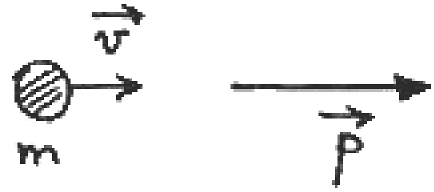
Momentum of a Particle

Newton's laws are more precisely stated in terms of <u>momentum</u>.

The momentum of a particle of mass m, moving with velocity $\vec{v}$ is defined to be:

$$\vec{p} = m\vec{v} \qquad \left[\frac{kg \cdot m}{s}\right]$$



– same direction as the velocity.

Law - I

When no forces are acting, the momentum of a particle is constant. [$F = 0 \Rightarrow V =$ constant]

$$\boxed{p = \text{constant}} \longleftarrow \text{No external forces}$$

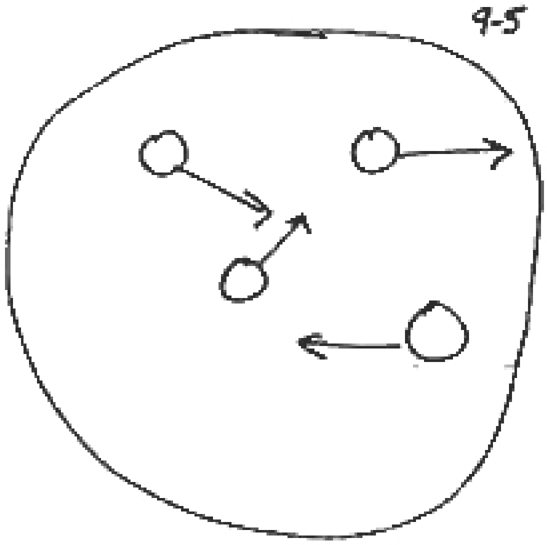
$\Rightarrow$ A conserved quantity

Conserved quantities in physics

- momentum
- energy
- angular momentum
- charge

In certain situations these basic conservation laws allow us to make general statements about the physics without doing detailed calculations.

System of Particles



Internally: collisions / interactions very complicated

Externally: No net forces

$\Rightarrow$ Total system linear momentum is conserved.

Law - II

Rate of change of linear momentum equals the applied force.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

$$= \left(\frac{dm}{dt}\right)\vec{v} + \underbrace{m\left(\frac{d\vec{v}}{dt}\right)}_{m\vec{a}}$$

$\vec{F} = m\vec{a}$, If $\left(\frac{dm}{dt}\right) = 0$

$\vec{F} = \left(\frac{d\vec{p}}{dt}\right)$ Most general statement. Can handle problems where mass may also change eg. Rockets, etc.

Law - III

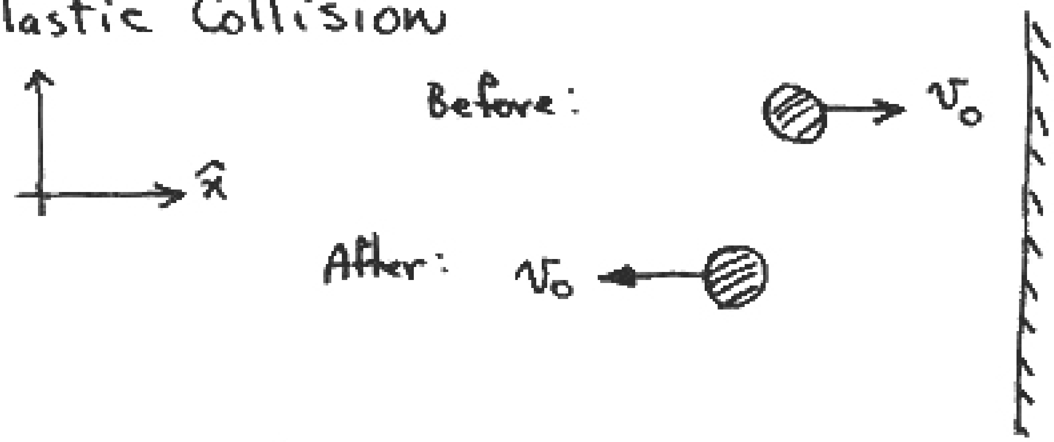
Action force is exactly opposite to reaction force.

$\therefore$ Rate of change of momentum generated by an action force on one body is exactly opposite to the rate of change of momentum generated by the reaction force on the other body.

