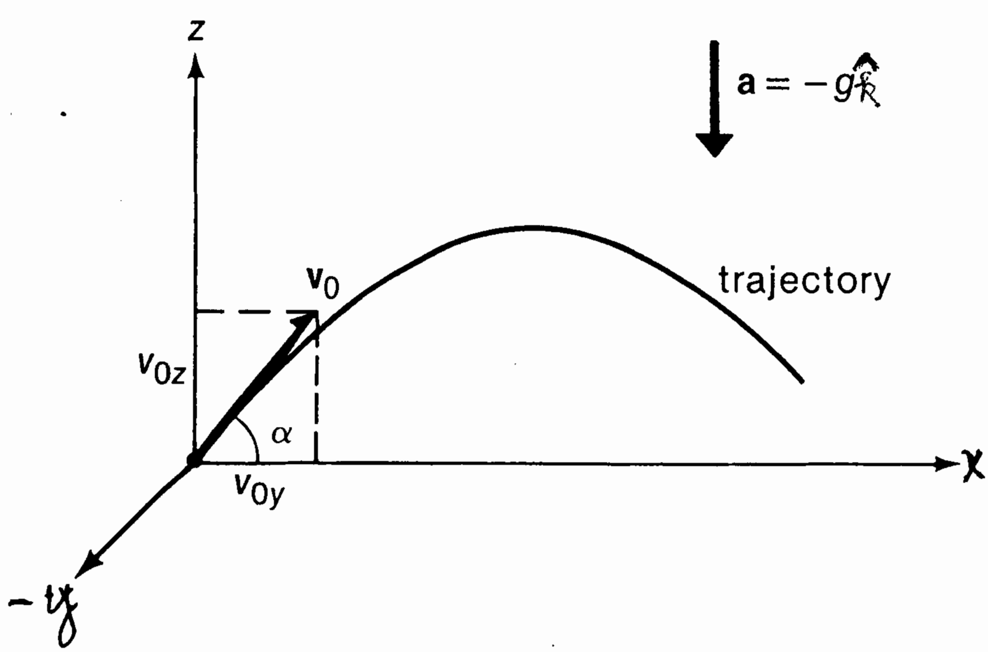
Motion of Projectiles

- Study of motion of a body which is given some initial velocity and starts from some initial position and follows a path determined by the effect of the gravitational acceleration and by air resistance. Projectile path is called its trajectory.
- Freely falling body near the earth's surface experiences downward acceleration with

$$g = 9.81 \text{ m/s}^2$$

Ideal Model:

- Trajectories of short range so that $\vec{g}$, magnitude and direction are constant.
- Neglect air resistance
- Neglect effects due to rotation of earth
- Constant acceleration motion !!



Assume

- Motion is in the xz-plane
- z-axis is in the direction of the upward vertical
- x-axis is in the direction of the horizontal velocity

$$a_x = 0$$
$$a_y = 0$$
$$a_z = -g = -9.81\ m/s^2 \quad [\text{acceleration opposite to } +z]$$

$$v_{oy} = 0$$

<u>Equations of Motion</u>:

$$x(t) = x_0 + v_{ox}t \qquad v_x(t) = v_{ox}$$
$$y(t) = 0 \qquad v_y(t) = 0$$
$$z(t) = z_0 + v_{oz}t - \frac{1}{2}gt^2 \qquad v_z(t) = v_{oz} - gt$$

- Motions are decoupled. Motion along each axis is independent of motions along other axes. (Experimental Fact). Can treat them separately.
- Chosen coordinates such that y, v_{oy} and a_y are initially zero and remain that way.
- 2-D motion.
- 1-D Vertical motion is a special case.

$x(t)$, $y(t)$, and $z(t)$ give complete <u>trajectory</u> of particle as a <u>function of time</u>. i.e. World lines along each axis.

What is the mathematical form of a ballistic trajectory?

$$x(t) = x_0 + v_{0x} t$$

Solve for t:

$$t = \frac{x - x_0}{v_{0x}}$$

Substitute for t in $z(t)$

$$z = z_0 + v_{0z}\left(\frac{x - x_0}{v_{0x}}\right) - \frac{1}{2} g \left(\frac{x - x_0}{v_{0x}}\right)^2$$

This can be written as

$$z = A + Bx + Cx^2$$

A, B, C constants

<u>"Eq. of a Parabola"</u>

Ballistic Motion

- Missile
- Bullet
- Ball
- Bomb

$\left.\right\}$ Neglect air resistance.

