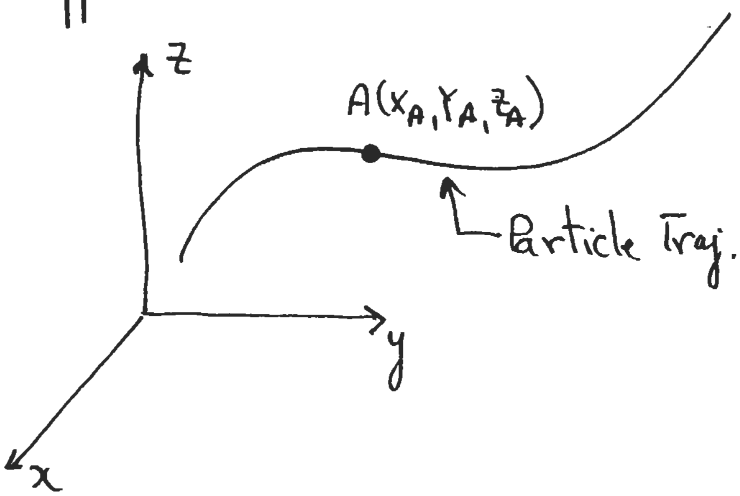
Vectors

Vectors are mathematical quantities that are useful to describe physical quantities that have both a magnitude and direction associated with them.

We will review the operations of vector addition, subtraction and multiplication by a number. We will show two types of vector multiplication.

We will deal with vectors in two different ways:

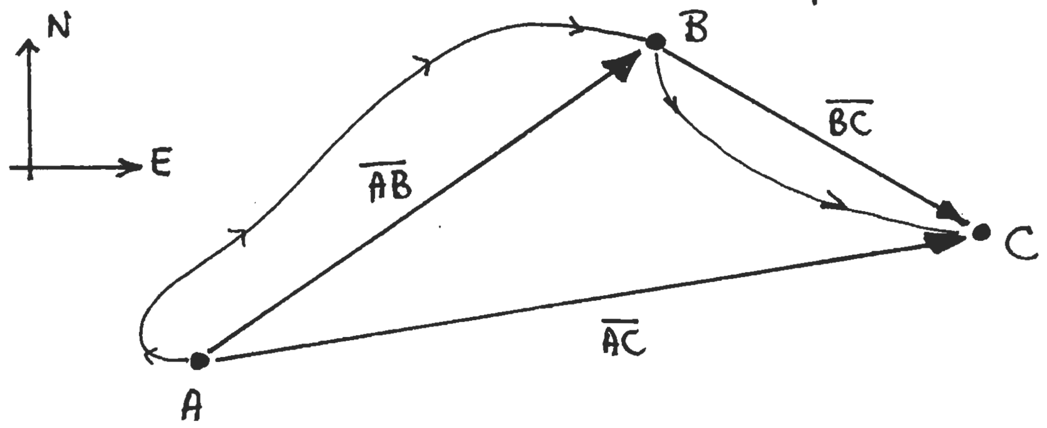
- Geometric approach
- Algebraic approach



- Want to describe motion of an ideal particle
- Choose a coordinate system - e.g. Cartesian x, y, z
- Want to describe the motion in a general way.
- Note that the "laws of nature" must be invariant to the choice of an inertial (non-accelerated) coordinate system.

Displacement

- Consider two successive displacements



- Start at A and move to B
- The directed line segment from A to B is called a displacement vector. Arrow indicates direction of motion.
- Line does not represent actual path followed just the final result.
- Move from B → C.
- $\overline{AC}$ represents result of both moves. Net direction ; Net displacement.

- "Addition of Vectors"

<u>Vector</u>: Any quantity that has a <u>magnitude</u> and <u>direction</u> and behaves like the displacement vector.

e.g.
- displacement
- velocity
- acceleration
- force
- momentum
- torque

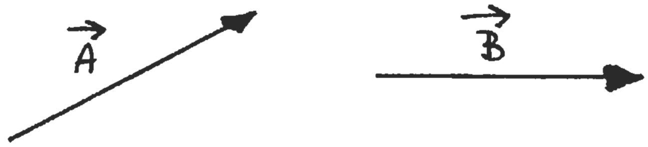
Scalar: Any quantity that has a magnitude but no direction.