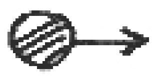
Whenever two bodies interact the resulting changes in momentum are equal and opposite.

$\Longrightarrow$ Law of Conservation of Momentum.

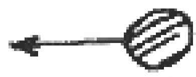
Example: Elastic Collision



Ball of $m_B = 100g$ strikes a wall with a speed of $50\,m/s$. Rebounds with the same speed. What is change in momentum of the ball?



$\vec{P}_i = m_B v_0 \hat{x}$



$\vec{P}_f = -m_B v_0 \hat{x}$

$\Delta\vec{p} = \vec{P}_f - \vec{P}_i$

$= -2 m_B v_0 \hat{x}$

$= -2 \times 0.1 \times 50 \hat{x} = -10\,kg \cdot m/s$

What happened to the missing momentum: $2 m_0 v_0 \hat{x}$

wall absorbed change in $\vec{p}$: $\vec{a}_W \sim \Delta p / \Delta t$ [Very tiny !!!]

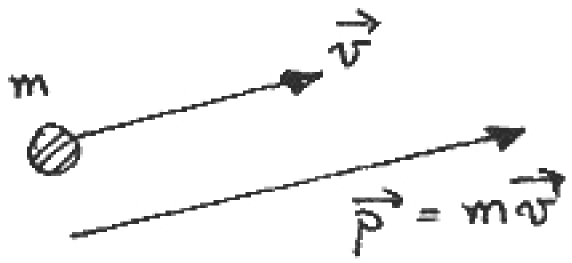
Systems of Particles: Energy and Momentum

- up to now we have studied motion of a single particle.
- We now want to look at a system of particles to see what we can learn about the motion.
- In general the solutions of the equations of motion are impossible.
- We will study how the conservation laws:
    - Energy
    - Momentum
    - Angular Momentum

  apply to a system of particles.

Momentum

Single Particle: $\vec{p} = m\vec{v}$



Total momentum for a system of particle is the sum of the individual momenta.

$$\vec{p}_1 = m_1 \vec{v}_1$$
$$\vec{p}_2 = m_2 \vec{v}_2$$
$$\vdots$$
$$\vec{p}_n = m_n \vec{v}_n$$

$$\vec{P} = \vec{p}_1 + p_2 + \cdots \vec{p}_n = \sum^{N} \vec{p}_i$$

2-Particle System

- simplest many-particle system.
- Particles exert forces on each other.

By Newton's 3'rd Law
$$\vec{F_1} = -\vec{F_2}$$

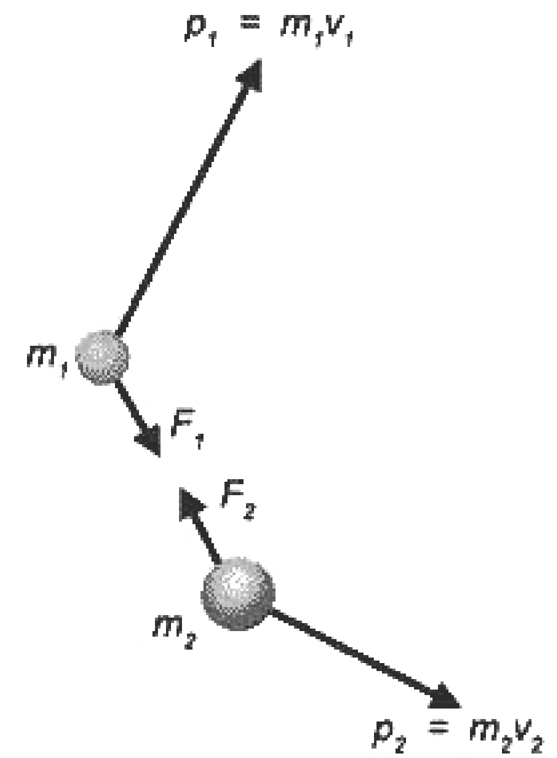
Eq. of motion:
$$\frac{d\vec{p_1}}{dt} = \vec{F_1}$$
$$\frac{d\vec{p_2}}{dt} = \vec{F_2}$$

$$\frac{d\vec{p_1}}{dt} + \frac{d\vec{p_2}}{dt} = \vec{F_1} + \vec{F_2} \equiv 0.$$

