Parallel-Axis Theorem

The moment-of-inertia depends on the location of the axis of rotation.

KE for a body about a fixed axis is

$$K = \frac{1}{2} I_z \omega^2$$

We had before that the KE is the sum of the translational energy of motion of the cm and the internal energy of motion relative to the cm.

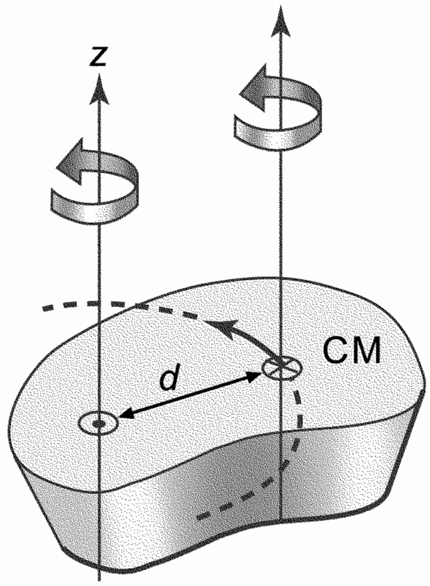
$$K = \frac{1}{2} M v_{cm}^2 + K_{Int}.$$

Consider rotation about z-axis <u>not</u> through cm.

cm moves in a circle of radius d around this axis.

$$\therefore v_{cm} = d\omega$$

$$\therefore \frac{1}{2} M v_{cm}^2 = \frac{1}{2} M d^2 \omega^2$$



Two alternative parallel axes of rotation of a rigid body. The z axis is fixed. The center-of-mass axis moves along a circle of radius *d* around the z axis. The body is in rotational motion relative to each of these axes.

The rotation of the body with angular velocity ω about a fixed z-axis is also a rotation about a parallel axis through the cm with the same angular velocity ω. One turn around z corresponds to one turn around axis through cm.

The KE associated with rotational motion around axis through cm is

$$K_{Int} = \frac{1}{2} I_{cm} \omega^2$$

$$\therefore K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M d^2 \omega^2$$

$$\frac{1}{2} I_z \omega^2 = \frac{1}{2} \left[I_{cm} + M d^2 \right] \omega^2$$

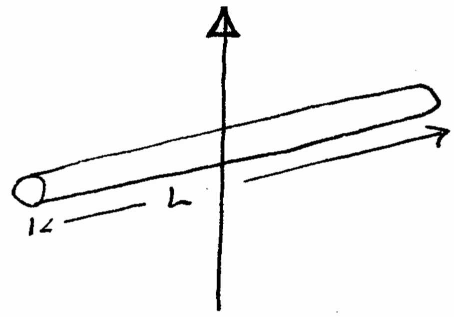
Comparing, we must have

$$I_z = I_{cm} + M d^2$$

[Parallel Axis Theorem]

Example: Rod

- Thin rod axis through its midpoint



$$I_{cm} = \frac{1}{12} mL^2$$

- what is I about an axis through its end?

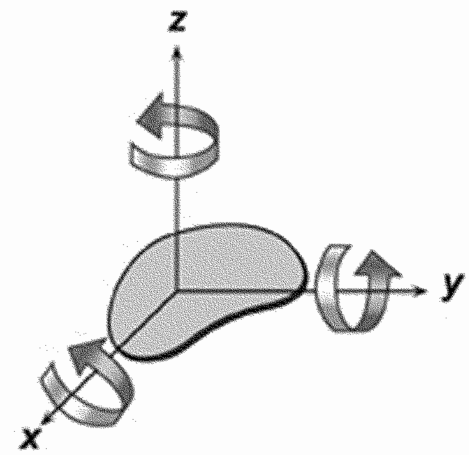
$$d = L/2$$

$$I = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2$$

$$I = \frac{mL^2}{3}$$

Perpindicular - Axis Theorem.

Relates moments-of-inertia of a thin flat plate about three mutually perpindicular axes.



A thin, flat plate. The plate may rotate about either the the x axis, the y axis, or the z axis.

