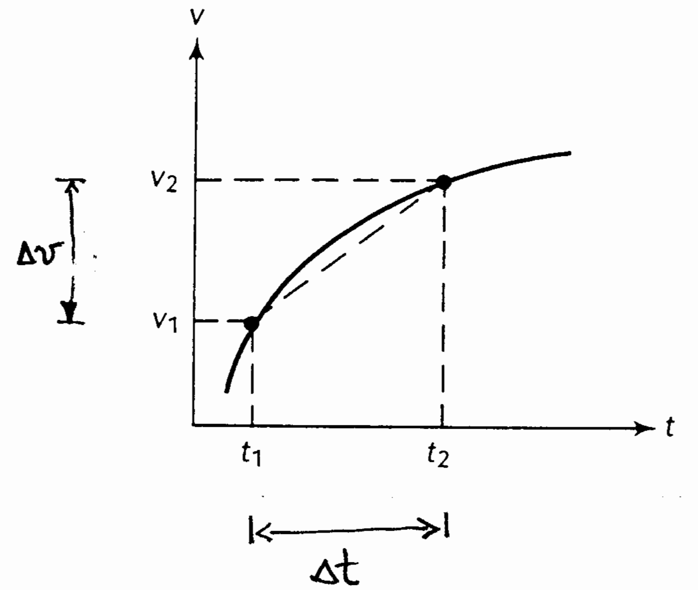
Acceleration

Both the velocity and the position of a particle may be functions of time.

particle speeds up / slows down } velocity changes → accelerated motion

Acceleration ⇒ rate of change of velocity

If $v = v_1$ at $t = t_1$
and $v = v_2$ at $t = t_2$



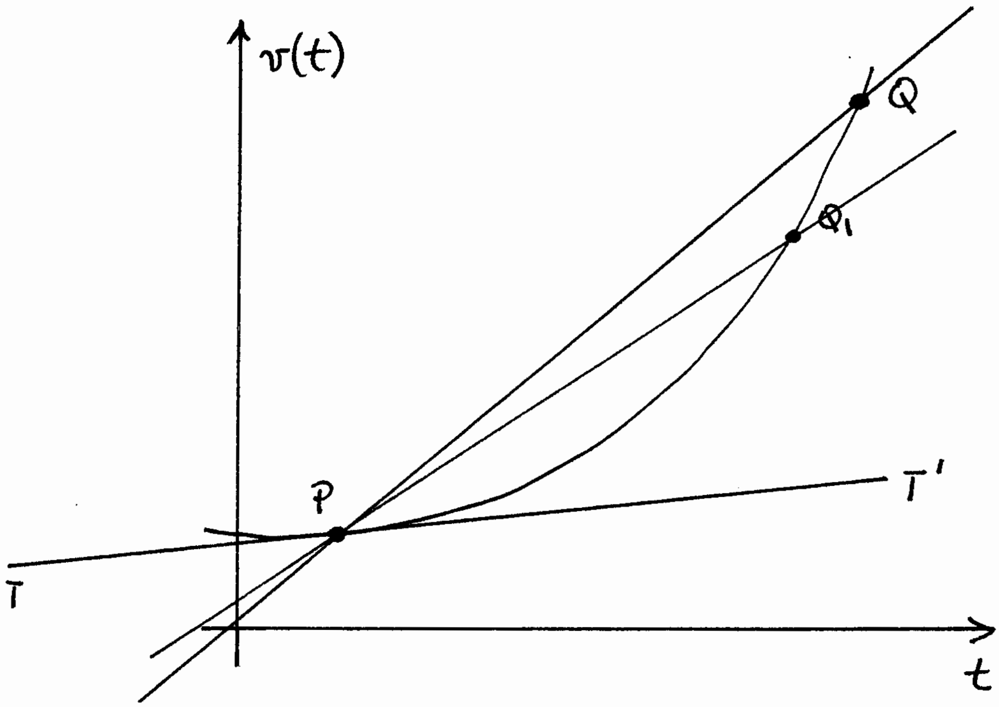
Average Acceleration:

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \quad (m/s^2)$$

$$\frac{v(t+\Delta t) - v(t)}{\Delta t}$$

$\bar{a} \equiv$ slope of straight line connecting the points (v_1, t_1) and (v_2, t_2).