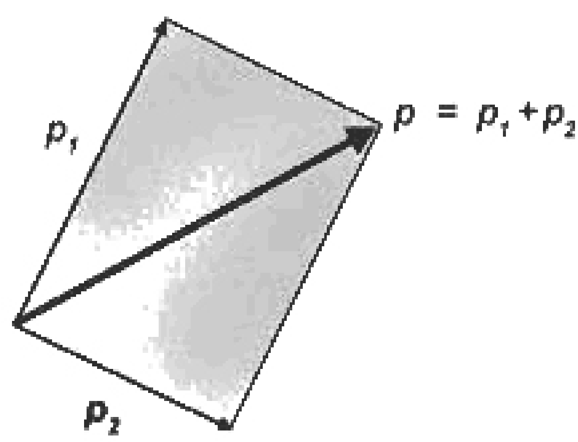
$$\therefore \frac{d}{dt}(p_1 + p_2) = 0$$

$$\vec{P} = \vec{p_1} + \vec{p_2} = \text{constant.}$$

- Particles exchange momentum as they interact.
- If only <u>internal forces</u> act the total linear momentum is conserved.



(a) At some instant, the momentum of m_1 is $p_1 = m_1v_1$ and the momentum of m_2 is $p_2 = m_2v_2$. If the particles are isolated, $F_1 = -F_2$.



(b) The total momentum of the system, P, is equal to the vector sum $p_1 + p_2$

2-Particles / External Forces

- consider 2-particles with external forces acting on them.
- e.g. gravitational force.
- Let $\vec{F}$ be the internal force.

$$\frac{d\vec{p_1}}{dt} = \vec{F} + \vec{F}_{1,ext}$$

$$\frac{d\vec{p_2}}{dt} = -\vec{F} + \vec{F}_{2,ext}$$

$$\frac{d}{dt}\underbrace{(\vec{p_1} + \vec{p_2})}_{\vec{P}} = \underbrace{\vec{F}_{1,ext} + \vec{F}_{2,ext}}_{\vec{F}_{ext}\text{: Total external force on system.}}$$

In general for many particles:

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

Eqn. of motion for the system. Determines overall translational motion.

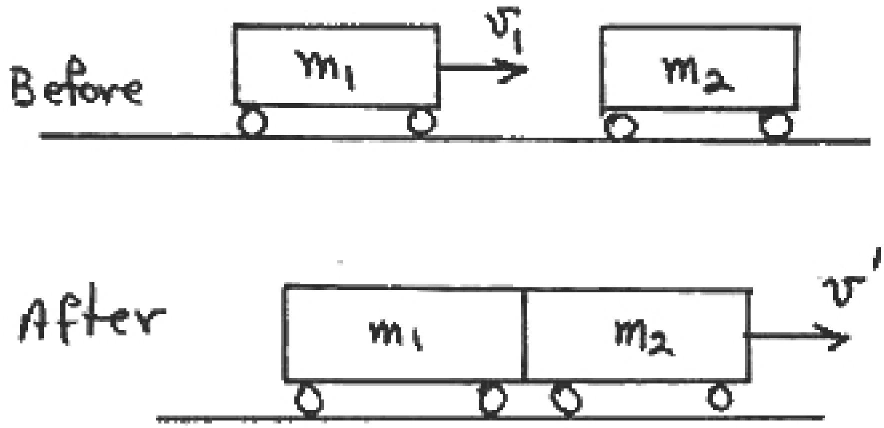
If $\vec{F}_{ext} = 0$

$$\frac{d\vec{P}}{dt} = 0 \implies \vec{P} = [\text{constant}]$$

Note:
In some problems external forces may vanish on one or more axes but not on all. Linear momentum is a constant for those components having vanishing $\vec{F}_{ext}$.

Example: Inelastic Collision

$m_1 = m_2 = 10{,}000\,\mathrm{kg}$
$v_1 = 24\,\mathrm{m/s}$



- Railroad cars collide and stay coupled
- What is common speed v'?
- Only internal forces act on system.