Consider a thin plate which can rotate about any of three $\perp$ axes

$$\left.\begin{array}{l} I_x \\ I_y \\ I_z \end{array}\right\} \text{corresponding moments-of-inertia}$$

Let plate be in xy-plane. The distance from z-axis to reference point P is

$$R = \sqrt{x^2 + y^2}$$

$$I_z = \int \rho R^2 dV = \int \rho (x^2 + y^2)\, dV$$

$$I_x = \int \rho y^2 dV$$

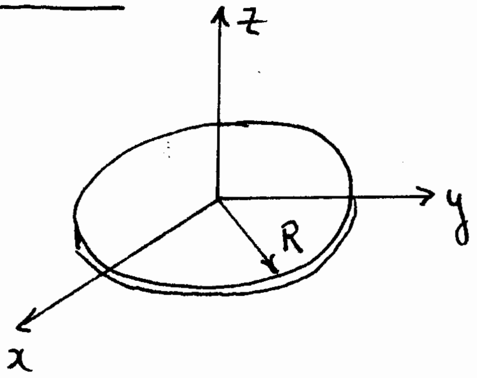
$$I_y = \int \rho x^2 dV$$

$$\therefore \boxed{I_z = I_x + I_y}$$

[Perpindicular-Axis Thm]

Example

i) Disk



• Disk in xy-plane

$$I_z = \frac{1}{2} M R^2$$

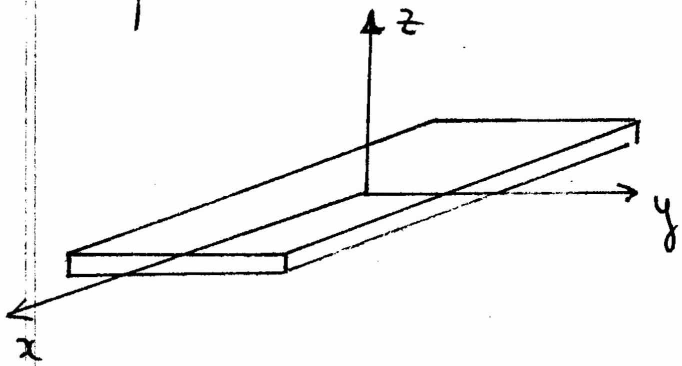
By symmetry

$$I_x = I_y$$

$$\therefore \quad I_z = 2 I_x$$

$$I_x = I_y = \frac{I_z}{2} = \frac{MR^2}{4}$$

ii) Square Plate.



Side = a

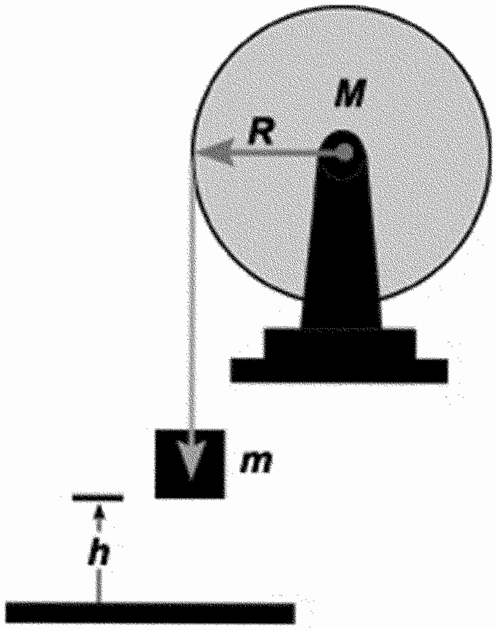
By symmetry

$$I_x = I_y.$$

$$2 I_x = I_z = \frac{1}{6} M a^2$$

Example

A light rope is wrapped around a solid cylinder of mass M and radius R.
Mass m tied to rope and released a height h above the floor. Assuming frictionless motion, what is speed of m and angular velocity of cylinder when m strikes floor?



As the cylinder rotates, the rope unwinds and mass *m* drops.

Ans: System initially has no kinetic energy but has PE.
Finally both m and M have KE and the PE of m is decreased.

$$E_1 = K_1 + U_1$$
$$= 0 + mgh$$

$$E_2 = K_2 + U_2$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$v = R\omega$ [Related by geometry - see Kinematics]

$I = \frac{1}{2}MR^2$ [Solid cylinder]

$E_1 = E_2$ conservation of energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\tfrac{1}{2}MR^2\right)(v/R)^2 = \tfrac{1}{2}(m + M/2)v^2$$

$$v = \sqrt{2gh/(1 + M/2m)}$$

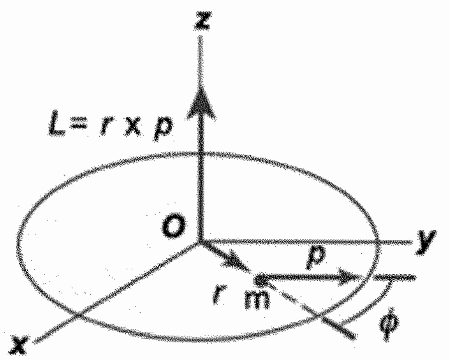
Angular Momentum of a Particle

- For a particle, linear momentum is not conserved if a force acts on it.