- Independent of the initial conditions the trajectory will be part of a parabola.

Calculate:

- Maximum Height
- Time-of-Flight
- Range

Assume:

$z(0) = 0$, $x(0) = 0$ } choose coordinates so particle at origin at $t=0$.

$v_z(0) = v_{z0}$ - Initial vertical velocity component

$v_x(0) = v_{x0}$ - Initial horizontal velocity component

- Motion in the xz-plane.

When projectile is at maximum height, $v_z = 0$. It is moving horizontally.

$$\therefore \quad 0 = v_{0z} - g t_{max}$$

$$t_{max} = \left(\frac{v_{0z}}{g}\right) \qquad \text{[Time to maximum height]}$$

$$z(t_{max}) = z_{max} = v_{0z} t_{max} - \frac{1}{2} g t_{max}^2$$

$$= v_{0z}\left(\frac{v_{0z}}{g}\right) - \frac{1}{2} g \left(\frac{v_{0z}}{g}\right)^2$$

$$z_{max} = \frac{1}{2} \frac{v_{0z}^2}{g}$$

What is the Range?

$$0 = v_{0z} t - \frac{1}{2} g t^2 \qquad [z = 0 \text{ at impact}]$$

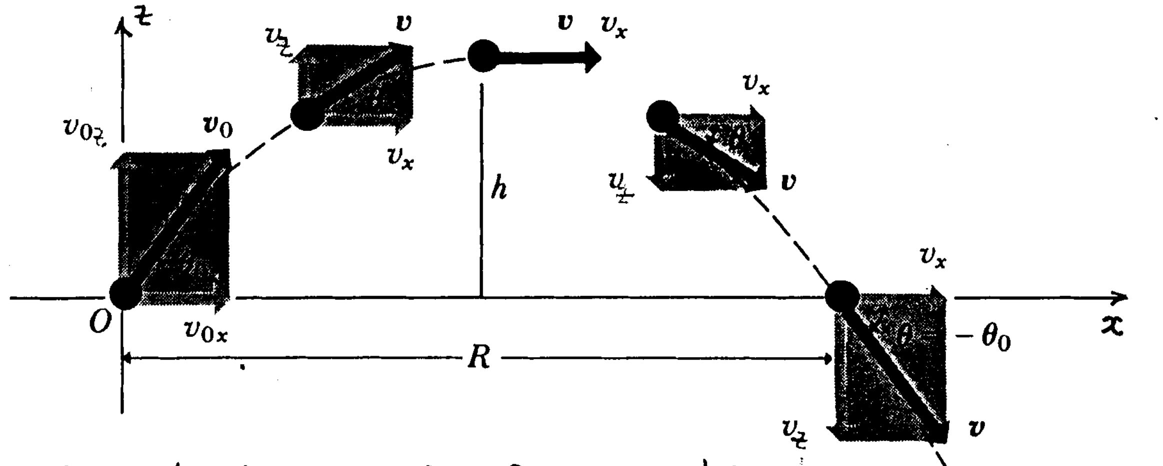
solving $\quad t = 0 \quad \leftarrow$ Starting Time

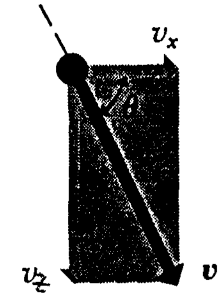
$\quad t_F = \dfrac{2 v_{0z}}{g} \quad \leftarrow$ Flight Time $= 2 t_{max}$

Range = Horizontal Velocity x Flight Time

$$R = v_{0x} t_F$$

$$= \frac{2 v_{0x} v_{0z}}{g}$$





What is the velocity, v, of the projectile at any point in the flight?