Initial Momentum: $m_1 v_1 \hat{x}$

Final Momentum: $(m_1+m_2)\, v' \hat{x}$

Conservation of $\vec{P}$:

$$m_1 V_1 = (m_1 + m_2)\, v'$$

$$v' = \frac{m_1}{m_1+m_2} V_1$$

$$= \frac{10{,}000}{20{,}000} \times 24 = 12\,\mathrm{m/s}.$$

Energy is not conserved; collision is inelastic

$K = \frac{1}{2} m_1 v_1^2$

$$K' = \frac{1}{2}(m_1+m_2)\, v'^2 = \frac{m_1 v_1^2}{4} \ll K$$

$$= \frac{1}{2} \frac{m_1^2}{m_1+m_2} v_1^2$$

$m_1 = m_2$

Example:

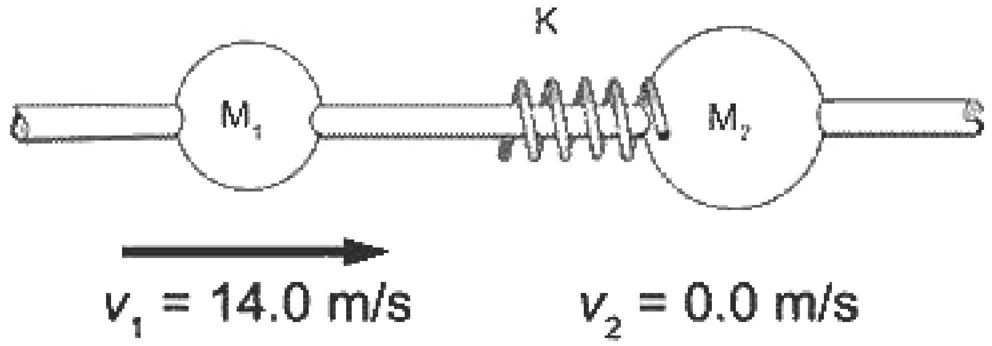
Two particles with mass $m_1 = 2.0\,kg$ and $m_2 = 5.0\,kg$ can slide on a frictionless rod. A spring with $k = 1000\,N/m$ is attached to m_2.

$v_1 = 14\,m/s$

$v_2 = 0.$

a) What is the maximum compression of the spring when the particles collide?

b) What are the final velocities of the particles?



- When the spring is under maximum compression, the relative velocity of the particles is zero.
- The system then has a velocity v_0.

Conservation of Linear Momentum: [At max. comp.]

$$m_1 v_1 + m_2 \times 0 = (m_1 + m_2) v_0$$

$$v_0 = \frac{m_1}{m_1 + m_2} v_1 = \frac{2 \times 14}{2+5} = 4\,m/s.$$

Initial KE, before collision

$$K_0 = \frac{1}{2} m_1 v_1^2 + \cancel{\frac{1}{2} m_2 \times 0} = \frac{1}{2} \times 2.0 \times (14)^2 = 196\,J$$

At maximum spring compression the remaining KE is

$$K = \frac{1}{2}(m_1+m_2)v_0^2 = \frac{1}{2}(2+5)\,4^2 = 56\text{ J}$$

Energy difference is stored as PE in the spring.

$$\frac{1}{2}kx^2 = K_0 - K$$

$$x = \sqrt{\frac{2(K_0-K)}{k}} = \sqrt{\frac{2\times(196-56)}{1000}} = 0.53\text{m}$$

b) When particles separate, the energy stored in the spring is returned to the particles and total energy and momentum is conserved.

① $$m_1v_1' + m_2v_2' = m_1v_1 + \cancel{m_2}\times 0$$

② $$\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 = \frac{1}{2}m_1v_1^2 + \cancel{\frac{1}{2}m_2\times 0^2}$$

③ $$2v_1' + 5v_2' = 28 \text{ kg.m/s}$$ [Sub. in Eq ①]

④ $$2v_1'^2 + 5v_2'^2 = 392 \text{ J}^2$$ [sub in eq. ②]

From ⑤ $v_2' = (28-2v_1')/5$

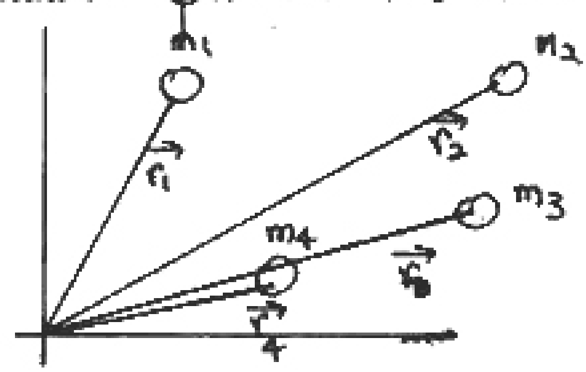
$$2v_1'^2 + 5\left[\frac{28-2v_1'}{5}\right]^2 = 392$$

	v_1'	v_2'
(NG)!!	14	0
	−6	8

Center-of-Mass

- Up to now we have ignored the size of objects
- We will now show that for an object of finite size, the center-of-mass mimics particle motion.