$$\frac{d\vec{P}}{dt} = \vec{F} \qquad \text{(Newton's 2'nd Law)}$$

- For some special types of forces the particle angular momentum is conserved. These forces are called central forces

$$\vec{F} = f\,\hat{r}$$



The angular momentum *L* of a particle of mass *m* and momentum *p* located at the position *r* is a vector given by *L = r x p*. Note that the value of L depends on the origin and is a vector perpendicular to both *r* and *p*.

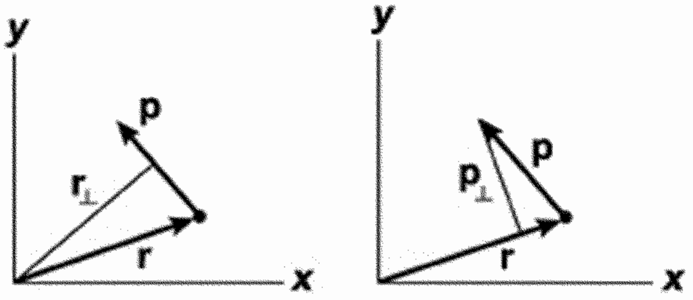
Consider a particle of mass m, located at the position vector $\vec{r}$, and moving with velocity $\vec{v}$.

The instantaneous angular momentum $\vec{L}$ of the particle relative to the origin 'O' is defined by the cross product of its instantaneous position vector and its instantaneous linear momentum $\vec{p}$:

$$\vec{L} = \vec{r} \times \vec{p} = rp \sin\phi$$

$$[L] = kg \cdot m/s^2$$

- First use of vector cross-product
- Magnitude and direction of L depend on choice of coordinate system
- Direction of $\vec{L}$ is $\perp$ to plane containing $\vec{r}$ and $\vec{p}$. Lies on RH normal to this plane.
- $\vec{L} \equiv 0$ if $\vec{r}$ parallel to $\vec{p}$.



$L_z = r_\perp p$ $\qquad$ $L_z = r p_\perp$

Geometrically:

$r_\perp \equiv$ perpindicular distance between the origin and the line of $\vec{p}$.

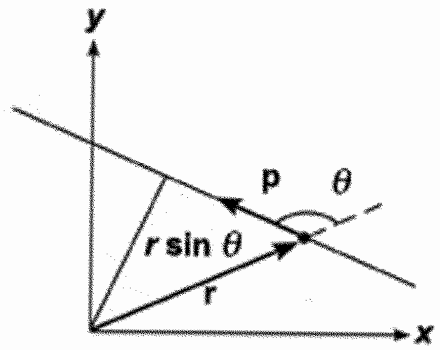
$p_\perp \equiv$ component of $\vec{p} \perp \vec{r}$.

$\vec{L} = \vec{r} \times \vec{p} \qquad L_z = r_\perp p = r p_\perp$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ p_x & p_y & p_z \end{vmatrix}$$

i) Particle moving in a straight Line

- Assume $\vec{F} = 0$, $\vec{v} = \text{const.}$
- Direction of $\vec{L} = \text{const.}$
- Magnitude of $\vec{L} = \text{const.}$



The distance between the origin and the line of motion is $r \sin \theta$.

$$\vec{L} = rp \sin\theta \ \hat{k}$$

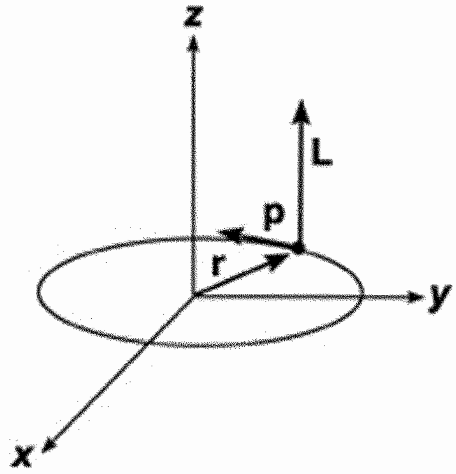
If $\vec{r}$ and $\vec{p}$ are in xy-plane, $\vec{L}$ is along $\hat{k}$

A particle will have non-zero angular momentum about some origin if its position vector measured from this point appears to rotate about this point.