$$\vec{v}(t) = v_x(t)\,\hat{i} + v_z(t)\,\hat{k}$$

$$|v| = \sqrt{v_x^2 + v_z^2} \qquad \text{[Magnitude]}$$

$$\tan\theta = \frac{v_z(t)}{v_x(t)} \qquad \text{[Direction]}$$

__$\vec{v}$ at Maximum Height:__

$$v_x(t) = v_{0x} \qquad \text{[A constant during motion]}$$

$$v_z(t = t_{max}) = v_{0z} - g\left(\frac{v_{0z}}{g}\right)$$

$$= 0 \qquad \text{[Trajectory is horizontal]}$$

$$\vec{v} = v_{0x}\,\hat{i}$$

__$\vec{v}$ at Maximum Distance = Range:__

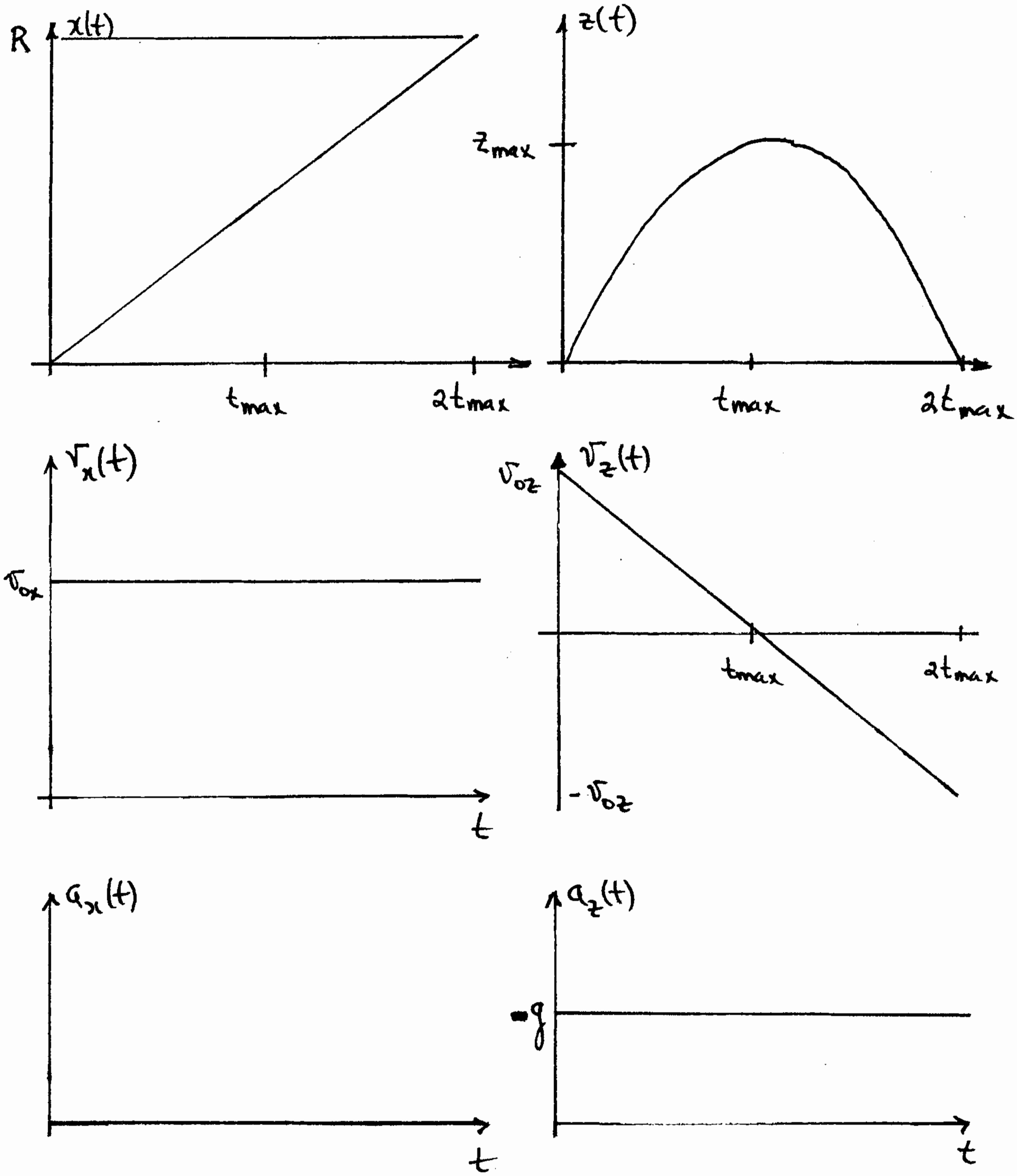
$$v_x(t) = v_{0x}$$

$$v_z\left(t = \frac{2v_{0z}}{g}\right) = v_{0z} - g\left(\frac{2v_{0z}}{g}\right) = -v_{0z}$$

$$\vec{v} = v_{0x}\,\hat{i} - v_{0z}\,\hat{j}$$

$|v| \equiv$ same as at the origin at $t=0$.

