Uniform Circular Motion

- Particle moves with constant speed along a circle of radius R.
- Simple relation between normal component of acceleration, the speed of the particle and the radius of the circle.

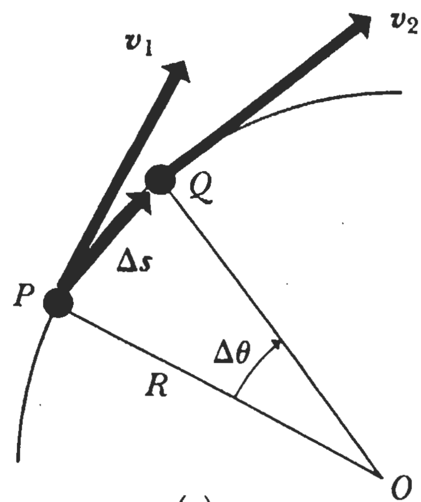
$$\vec{a} = \frac{d\vec{v}}{dt}$$

- If velocity changes in magnitude or direction the particle is accelerated.
- For motion in a circle the speed $= |v| =$ constant but the direction continously changes.

Consider particle at two times Δt, a short interval apart.

$$|\vec{v}_1| = |\vec{v}_2| = v \quad \text{[Constant speed]}$$

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1 \quad \text{[Diff. in velocity]}$$

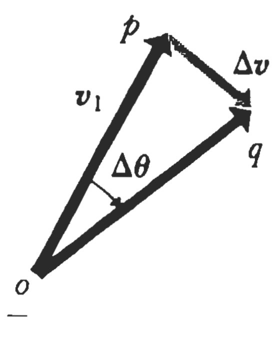


(a)

If Δt is small then

$$|\Delta\vec{v}| \simeq \underbrace{v\,\Delta\theta}$$

measures arc length but becomes exact as $\Delta t \to 0$.



$$a = \frac{|\Delta v|}{\Delta t} \sim v\left(\frac{\Delta\theta}{\Delta t}\right)$$

rad/s

What is $\left(\frac{\Delta\theta}{\Delta t}\right)$?

In a time Δt the particle moves a distance $(v\Delta t)$

But this must be equal to $(R\Delta\theta)$

Angle in radians

Radius of circle

$$\therefore\ R(\Delta\theta) = v(\Delta t)$$

$$\left(\frac{\Delta\theta}{\Delta t}\right) = \frac{v}{R} = \omega = \text{constant}$$

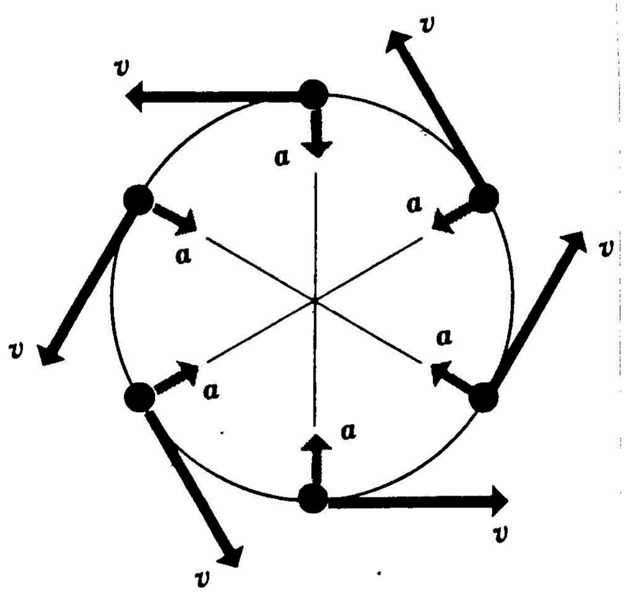
Angular velocity (rad/s)

$$\therefore\ a = v\left(\frac{v}{R}\right)$$

$$\boxed{a_\perp = \frac{v^2}{R} = \omega^2 R} \qquad m/s^2$$

Direction of a?

We had $\vec{a} = \dfrac{\Delta\vec{v}}{\Delta t}$



Velocity and acceleration vectors of a particle in uniform circular motion.

In the limit as $\Delta t \to 0$
$\Delta\theta \to 0$

$\Delta\vec{v}$ is $\perp$ $\vec{v}$

$\therefore$ $\vec{a}$ is $\perp$ velocity at every point.

Since the velocity vector is tangential, the acceleration points inward towards the center of the circle.

Sometimes called centripetal acceleration
$\Rightarrow$ center seeking

Can express magnitude of $\vec{a}$ in terms of the period τ of the motion.

$\tau \Rightarrow$ time for one revolution.