Instantaneous Acceleration



Instead of an average acceleration over some time interval Δt, we want to be able to calculate the instantaneous acceleration $\longleftrightarrow$ acceleration at any time t.

It is defined as the limiting process for $\Delta t \to 0$

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} \quad \text{(calculus)}$$

= derivative of the velocity with respect to time

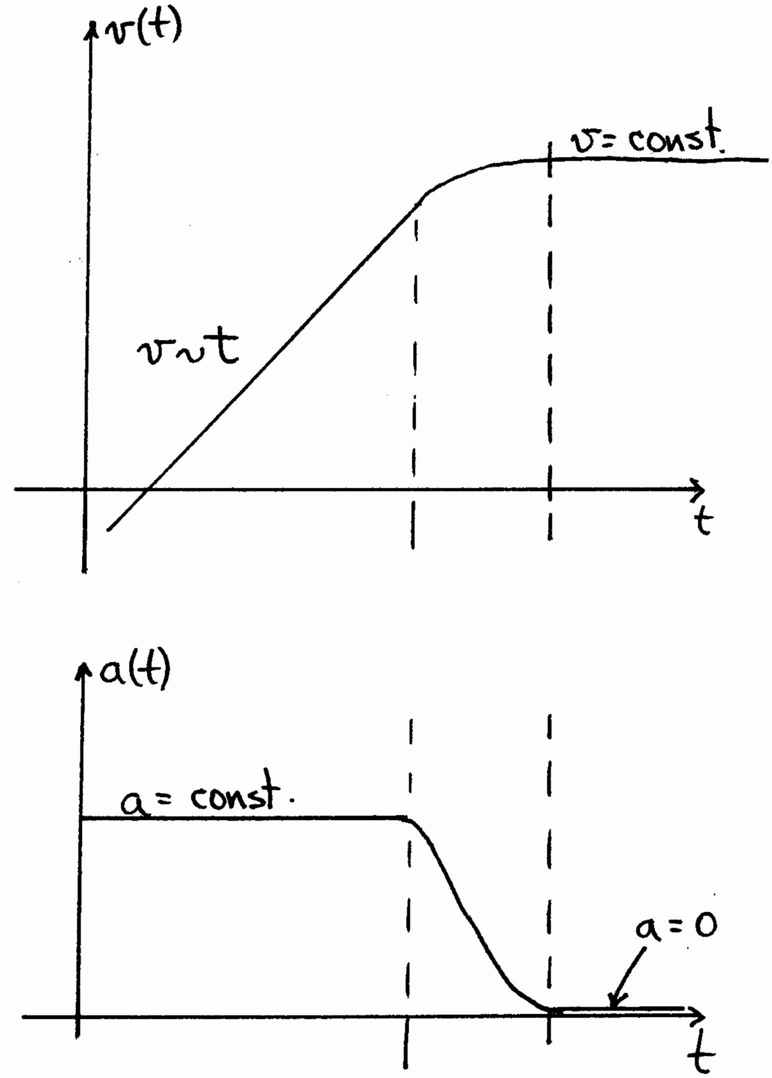
For point P above : $a(t) \equiv$ slope TT' in the limit

Since $v(t) = \frac{dx}{dt}$

$$a(t) = \frac{d\,v(t)}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

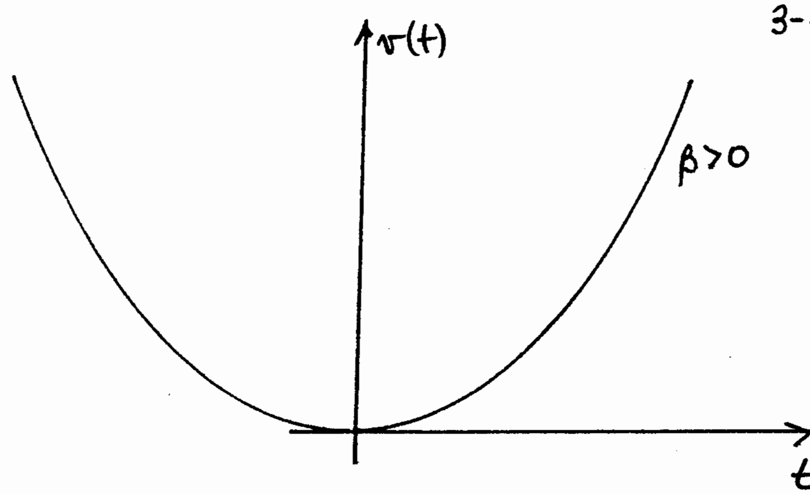
<u>Example</u>

Note: Even if $v(t)=0$, $a(t)$ is not necessarily zero !!