If the position vector only increases or decreases in magnitude, the particle is moving along a line passing through the origin and it therefore has zero angular momentum wrt this origin.

"Choice of coordinate system is crucial. Must define an origin befor calculating angular momentum".

ii) Particle in uniform circular motion.



A particle in uniform circular motion. The vector L is perpendicular to the plane of the circle.

- ball at the end of a string.
- earth and planets around sun.

- choose origin at centre of circle.

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{p} \neq$ constant.

$$\vec{L} = rp\,\hat{z} = mrv\,\hat{z} = (mr^2)\omega\,\hat{z} = I\omega\,\hat{k}$$

magnitude is constant since r and p are always $\perp$.

Direction is $\perp$ to plane of circle and is fixed.
L remains constant in <u>magnitude</u> and <u>direction</u>.

In this example there <u>are</u> forces acting.

$$a_c = \frac{mv^2}{r} \qquad \text{and} \qquad F_c = \frac{mv^2}{r}$$

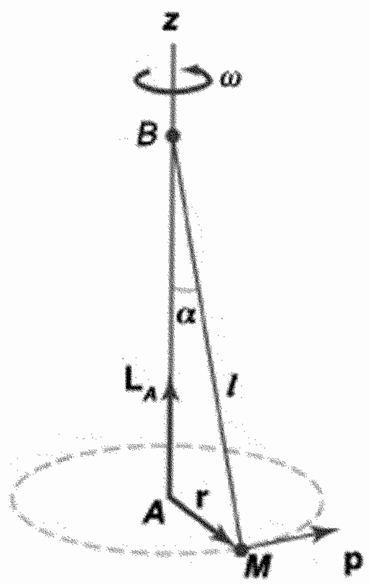
Force acts towards center of circle (origin).
If we had chosen some other origin, $\vec{L}$ would not have been a constant !!!

Angular Momentum of a Conical Pendulum

- Assume circular motion with constant $\underline{\omega}$.

(a)

- choose origin at A :



$$\vec{L}_A = \vec{r} \times \vec{p}$$

$$= r p \hat{k}$$

r = radius of circle

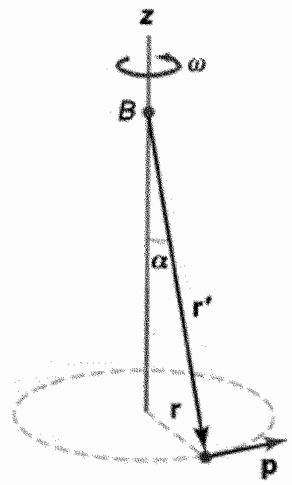
$$p = mv = mr\omega$$

$$\vec{L}_A = mr^2\omega \hat{k}$$

$\vec{L}_A$: constant in magnitude
constant in direction

(b) Chose origin at B at the pivot:

$$|L_B| = |\vec{r}' \times \vec{p}|$$
$$= |\vec{r}'||\vec{p}|$$

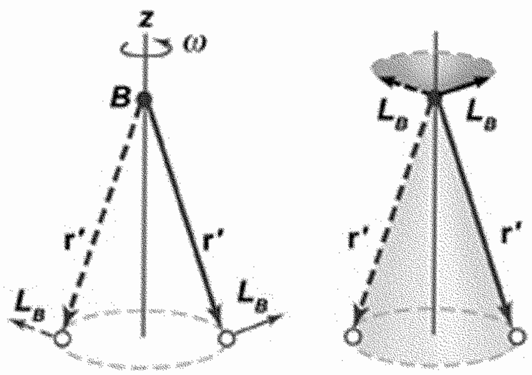


$$|L_B| = Mv L = MrL\omega$$

$|\vec{r}'| = L$; length of string

$\vec{L}_B$: magnitude not constant, depends on location B. direction of $\vec{L}_B$ is also not constant.

· For fixed B, magnitude is constant.
· Direction sweeps through a cone for each rotation.



· z-component of $\vec{L}_B$ is constant.
· Horizontal component travels in a circle with angular velocity ω.
· Dynamical consequences !!

Angular Momentum ⟷ Forces

"The angular momentum of a particle is conserved whenever the force acting on the particle is a central force: i.e. a force that points towards or away from a central point".

- a ball on a string.
- sun pulling on the planets - gravitational force.