If particle travels once around the circle, a distance of $2\pi R$, in a time τ, its speed is

$$v = \frac{2\pi R}{\tau}$$

$$\therefore \quad a_\perp = \frac{4\pi^2 R}{\tau^2} \quad , \quad m/s^2$$

Example

Carnival Ride: Passengers move in a circle (horizontal) of radius 5.0m, making a complete circle in 40s. What is $a_\perp$?

$$v = \frac{2\pi R}{\tau} = \frac{2\pi(5.0)}{40} = 7.85\ m/s$$

$$a_\perp = \frac{v^2}{R} = \frac{7.85^2}{5.0} = 12.3\ m/s^2$$

$$\left(\frac{a_\perp}{g}\right) = 1.25 \text{ gees.}$$

Constant $\vec{a}$?

a) Free-Fall Problems

$\vec{a}$ is constant in <u>magnitude</u> and <u>direction</u>

$$\frac{d\vec{a}}{dt} \equiv 0. \quad \Rightarrow \quad \vec{a} \equiv \text{constant}$$

b) Uniform Circular Motion

$\vec{a}$ is constant in <u>magnitude</u> but not in <u>direction</u>

Direction is always towards the center and continously changes direction.

$$\frac{d\vec{a}}{dt} \neq 0. \quad \Rightarrow \quad \vec{a} \neq \text{constant}$$

<u>Note</u>: Cannot in this case use equations of motion developed for constant acceleration!!

Non Uniform Circular Motion

This discussion has assumed uniform circular motion or constant particle speed. If the speed varies, there will remain an $a_\perp$ as calculated but in addition there will <u>also</u> be a tangential component of acceleration:

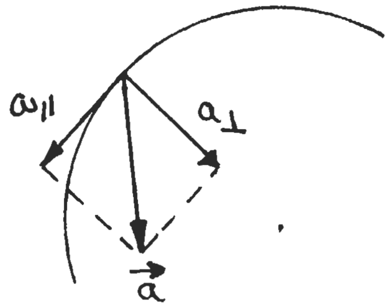
$$a_{\parallel} = \lim_{\Delta t \to 0} \frac{\Delta v_{\parallel}}{\Delta t} = \frac{dv_{\parallel}}{dt}$$

- related to change in magnitude of velocity

$$a_\perp = \frac{v^2}{R}$$

- related to change in direction of velocity

$$a = \sqrt{a_\perp^2 + a_\parallel^2} \quad \text{(magnitude)}$$



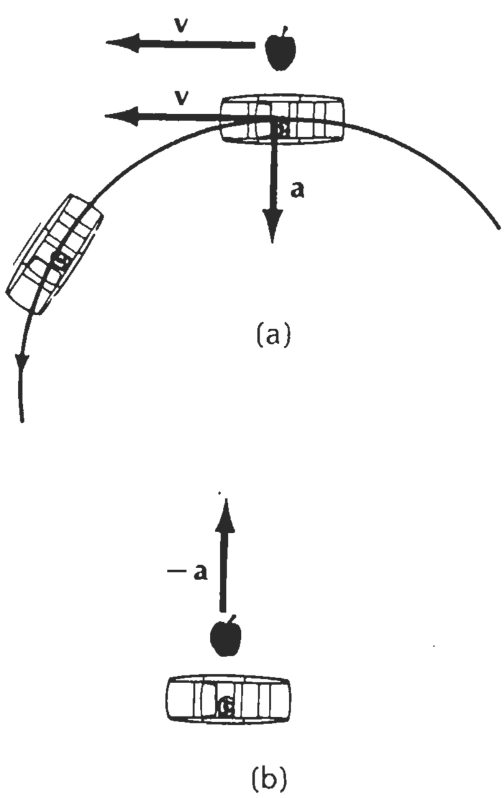
When object is first started to move in a circle need tangential acceleration to increase speed from zero to constant value, v.

<u>Centrifugal</u>

- Acceleration away from the center.