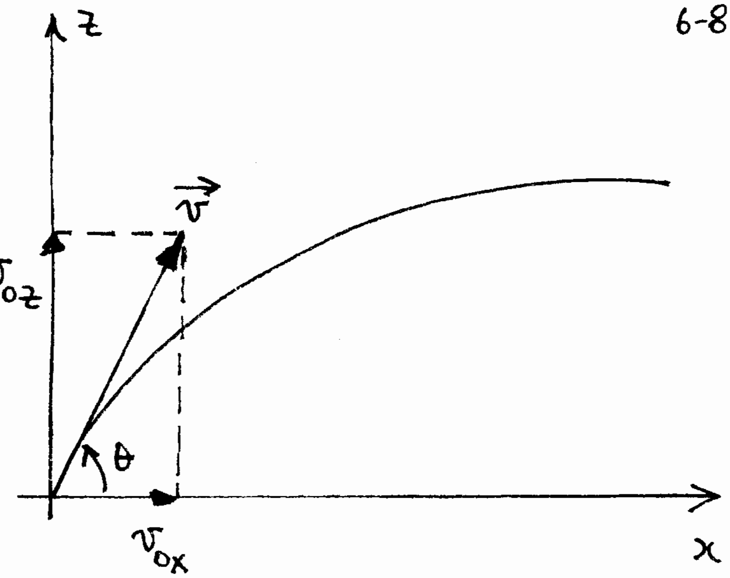
Direction $\theta = -\theta_0$ instead of $+\theta_0$ at $t=0$



Where is $|v|$ the least during the motion?

Example

- Projectile launched
  - Elevation angle θ
  - Initial speed v_0



$$v_{ox} = v_0 \cos\theta$$
$$v_{oz} = v_0 \sin\theta$$

Ballistic Eq. of Motion:

$$x(t) = (v_0 \cos\theta)\, t + x_0$$
$$z(t) = z_0 + (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$$

$$v_x(t) = \frac{dx}{dt} = v_0 \cos\theta$$ [constant]

$$v_z(t) = \frac{dz}{dt} = v_0 \sin\theta - gt$$ [varies in flight]

$$z_{max} = \frac{v_0^{\,2} \sin^2\theta}{2g}$$ Maximum Height

$$t_{Flight} = \frac{2 v_0 \sin\theta}{g}$$ Flight - Time

$$x_{max} = \frac{2 v_0^{\,2} \sin\theta \cos\theta}{g} = \frac{v_0^{\,2} \sin 2\theta}{g}$$ Range

- All these specific results (height, time, range) apply only if launch and impact points are at the same height, z.
- Special cases must be treated carefully.

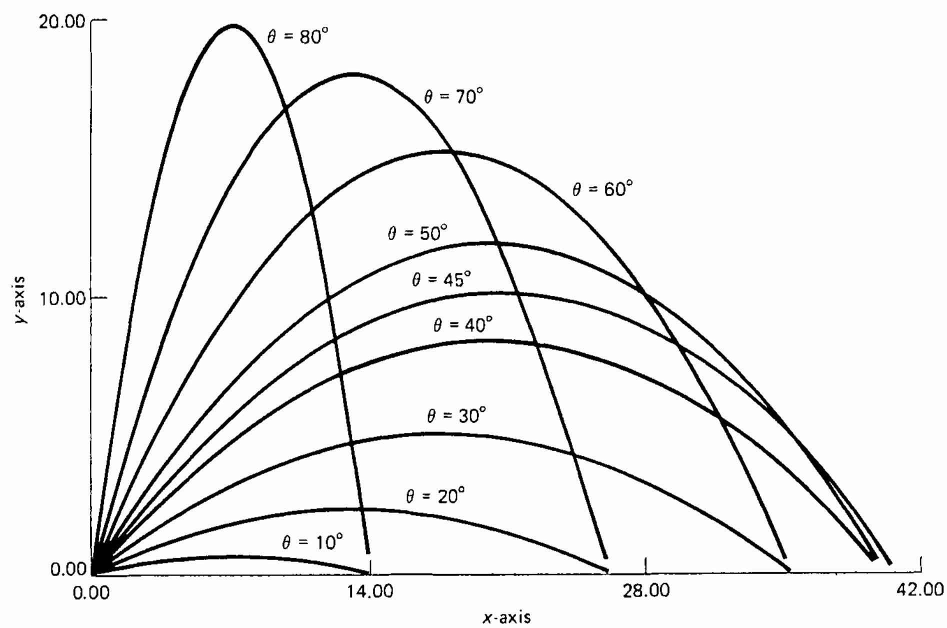
What angle θ gives maximum range?

$$R = \frac{v_0^2 \sin 2\theta}{g}.$$

Max value of $\sin 2\theta = 1$

$$2\theta = 90^\circ$$

$$\theta = 45^\circ$$



Figure

These parabolic trajectories are for the same initial speed (20 m/s) and launch angles at 10° intervals.