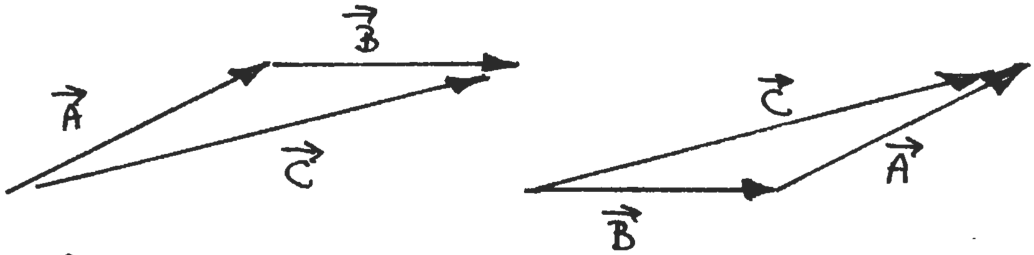
e.g.
- length
- time
- mass
- area
- volume
- density
- energy
- temperature

$\vec{A} = \vec{B}$

- only if $|A| = |B|$ magnitude
- only if directions are equal
- location and starting point do not matter.
- units must be the same.

Vector Addition





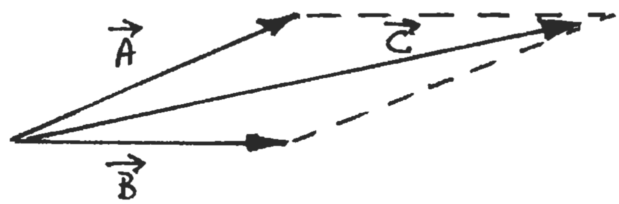
$\vec{C} = \vec{A} + \vec{B}$ $\qquad$ $\vec{C} = \vec{B} + \vec{A}$

$$\therefore \quad \vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

↳ commutative Law

- order of addition does not matter!
- addition makes sense only for same kinds of vectors.

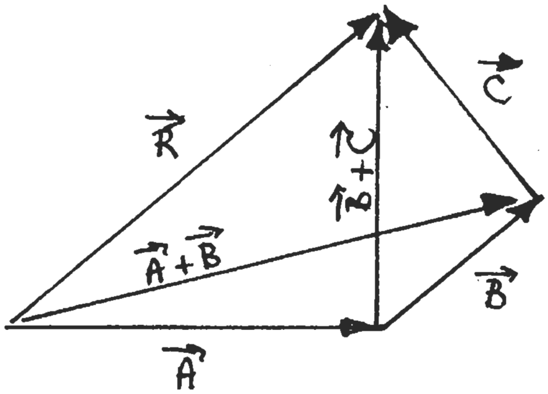
Parallelogram:



3-or More Vectors:

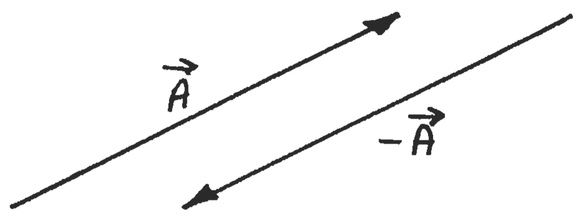
$\vec{R} = (\vec{A} + \vec{B}) + \vec{C}$