<u>Ground Reference Frame</u>

- Object released
- Moves in st. line rel. to lab.
- Automobile has centrepetal acceleration away from object.
  $a_\perp$

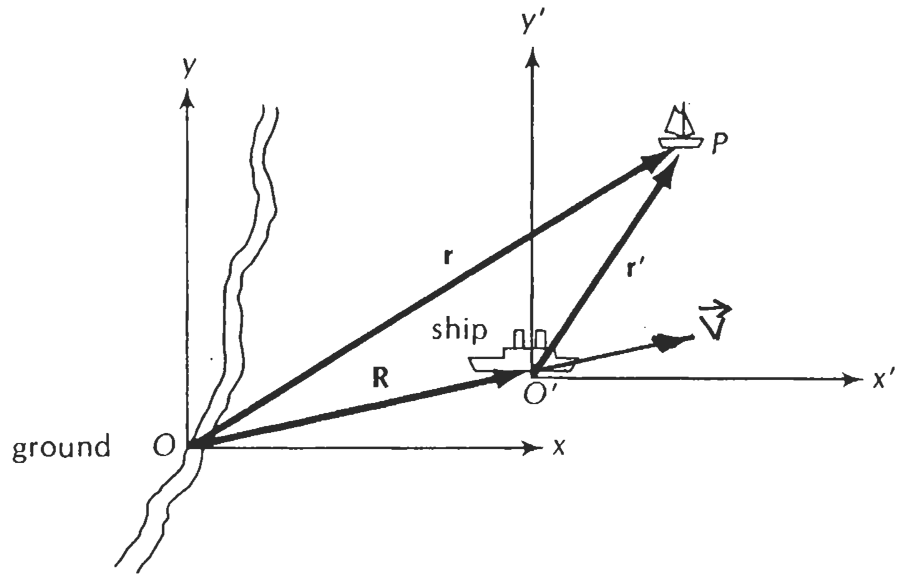


<u>Car Reference Frame</u>

- Object released
- Appears to accelerate away from car : $a_\perp$.
  - ⇒ centrifugal acceleration
  - ⇒ Fictitious forces as seen in accelerated frame of car ⇒ Non-Inertial Frame !!
    - ↳ Description of Mechanics is modified in such accelerated frames.

Relativity of Motion

- Motion is relative
  - velocity and acceleration depend on the frame of reference used for calculation.



Ground Reference Frame: x, y, z, t
Ship Reference Frame : x', y', z', t'

Since time is Absolute (Newtonian Mechanics)
$t = t'$ $v \ll c$

$\vec{r} \equiv$ position vector of sailboat relative to ground.
$\vec{r}' \equiv$ position vector of sailboat relative to ship

Assume:
- Coordinat systems coincide at $t = t' = 0$
- Ship coordinates moving with velocity $\vec{V}$ along $\vec{R}$

Then at any time t

$$\vec{r} = \vec{r}\,' + \vec{R} \qquad ①$$

$\vec{R}$ ← Ground frame

$\vec{r}\,'$ ← Measured in ship frame.

- can add because in Newtonian mechanics at low velocities, $v \ll c$, length is Absolute.
- same scales apply.

$$\vec{R} = \vec{V} t \qquad ②$$

$$\therefore \quad \vec{r} = \vec{r}\,' + \vec{V} t \qquad ③$$

$$\vec{r}\,' = \vec{r} - \vec{V} t \qquad ④$$

Special Case:

$\vec{V}$ along x-axis

$$\vec{r}\,' = \vec{r} - Vt\,\hat{i} \qquad ⑤$$

$$\left.\begin{aligned} x' &= x - Vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \quad \text{Galilean Transformation}$$

$v \ll c$ Length, Time ≡ Absolute

Velocities

Differentiate Eq ④

$$\vec{v}' = \frac{d\vec{r}'}{dt'} = \frac{d\vec{r}'}{dt} = \frac{d}{dt}(\vec{r} - \vec{V}t) = \frac{d\vec{r}}{dt} - \vec{V}$$

$$\therefore \quad \vec{v}' = \vec{v} - \vec{V} \quad \text{and} \quad \vec{v} = \vec{v}' + \vec{V}$$

vel. of ship rel. to ground

vel. of sailboat rel. to ground

- Velocity of sailboat relative to ship is the difference between the velocity of the sailboat relative to ground and velocity of ship relative to ground.

For $\vec{V}$ along x-axis.

$$\left.\begin{aligned} v_x' &= v_x - V \\ v_y' &= v_y \\ v_z' &= v_z \end{aligned}\right\} \text{Velocity Transformations}$$

Accelerations

$$\vec{a}' = \frac{d\vec{v}'}{dt'} = \frac{d\vec{v}'}{dt} = \frac{d}{dt}(\vec{v} - \vec{V})$$

Assume constant

$$\vec{a}' = \frac{d\vec{v}}{dt}$$

$$\vec{a}' = \vec{a}$$

Acceleration is absolute relative to inertial frames.

The accelerations are identical because the relative velocity, $\vec{V}$, of the two frames is a constant. If the relative velocities were not a constant then the accelerations would differ by

$$\left(\frac{d\vec{V}}{dt}\right)$$

Relative Velocity

Relative velocities are used to describe the motion of reference frames with respect to each other. Easy to make a mistake by adding or subtracting wrong velocity.