Calculus: $\frac{dx_{max}}{d\theta} = \frac{v_0^2}{g} 2 \cos 2\theta = 0 \implies \theta = 45^\circ$

Look at $d^2x/d\theta^2$ to see if max. or min.

Example

What is the range of a 22-calibre rifle bullet?

$v_0 \approxeq 330\ \mathrm{m/s}$ [Expt. in class]

$\theta = 45°$ [Maximizes range]

$$x_{max} = \frac{v_0^2 \sin 2\theta}{g} = \frac{330^2 \times 1}{9.81} = 11.1\ \mathrm{km}.$$

- Air resistance neglected. Poor assumption for a light high speed projectile.

Example: Baseball

$v_0 \sim 90\ \mathrm{mi/h}$ [60 mph = 88 ft/s]

$\sim \frac{90}{60} \times 88 = 132\ \mathrm{ft/s}.$

$\theta = 45°$

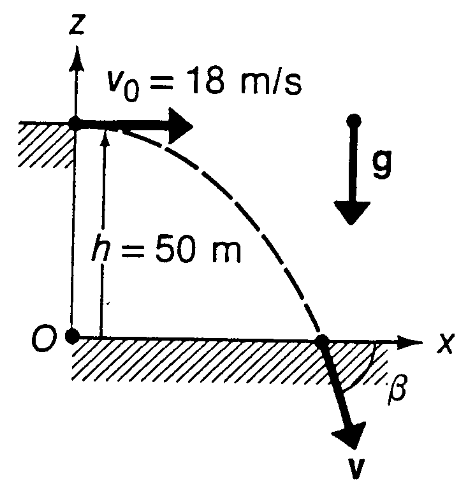
$$x_{max} = \frac{v_0^2 \sin 2\theta}{g} = \frac{132^2 \times 1}{32.2} = 541\ \mathrm{ft}.$$

- Neglects air resistance.

Example

Ball kicked horizontally at 18 m/s off a 50 m high cliff.

- Time to impact?
- Speed at impact?
- Impact point?
- Angle at impact?



$x(t) = v_0 \cos\theta \, t$

$= v_0 t$ ① $\quad \theta = 0°$

$z(t) = z_0 + v_{0z} t - \frac{1}{2} g t^2$ ②

$= H - \frac{1}{2} g t^2$ ② $\quad v_{0z} = 0, \; z_0 = H$ (see coord. system)

At impact we must have $z = 0$.

Solving Eq.② $\quad T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 50}{9.81}} = 3.19$ s.

$x(T) = 18 \times 3.19 = 57.42$ [Eq. ①]

$v_x(T) = \frac{dx}{dt} = v_0 = 18$ m/s [Independent of Time]

$v_z(T) = \frac{dz}{dt} = -gT = -9.81 \times 3.19 = -31.26$ m/s.

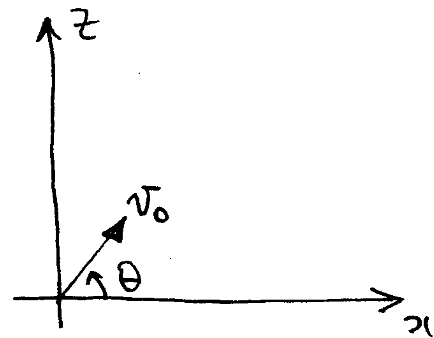
$\tan\beta = \frac{v_z}{v_x} = -\frac{31.26}{18.0} \quad \beta = -60.1°$

$|v| = \sqrt{v_x^2 + v_z^2} = \sqrt{18^2 + 31.26^2} = 36.1$ m/s [Speed]