$\quad = \vec{A} + (\vec{B} + \vec{C})$



Negative of Vector

- same magnitude
- opposite direction

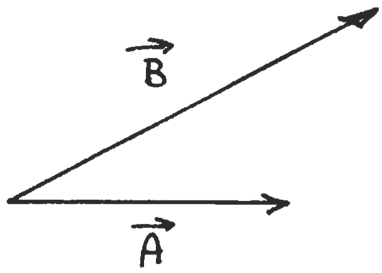


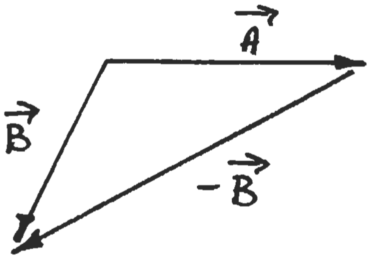
$\vec{A} + (-\vec{A}) = 0$

Subtraction

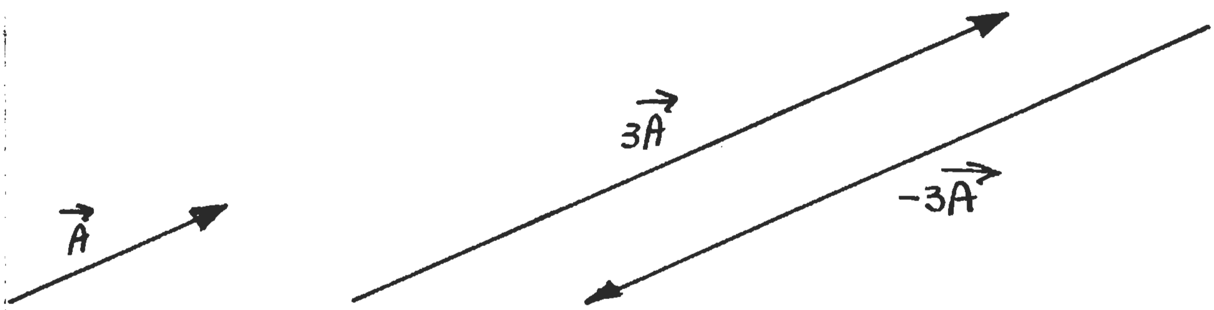
- The subtraction of two vectors $\vec{A}$ and $\vec{B}$ is defined as the sum of $\vec{A}$ and $-\vec{B}$.

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$





Scalar × Vector



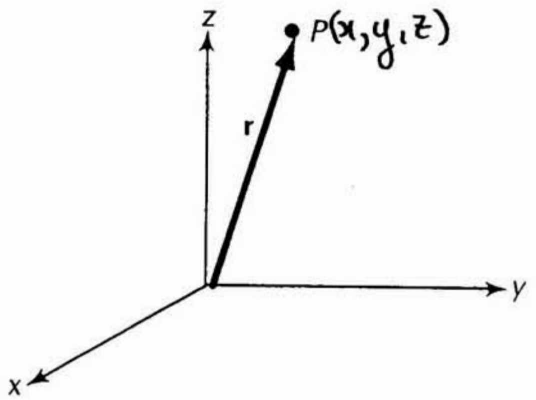
- magnitude multiplied by scalar.

same direction

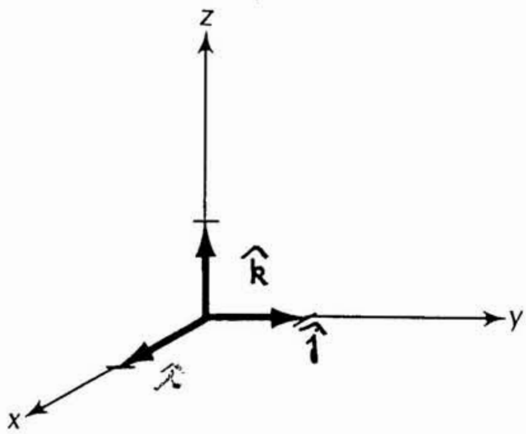
ex: $\vec{F} = m\vec{a}$ different physical quantities: m has own units.

Vector Components

- A vector is completely described by its components.
- Useful for vector algebra
- choose a coordinate system
- choose an origin at foot-of-vector



The position vector r of the point P.



The unit vectors.

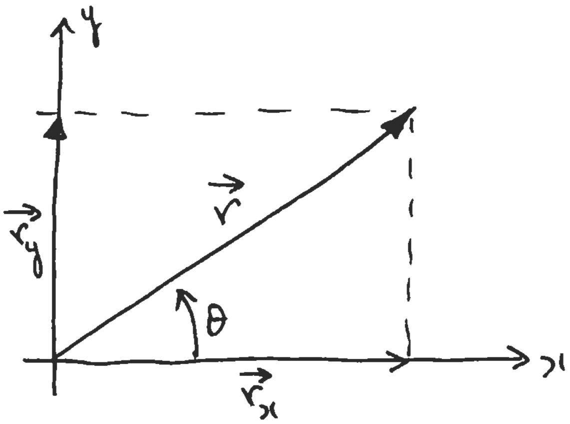
- $P(x,y,z)$ is an arbitrary point with coordinates x, y, z.
- $\vec{r}$ is a position vector from the origin to the point x, y, z.

Unit Vectors

$\hat{i}$, $\hat{j}$ and $\hat{k}$ are three vectors of magnitude $= 1$ unit (dimensionless) pointing along the coordinate axes.