How do we test if something is conserved? Differentiate with respect to time. If the time derivative is zero, the quantity does not change and is conserved.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p})$$

$$= \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt}$$

$$= \vec{v}\times(m\vec{v}) + \vec{r}\times\frac{d\vec{p}}{dt}$$

$\vec{v}\times\vec{v} \equiv 0$ $\quad$ $\vec{F}$ (Newton's Law)

$$\boxed{\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}}$$

If $\frac{d\vec{L}}{dt} = 0$ we must have $\vec{r}\times\vec{F} \equiv 0$

- $\vec{F}$ must be parallel to $\vec{R}$

i.e. $\vec{F}$ must point towards or away from the origin. ($\vec{F}$ is central!!)

If $\vec{F}$ is central:

$$\frac{d\vec{L}}{dt} = 0$$

[many forces in nature are central]

or $\vec{L}$ = constant.

<u>Law of Conservation of Ang. Mom.</u>

If $\vec{F}$ is non-central $\rightarrow$ $\vec{L}$ <u>is not</u> conserved.

$$d\vec{L}/dt = \vec{r} \times \vec{F}$$

The quantity $\vec{r} \times \vec{F}$ is called a <u>torque</u>

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\therefore \boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

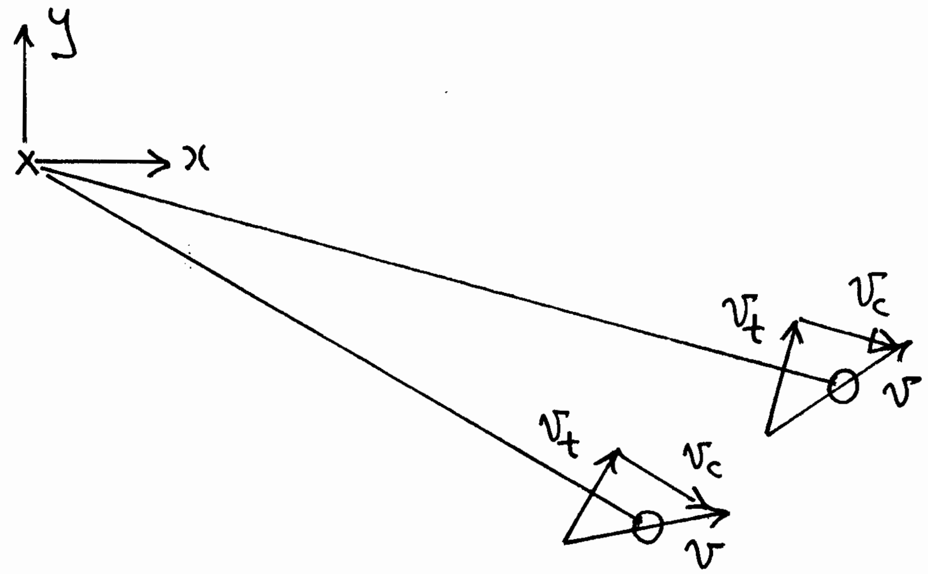
Rigid Body Dynamics

$\vec{\tau}$ + $\vec{L}$ must both be refered to the <u>same origin</u> of an inert. coor. system

The rate of change of angular momentum equals the torque.

- Analagous to rate-of-change of linear momentum which is equal to the net force.

$\vec{L}$ and Central Forces



$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{p}}{dt} = F \qquad \text{Newton's 2'nd Law}$$