- The position of the center-of-mass is the average position of the mass of the system.



$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots m_n\vec{r}_n}{m_1 + m_2 + \cdots m_n}$$

$$\vec{r}_{cm} = (m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots m_n\vec{r}_n)/M = \frac{\sum m_i\vec{r}_i}{M}$$

$M \equiv$ Total Mass

In terms of vector components:

$$x_{cm} = \frac{1}{M}[m_1x_1 + m_2x_2 + \cdots m_nx_n] = \frac{1}{M}\sum m_i x_i$$

$$y_{cm} = \frac{1}{M}[m_1y_1 + m_2y_2 + \cdots m_ny_n] = \frac{1}{M}\sum m_i y_i$$

$$z_{cm} = \frac{1}{M}[m_1z_1 + m_2z_2 + \cdots m_nz_n] = \frac{1}{M}\sum m_i z_i$$

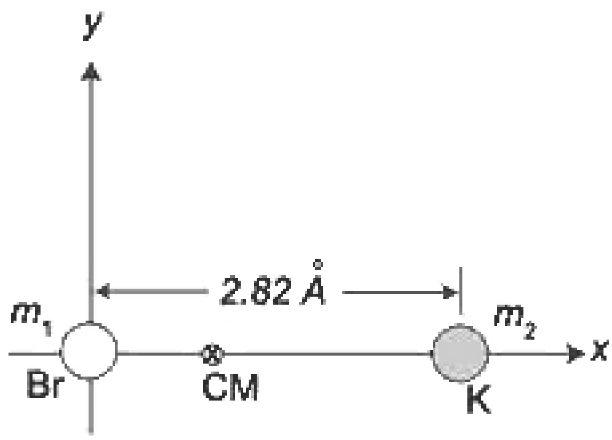
$$\vec{r}_{cm} = x_{cm}\hat{x} + y_{cm}\hat{y} + z_{cm}\hat{z}$$

Example: CM of Two Particles.

[Potassium Bromide KBr]

$$y_{cm} = 0$$

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



Atoms of bromine (Br) and potassium (K), regarded as particles.

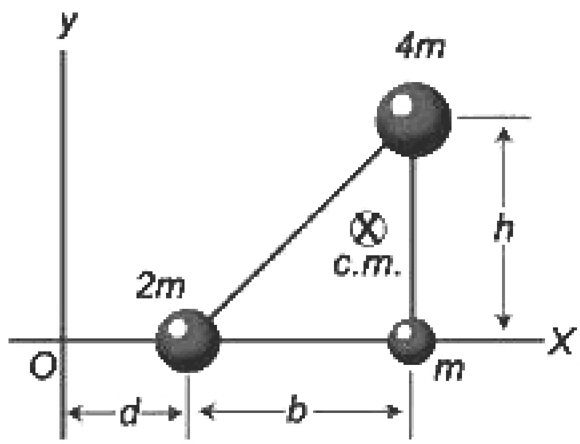
$x_1 = 0$ $\quad$ $x_2 = 2.82\,A^\circ$

$m_1 = 79.9\,u$ $\quad$ $m_2 = 39.1\,u$

$$x_{cm} = \frac{39.1 \times 2.82\,A^\circ}{79.9 + 39.1} = 0.93\,A^\circ$$

[Near heavier atom]