- Other vectors can be written in terms of them
- Called unit vectors.
- Carry no units

Components - 2 dimensions

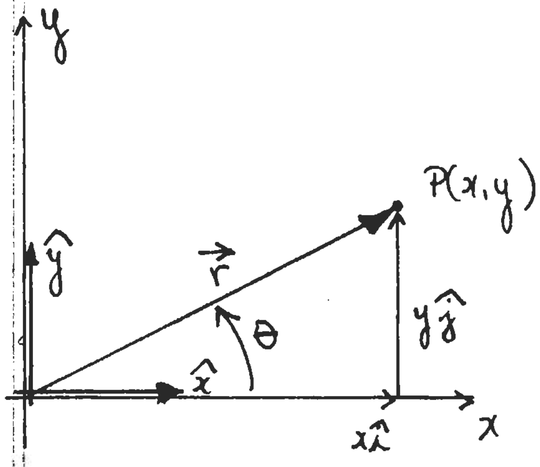


$$\vec{r} = \vec{r}_x + \vec{r}_y$$

$$r_x = r\cos\theta$$

$$r_y = r\sin\theta$$

- components can be used instead of vector itself.



$$\vec{r} = x\hat{i} + y\hat{j}$$

$$r = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

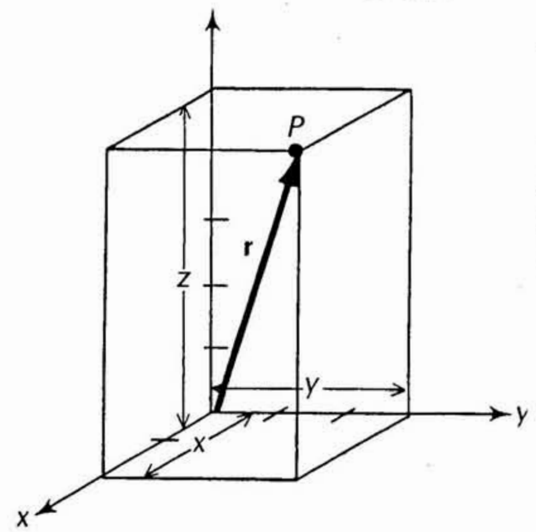
$$\tan\theta = \frac{y}{x}$$

What is

$$x\hat{i} + y\hat{j} + z\hat{k} = ?$$

- Move a distance x-units along x-axis
  y-units ✓ y-axis
  z-units ✓ z-axis

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



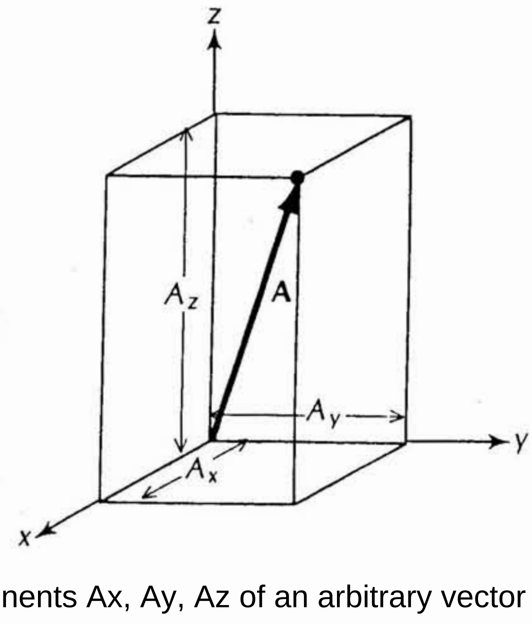
The components x, y, z of the position vector r.

Arbitrary Vector $\vec{A}$

- Coordinate system at the foot of vector $\vec{A}$.
- Drop perpindiculars from the tip to each of the coordinate axes.
- Intercepts give the components with values (numerical) A_x, A_y, A_z.

In terms of unit vectors

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

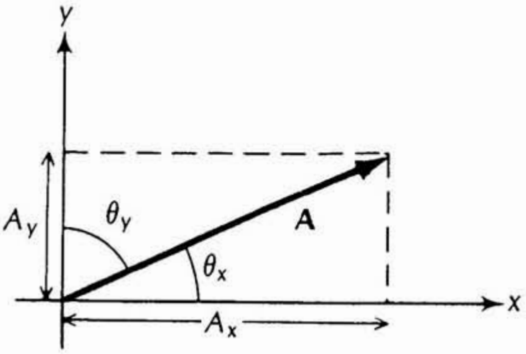


The components Ax, Ay, Az of an arbitrary vector A.