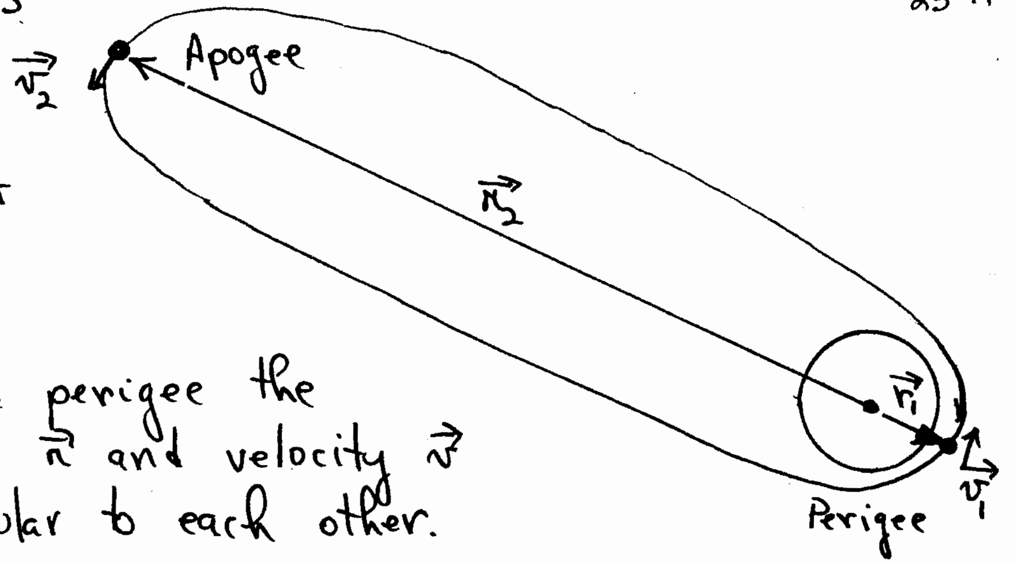
$$\vec{L} = m r v_t$$

If $\vec{F}$ is central,

v_c changes
v_t does not change

$\therefore \vec{L}$ is conserved for central forces

Example: Comets



Apogee: Farthest
Perigee: Closest

At apogee and perigee the position vector $\vec{r}$ and velocity $\vec{v}$ are perpindicular to each other.

Angular momentum about center of sun (magnitude)

$L_1 = m r_1 v_1$ (perigee)

$L_2 = m r_2 v_2$ (apogee)

$\vec{L} = \vec{r} \times \vec{p}$ (Direction out of page)

If forces are central (gravity ok)

$$L_1 = L_2$$

$$m r_1 v_1 = m r_2 v_2$$

$$v_2 = \left(\frac{r_1}{r_2}\right) v_1$$

Halley's Comet:

$r_1 = 8.75 \times 10^{10}$ m, $v_1 = 5.46 \times 10^4$ m/s } perigee

$r_2 = 5.26 \times 10^{12}$ m apogee

$$v_2 = \frac{8.75 \times 10^{10}}{5.26 \times 10^{12}} \times 5.46 \times 10^4 = 9.08 \times 10^2 \text{ m/s}$$

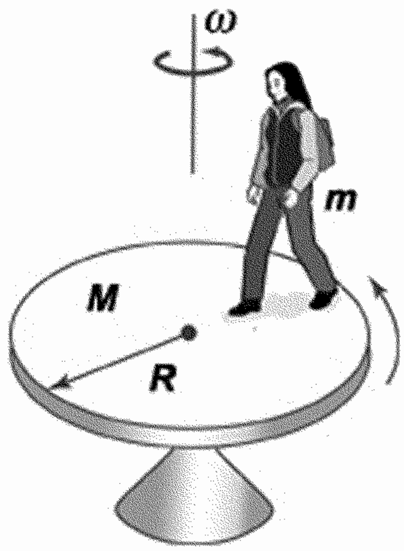
Example: Merry - Go - Round

Platform in form of circular disk rotates about a frictionless vertical axis. A student walks slowly towards center from the rim. The initial angular velocity (student at rim) is ω_i.

$R = 2.0\ m$

$M = 100\ kg \qquad m = 60\ kg.$

$\omega_i = 2\ s^{-1}$



a) What is ω, when student is at $r = 0.5\ m$ from center?

The initial moment-of-inertia is that of platform and student at the rim.

$$I_i = \frac{1}{2}MR^2 + mR^2$$

When student is at the position $r < R$, the moment of inertia is

$$I_f = \frac{1}{2}MR^2 + mr^2$$

There are no external torques on the system (platform + student) about the axis of rotation. As the student walks towards the center of the platform, the angular velocity must increase since the angular momentum must remain constant.

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

$$\left(\tfrac{1}{2} MR^2 + mR^2\right)\omega_i = \left(\tfrac{1}{2} MR^2 + mr^2\right)\omega_f$$

$$\omega_f = \frac{(\tfrac{1}{2}MR^2 + mR^2)}{(\tfrac{1}{2}MR^2 + mr^2)}\,\omega_i$$

$$\omega_f = \left(\frac{200+240}{200+15}\right)(2) = 4.1 \text{ rad/s.}$$

b) what are the initial and final KE?

$$K_i = \tfrac{1}{2} I_i \omega_i^2 = \tfrac{1}{2}(440)\,2^2 = 880\text{ J}$$

$$K_f = \tfrac{1}{2} I_f \omega_f^2 = \tfrac{1}{2}(215)\,4.1^2 = 1800\text{ J.}$$

KE has increased!!! The student does work as he walks to center — the KE of system increases. Internal forces within system did work.

Student is in a rotating, noninertial frame of reference and senses an outward "centrifugal" force that varies with r. He exerts a counter-acting force and does work — exerts energy.