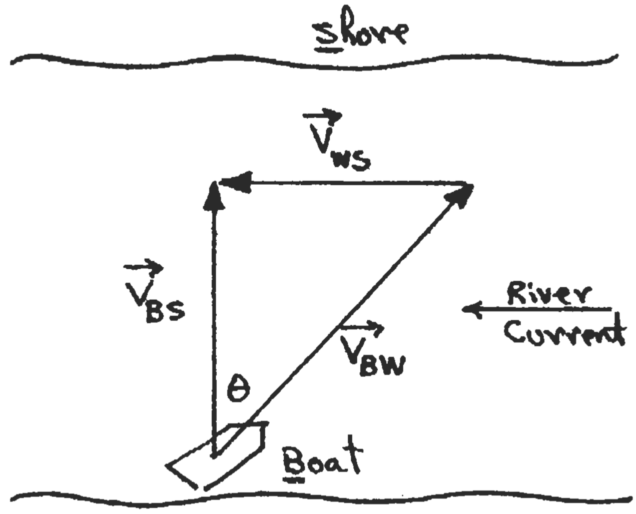
Problem-Solving Strategy:

- Label all velocities carefully.
- Use two subscripts to label all velocities.
- First subscript refers to object.
- Second subscript refers to reference frame where it has a given velocity.
  i.e. $v_{AB} \rightarrow$ velocity of A relative to B.
- In Eq. involving velocities, the first subscript on the LHS of the Eq. should be the same as the first subscript in the first term on the RHS and the second on the LHS is the same as the second in the last term on the RHS. Adjacent subscripts in adjacent terms must match.

Note: For any two objects or reference frames A and B, the vel. of A relative to B has the same magnitude but opposite direction as the velocity of B relative to A.

$$\vec{V}_{BA} = -\vec{V}_{AB}$$

Example



$$\therefore \quad \vec{V}_{BS} = \vec{V}_{BW} + \vec{V}_{WS}$$

same

$\vec{V}_{BW}$ = velocity of Boat relative to Water

$\vec{V}_{BS}$ = velocity of Boat relative to Shore

$\vec{V}_{WS}$ = velocity of Water relative to Shore

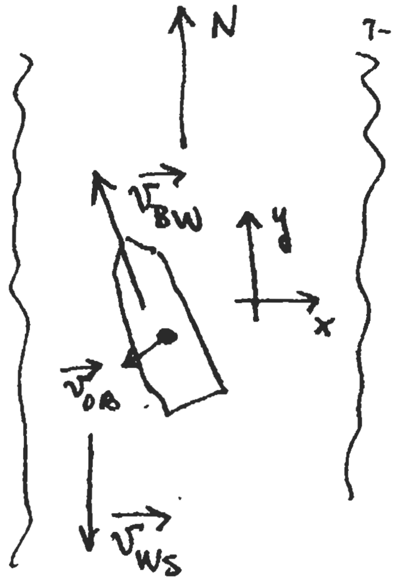
V_{BW} = 20 km/h : Boat speed in still water.
V_{WS} = 12 km/h : River current rel. to shore.

What is proper heading for boat to travel straight across the river?
Head in such a direction θ, that

$$\sin\theta = \frac{V_{WS}}{V_{BW}} = \frac{12}{20} = 0.60$$

$$\theta = 36.9^\circ \quad \text{(upstream)}$$

Example : Relative Velocities



Hydrofoil: 30° west-of-North at 30 m/s relative to water

Water: south at 5 m/s relative to shore

Object: 30° south-of-west at 6 m/s relative to boat.

What is velocity of object relative to river banks (shore)?

$$\vec{v}_{OS} = \vec{v}_{OB} + \vec{v}_{BW} + \vec{v}_{WS}$$

$$\vec{v}_{OB} = 6\left[-\cos 30^\circ\,\hat{i} - \sin 30^\circ\,\hat{j}\right]$$

$$\vec{v}_{BW} = 30\left[-\sin 30^\circ\,\hat{i} + \cos 30^\circ\,\hat{j}\right]$$

$$\vec{v}_{WS} = -5\,\hat{y}$$

$$\vec{v}_{OS} = (-6\cos 30^\circ - 30\sin 30^\circ)\,\hat{i} + (-6\sin 30^\circ + 30\cos 30^\circ - 5)\,\hat{j}$$

$= -20.2\,\hat{i} +$ [redacted] $\hat{j}$ [41.7° N-of-W]

$|\vec{v}_{OS}| = 27.05$ m/s