Example: CM of Three Particles



$$x_{cm} = \frac{\sum m_i x_i}{M} = \frac{2md + m(d+b) + 4m(d+b)}{7m}$$

$$= d + \frac{5}{7} b$$

$$y_{cm} = \frac{\sum m_i y_i}{M} = \frac{2m(0) + m(0) + 4mh}{7m}$$

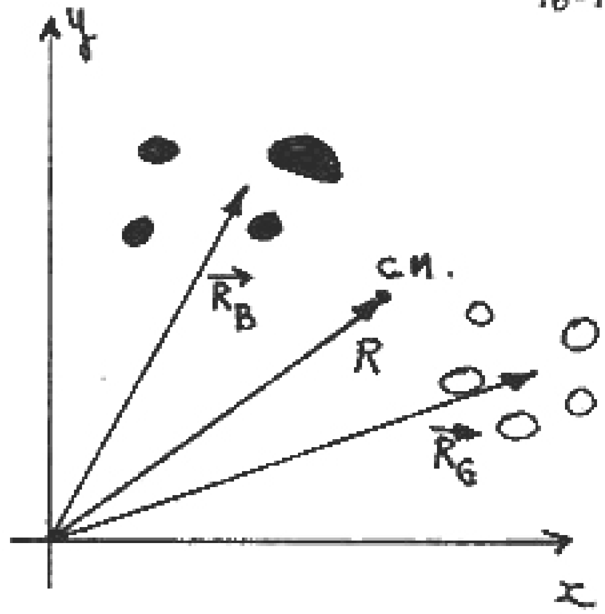
$$= \frac{4}{7} h$$

The position vector of the cm

$$r_{cm} = x_{cm}\,\hat{x} + y_{cm}\,\hat{y} = \left(d + \frac{5}{7} b\right)\hat{x} + \frac{4}{7} h\,\hat{y}$$

Groups of Particles

- Divide system of particles into groups.
- Find cm of each group.
- Treat each group as a particle at its cm and find cm of the combined groups.



$$MR = \sum_{i=1}^{l} m_i \vec{r}_i + \sum_{j=l+1}^{n} m_j \vec{r}_j$$

Let $\vec{R}_B = \frac{1}{m_B} \sum_{i=1}^{l} m_i \vec{r}_i$ $\qquad m_B = \sum_{i=1}^{l} m_i$

$\vec{R}_G = \frac{1}{m_G} \sum_{j=l+1}^{n} m_j \vec{r}_j$ $\qquad m_G = \sum_{j=l+1}^{n} m_j$

$$m\vec{R} = m_B \vec{R}_B + m_G \vec{R}_G$$

$$m = m_B + m_G$$

$$\vec{R} = \frac{1}{m}\left[m_B \vec{R}_B + m_G \vec{R}_G\right]$$

CM of Solid Bodies

- Consider objects with continuous distributions of mass.
- Divide body up into elements of mass Δm_i with coordinates $x_i\ y_i\ z_i$.

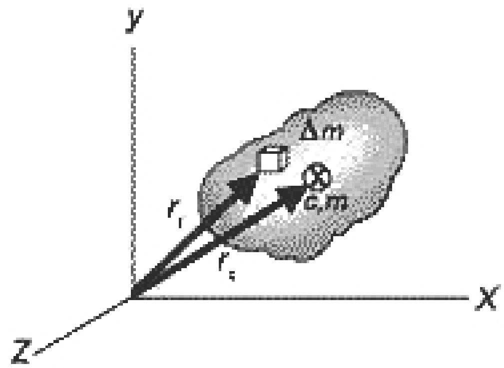
The x-coordinate of the CM becomes

$$x_c \approx \frac{\sum x_i \Delta m_i}{M}$$

Let the number of elements approach infinity.

Then

$$x_c = \lim_{\Delta m_i \to 0} \frac{\sum x_i \Delta m_i}{M} = \frac{1}{M}\int x\,dm$$



A rigid body can be considered a distribution of small elements of mass Δm_i. The center of mass is located at the vector position r_c, which has coordinates x_c, y_c, and z_c.

Also $$y_c = \frac{1}{M}\int y\,dm$$

$$z_c = \frac{1}{M}\int z\,dm$$ [First moments of mass distr.]

For the position vector of the cm

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

Note: From above it follows that the CM of homogeneous, symmetric bodies must lie on an axis of symmetry.

If an object has a <u>point</u>, <u>line</u> or <u>plane</u> of symmetry, the <u>cm</u> must lie on <u>that point</u>, on <u>that line</u> or on <u>that plane</u>.

No particle need be at cm. e.g. donut.

It is often convenient to express the mass distribution in terms of the <u>local density</u> and an element of volume:

$$dm = \rho\, dv \qquad \rho = \rho(x, y, z).$$

Then $$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,\rho\, dv$$

$$x_c = \frac{1}{M}\int x\rho\, dv$$

$$y_c = \frac{1}{M}\int y\rho\, dv$$

$$z_c = \frac{1}{M}\int z\rho\, dv$$

This is a general result even if $\rho(xyz)$ varies throughout the volume. If ρ is a constant then cm is often easily obtained by the symmetry of the object volume.

⇒ First moments of the volume distribution if ρ = constant.
Integrals are evaluated over the entire volume.

$$dv = dx\,dy\,dz$$