- Origin at A

- Consider mass particle m_j with position vector $\vec{r}_j$.





$$\vec{\tau}_j = \vec{r}_j \times m_j \vec{g}$$

$$\vec{\tau} = \sum \vec{\tau}_j = \sum \vec{r}_j \times m_j \vec{g}$$

$$= \sum (m_j \vec{r}_j) \times \vec{g}$$

But $\sum m_j \vec{r}_j = M\vec{R}$

where $\vec{R} \equiv$ position vector of CM

$$\therefore \vec{\tau} = M\vec{R} \times \vec{g}$$

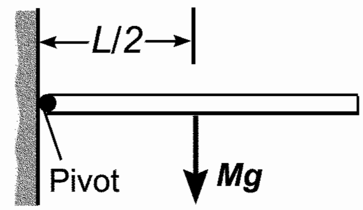
$$= \vec{R} \times M\vec{g}$$

$$= \vec{R} \times \vec{W}$$

To balance an object (i.e. $\vec{\alpha} \equiv 0$) the pivot must be at CM.

Example: Rotating Rod

A uniform rod of length L and mass M is free to rotate about pivot at left end.



Weight Mg acts at cm.
The torque due to the weight about the pivot is,

$$\tau = \frac{MgL}{2}$$

Forces at the pivot can produce no torques about the pivot.

$$I = \frac{1}{3} ML^2 \qquad [\text{M-of I about end of rod}]$$

$$\therefore \quad I\alpha = Mg\frac{L}{2}$$

$$\alpha = \frac{Mg(L/2)}{\frac{1}{3}ML^2} = \frac{3}{2}\frac{g}{L}$$

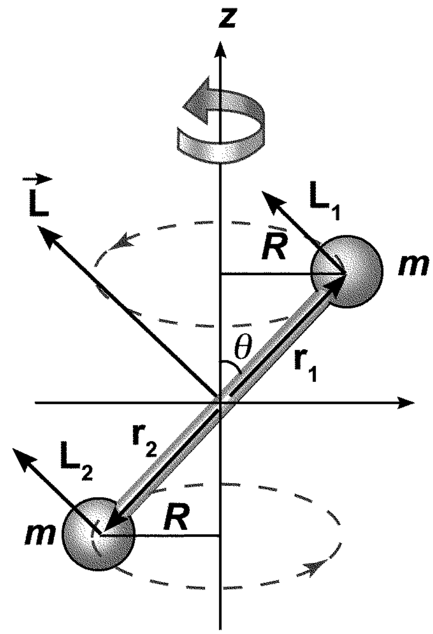
The linear acceleration for points on the rod a distance r from the pivot is

$$a = r\alpha = \frac{3}{2}\frac{gr}{L}$$

For $r > \frac{2}{3}L$ we have that $a > g$!!!

Example

Two particles of mass m at the ends of a light rod. Rod makes an angle θ with z-axis (axis of rotation).



Two particles of mass *m* at the ends of a rod that makes an angle θ with the axis of rotation. At the instant shown here the angular momentum vectors L_1 and L_2 of the two particles are both in the *y-z* plane.

$$R = r\sin\theta$$
$$v = \omega R = \omega r \sin\theta$$

Ang. momentum for each particle has magnitude.

$$m|\vec{r}\times\vec{v}| = mrv = m\omega r^2 \sin\theta$$

Individual L's are in same direction. Total ang. momentum

$$L = 2m\omega r^2 \sin\theta$$

It makes an angle $(90° - \theta)$ wrt z-axis and precesses about axis as particles move.

$$L_z = L\cos(90° - \theta) = 2m\omega r^2 \sin^2\theta$$

$$L_z = 2m\omega R^2$$

$$L_z = I\omega$$

$\vec{L} = I\vec{\omega}$

- Angular momentum varies in proportion to the angular velocity ω. $\underline{I}$ is an inertial property of the body and measures the resistance of the body to changes in its angular momentum.

$\vec{p} = m\vec{v}$

- Momentum varies in proportion to the velocity $\vec{v}$. The mass m measures the resistance of the body to a change in its velocity.

$$K = \frac{L_z^2}{2I} \qquad\qquad K = \frac{p^2}{2m}$$

Suppose L_z = constant

If system contracts and I decreases, KE must increase. For this to be possible, there must be a source of energy.

- For collapsing galaxies or stars, this source is gravity. Gravitational PE is negative and becomes more negative as object contracts.
- Spinning ballerina or ice-skater does work as arms and legs are pulled inward.

Conservation of Angular Momentum

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$ Rate of change of ang. mom.

Total Torque

$\vec{L}$ = Constant if $\vec{\tau} \equiv 0$.

$$\vec{\tau}_{total} = \vec{\tau}_{Internal} + \vec{\tau}_{External}$$

Always add up to zero (Expt. Evidence)

The total angular momentum changes only in response to an external torque. No amount of internally produced torques change the angular momentum.

"The total angular momentum of an isolated system is conserved"

For a rigid body rotating about a fixed axis (e.g. z-axis) with $\tau_z = 0$, the conservation of angular momentum reduces to

$$\frac{dL_z}{dt} = 0 \Longrightarrow \tau_z = 0$$

$$L_z = \text{constant} \Longrightarrow L_i = L_f$$

$$I\omega = \text{constant.}$$

$$\underline{I_i \omega_i = I_f \omega_f}$$