Projectile Problem-Solving-Strategy

$$x(t) = x_0 + (v_0 \cos\theta_0)t$$
$$z(t) = z_0 + (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

$$v_x(t) = v_0 \cos\theta_0$$
$$v_z(t) = v_0 \sin\theta_0 - gt$$

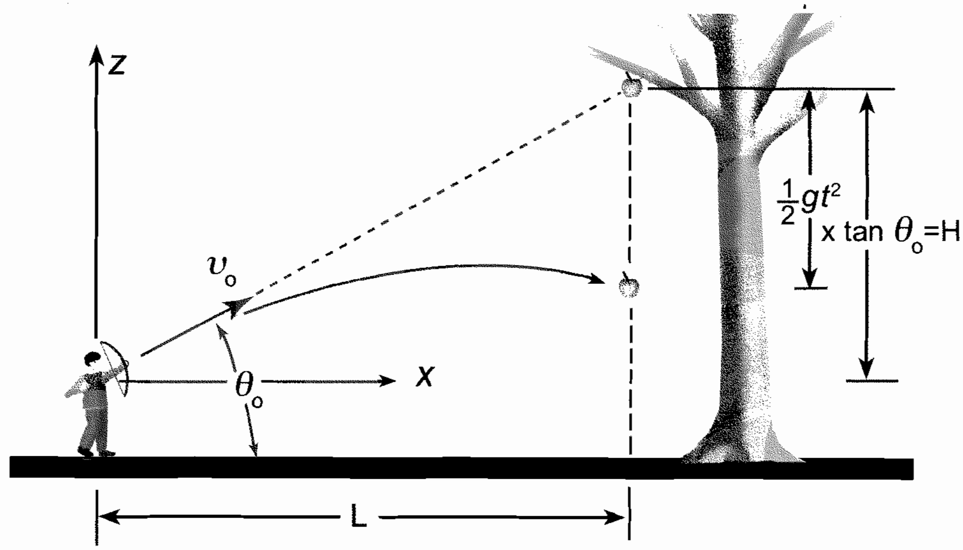


1. Define your coordinate axes. Sketch axes, label positive directions and show location of origin.
2. List known and unknown quantities. In some problems the initial velocity is given (mag. + dir.) and using. Eq. o Motion you can find coord. and velocity at any other time. In other problems you may know two points on the trajectory and asked to determine the initial velocity. Know what is given and what is unknown!!
3. Learn how to go from prose into symbols.
   When → Time, t
   Where → x, z
   Velocity → v_x, v_z, etc.
4. Have a picture of ballistic trajectory in mind.
   Highest point $v_z = 0$. → calculate t.
   Range $z = 0$ [or some fixed value] calculate t.

5. Take a hard look at your results. Do they seem to make sense? Are they within the general range of magnitudes you expected? Is the sign correct? Are the resulting units correct?

Example

An arrow is fired at an apple at the same instant it drops from the tree. How must the arrow be aimed so as to hit the apple?



Trajectory of an arrow shot directly at a freely falling apple.

$\left.\begin{matrix} x_D(t) \\ z_D(t) \end{matrix}\right\}$ Trajectory of dart. $\qquad \vec{r}_D = x_D(t)\,\hat{i} + z_D(t)\,\hat{k}$

$\left.\begin{matrix} x_A(t) \\ z_A(t) \end{matrix}\right\}$ Trajectory of apple. $\qquad \vec{r}_A = L\,\hat{i} + z_A(t)\,\hat{k}$

Let impact occur at time $t = T$

Then for dart to strike apple we must have

$$\left.\begin{aligned} z_D(T) &= z_A(T) \\ x_D(T) &= x_A(T) \end{aligned}\right\} \text{Conditions}$$

Eq's of motion

Apple:

$$x_A(t) = L \qquad \text{[Constant x-location]}$$
$$z_A(t) = H - \frac{1}{2}gt^2$$

Dart:

$$x_D(t) = v_{0x}t$$
$$z_D(t) = v_{0z}t - \frac{1}{2}gt^2$$

At time $t = T$ we must have that

$$L = v_{0x}T \qquad \text{[x-motion]}$$
$$\therefore\ T = \frac{L}{v_{0x}} \qquad \text{[Time to impact]}$$

Also $v_{0z}T - \frac{1}{2}gT^2 = H - \frac{1}{2}gT^2$ [z-motion]

substituting for T:

$$v_{0z}\frac{L}{v_{0x}} = H$$

$$\boxed{\frac{v_{0z}}{v_{0x}} = \frac{H}{L}}$$

$$v_{0x} = v_0 \cos\theta$$
$$v_{0z} = v_0 \sin\theta$$

$$\therefore\ \frac{H}{L} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

- Aiming is independent of v_0! However v_0 must be great enough to reach target range.
- Angle θ represents sighting at target before it starts to fall.