Example : 2-Dimensions

$A_x = A \cos \theta_x$

$A_y = A \sin \theta_x$



Vector A in two dimensions and its components.

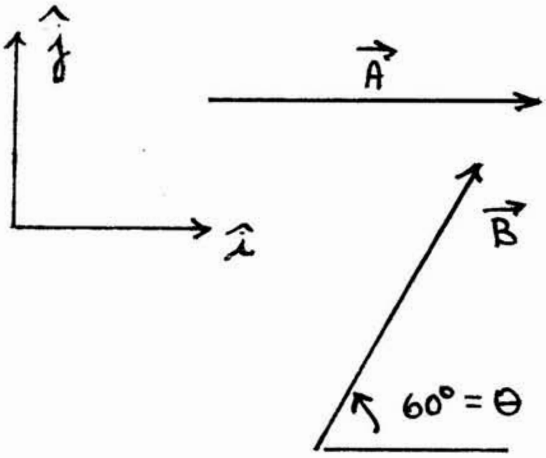
Magnitude of $\vec{A}$:

$$|A| = A = \sqrt{A_x^2 + A_y^2}$$

$$\vec{A} = A_x \hat{\imath} + A_y \hat{\jmath}$$

$$= A \cos\theta_x \hat{\imath} + A \sin \theta_x \hat{\jmath}$$

Example

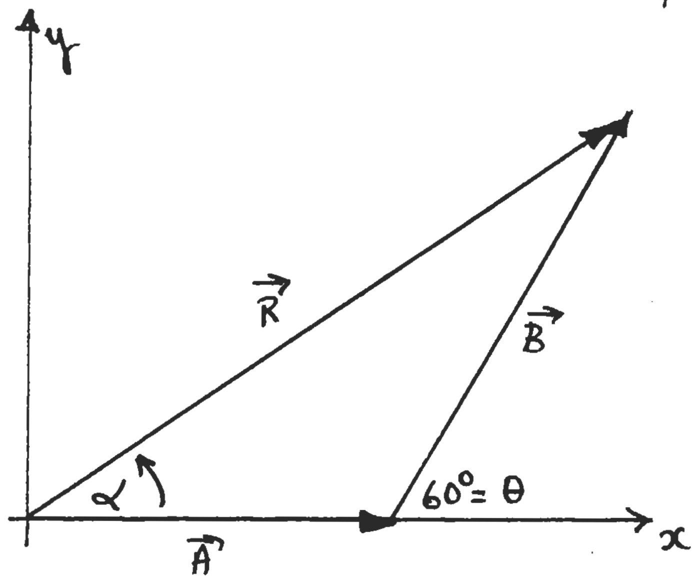


Want: $\vec{A} + \vec{B} = ?$

$$\left.\begin{array}{l}|A| = 3 \\ |B| = 4\end{array}\right\} \text{Given magnitudes}$$

- Select most convenient coordinate system
- choose one axis along one of the vectors.

Want:
Magnitude of $\vec{R}$
Direction of $\vec{R}$



$$\vec{A} = A_x \hat{i} + A_y \hat{j} = A\hat{i} + 0\hat{j}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} = B\cos 60° \hat{i} + B\sin 60° \hat{j}$$

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

$$= (A + B\cos 60°)\hat{i} + (0 + B\sin 60°)\hat{j}$$

$$= (3 + 4\cos 60°)\hat{i} + (4\sin 60°)\hat{j}$$

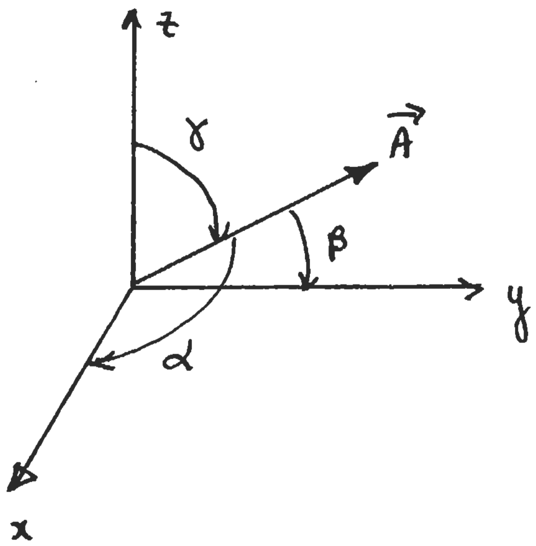
$$\vec{R} = 5\hat{i} + 3.46\hat{j}$$

$$|R| = \sqrt{R_x^2 + R_y^2} = \sqrt{5^2 + 3.46^2} = 6.08$$

$$\alpha = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{3.46}{5}$$

$$= 34.7°$$

<u>Example</u> - General 3-D Vector



$A_x = A\cos\alpha$
$A_y = A\cos\beta$
$A_z = A\cos\gamma$

$\alpha, \beta, \gamma \Rightarrow$ direction cosines

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
(not all independent)

Magnitude: $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$\cos\alpha = \frac{A_x}{A}$, etc.

Position vector $\vec{r}$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

distance from (x, y, z) to origin.

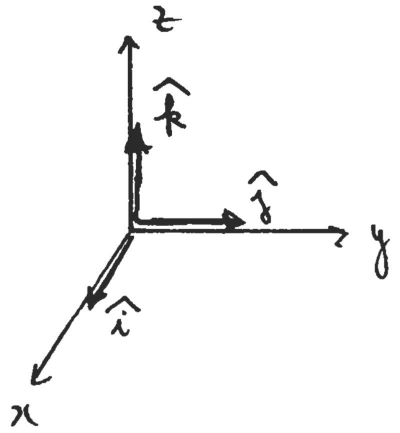
$$\vec{A} = A\cos\alpha\,\hat{i} + A\cos\beta\,\hat{j} + A\cos\gamma\,\hat{k}$$

Let
$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$
$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}$$

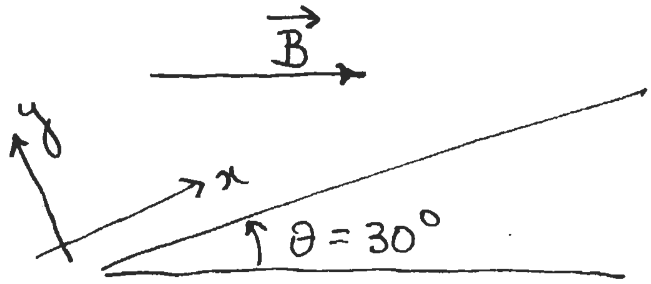
- Generalize to any number of vectors added or subtracted from each other.

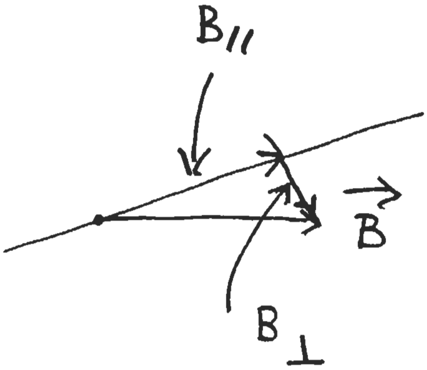


$$\left.\begin{array}{l} \hat{x} \leftrightarrow \hat{i} \\ \hat{y} \leftrightarrow \hat{j} \\ \hat{z} \leftrightarrow \hat{k} \end{array}\right\} \text{Alternate Notation}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

Inclined Plane





Decompose vector $\vec{B}$:

$B_{\parallel}$: Parallel to x.

$B_{\perp}$: Perpendicular to x; i.e. along $-y$.

$$\vec{B} = \vec{B_{\parallel}} + \vec{B_{\perp}}$$

$$= B_{\parallel}\,\hat{x} + B_{\perp}\,\hat{y}$$

$$B_{\parallel} = |\vec{B}| \cos 30°$$

$$B_{\perp} = -|\vec{B}| \sin 30°$$

Vector Multiplication

- Several ways to multiply vectors
- Need to take into account <u>magnitude</u> and <u>direction</u>

1. Dot Product

- scalar product
- inner product

The vectors A and B and the angle between them.

Vectors: $\vec{A}$ and $\vec{B}$

$$\vec{A} \cdot \vec{B} = A\,B\,\cos\phi \qquad \phi \le 180^\circ$$

– result is a scalar, number

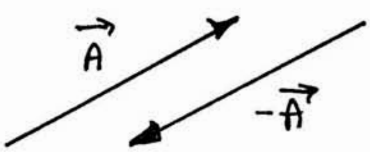
$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \quad \text{(commutative)}$$

$$\vec{A} \cdot \vec{A} = A\,A\cos 0^\circ$$

$= A^2$ [parallel vectors]

$= (\text{magnitude})^2$

$$\vec{A} \cdot (-\vec{A}) = -A^2$$



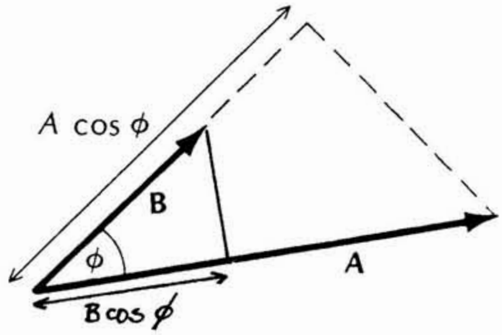
[antiparallel vectors]

$\vec{A} \cdot \vec{B} = A\,(B\cos\varphi)$

↑ Proj. of $\vec{B}$ on $\vec{A}$

$= B\,(A\cos\varphi)$

↑ Proj. of $\vec{A}$ on $\vec{B}$



The component of A along B is A cos ϕ
the component of B along A is B cos ϕ.

$0 < \varphi < 90^\circ \qquad (\vec{A} \cdot \vec{B}) > 0$

$90 < \varphi < 180^\circ \qquad (\vec{A} \cdot \vec{B}) < 0$

$\phi = 90^\circ \qquad (\vec{A} \cdot \vec{B}) = 0$

- $\vec{A}$ and $\vec{B}$ are $\perp$ to each other
- A good test for $\perp$ vectors.

$$\left.\begin{aligned} \hat{i} \cdot \hat{i} &= 1 \\ \hat{j} \cdot \hat{j} &= 1 \\ \hat{k} \cdot \hat{k} &= 1 \end{aligned}\right\} \text{Parallel unit vectors}$$

$$\left.\begin{aligned} \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} &= 0 \\ \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} &= 0 \\ \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} &= 0 \end{aligned}\right\} \text{Perpindicular unit vectors}$$

Distributive Law

To work out product of $\vec{A}\cdot\vec{B}$ we make use of the fact that vector multiplication obeys the 'distributive law':

$$(\vec{C}+\vec{D})\cdot\vec{E} = \vec{C}\cdot\vec{E} + \vec{D}\cdot\vec{E}$$

Proof: Product is the magnitude of $\vec{E}$ times component of $\vec{C}+\vec{D}$ along $\vec{E}$. But the component of $\vec{C}+\vec{D}$ along any direction is equal to the sum of the component of $\vec{C}$ plus the component of $\vec{D}$.

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_xB_x\,\hat{i}\cdot\hat{i} + \cancel{A_xB_y\,\hat{i}\cdot\hat{j}} + \cancel{A_xB_z\,\hat{i}\cdot\hat{k}}$$

$$+ \cancel{A_yB_x\,\hat{j}\cdot\hat{i}} + A_yB_y\,\hat{j}\cdot\hat{j} + \cancel{A_yB_z\,\hat{j}\cdot\hat{k}}$$

$$+ \cancel{A_zB_x\,\hat{k}\cdot\hat{i}} + \cancel{A_zB_y\,\hat{k}\cdot\hat{j}} + A_zB_z\,\hat{k}\cdot\hat{k}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Each term is a product of vectors which are either // or $\perp$

What is

$$\hat{j}\cdot\vec{A} = \hat{j}\cdot\left[A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right]$$

$$= A_y \quad !!$$

Component of A along y-axis.

First application of a scalar product will be the concept of work. The work done for a constant force $\vec{F}$ acting on a body which is displaced an amount $\vec{d}$, is given by

$$W = \vec{F}\cdot\vec{d}$$

Example : [Dot Product]

$$\vec{A} = 3\hat{i} + 7\hat{k}$$
$$\vec{B} = -\hat{i} + 2\hat{j} + \hat{k}$$

$A_x = 3 \qquad B_x = -1$

$A_y = 0 \qquad B_y = 2$

$A_z = 7 \qquad B_z = 1$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$
$$= (3)(-1) + (0)(2) + (7)(1)$$
$$= +4$$

$$\vec{A}\cdot\vec{B} = |A||B| \cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{4}{\sqrt{3^2+0^2+7^2}\ \sqrt{(-1)^2+2^2+1^2}} = \frac{4}{\sqrt{58}\ \sqrt{6}}$$

$\theta = 77.6^\circ$ [Angle between two vectors $\vec{A}, \vec{B}$]

Easy Way!!