Example

$$v(t) = \frac{1}{2}\beta t^2$$



What is $\bar{a}$ between $t=1$ and $t=3s$? $\Rightarrow$ $\Delta t = 2s$

$$v(t+\Delta t) = \frac{1}{2}\beta(t+\Delta t)^2$$

$$= \frac{1}{2}\beta t^2 + \beta t(\Delta t) + \frac{1}{2}\beta(\Delta t)^2$$

$$\bar{a} = \frac{v(t+\Delta t) - v(t)}{\Delta t} = \beta t + \frac{1}{2}\beta(\Delta t)$$

$t = 1s$; $\Delta t = 2s$

$$\bar{a} = \beta(1) + \frac{1}{2}\beta(2) = 2\beta \ m/s^2$$

Or:

$$v(t+\Delta t) = v(3) = \frac{1}{2}\beta(3)^2 = 4.5\beta$$

$$v(t) = v(1) = \frac{1}{2}\beta(1)^2 = 0.5\beta$$

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{(4.5-0.5)}{2}\beta = 2\beta \ m/s^2$$

Acceleration

$$a = \frac{dv}{dt} = \beta t$$

$$\left.\begin{array}{l} a(1) = \beta \\ a(2) = 3\beta \end{array}\right] \ \bar{a} = 2\beta \ m/s^2$$

Constant Acceleration

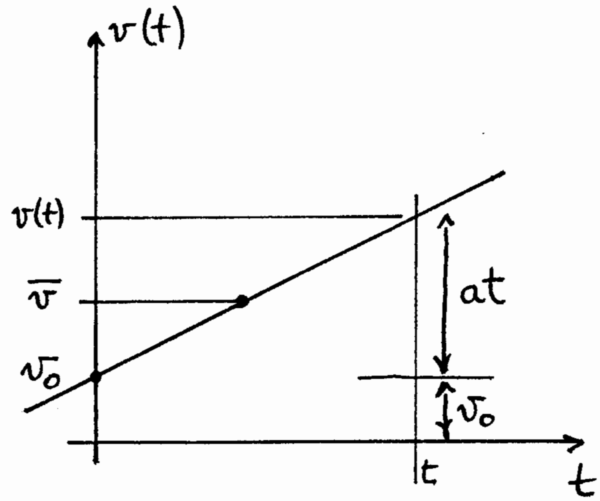
- An important special type of motion.

$a(t) = a$, a constant

$a > 0$ velocity increasing $+x$
$a < 0$ velocity decreasing

$a(t) = \frac{dv}{dt} = a$, a constant

$\therefore$ $v(t) \equiv$ straight line



For constant $a(t) = a$

$\bar{a} = a = \frac{v(t) - v_0}{t - 0}$

$v = v_0 \equiv$ velocity at $t = 0$.

$v > 0$ particle moving $+x$
$v < 0$ particle moving $-x$

$\therefore$ $\boxed{v(t) = v_0 + at}$ (1)

If particle is at x_0 at time $t=0$, After an elapsed time t it will be at

$$x = x_0 + \bar{v}t$$

Since $v(t)$ increases uniformly with t

$$\bar{v} = \frac{1}{2}\left[v_0 + v(t)\right] = \frac{1}{2}\left[v_0 + v_0 + at\right]$$

$$\bar{v} = v_0 + at/2$$

$$\therefore \quad \boxed{x(t) = x_0 + v_0 t + \tfrac{1}{2} a t^2} \qquad (2)$$

Original pos. at $t=0$.

change in position due to initial velocity

change in position due to changing velocity $\Rightarrow$ acceleration

Eq's (1) and (2) give $v(t)$ and $x(t)$ as functions of time.

From (1) $\quad t = \dfrac{v - v_0}{a}$

Substitute into (2)

$$x(t) - x_0 = v_0 \left(\frac{v - v_0}{a}\right) + \frac{1}{2} a \left(\frac{v - v_0}{a}\right)^2$$

After some algebra

$$\boxed{v^2 - v_0^2 = 2a(x - x_0)} \qquad (3)$$

Using calculus:

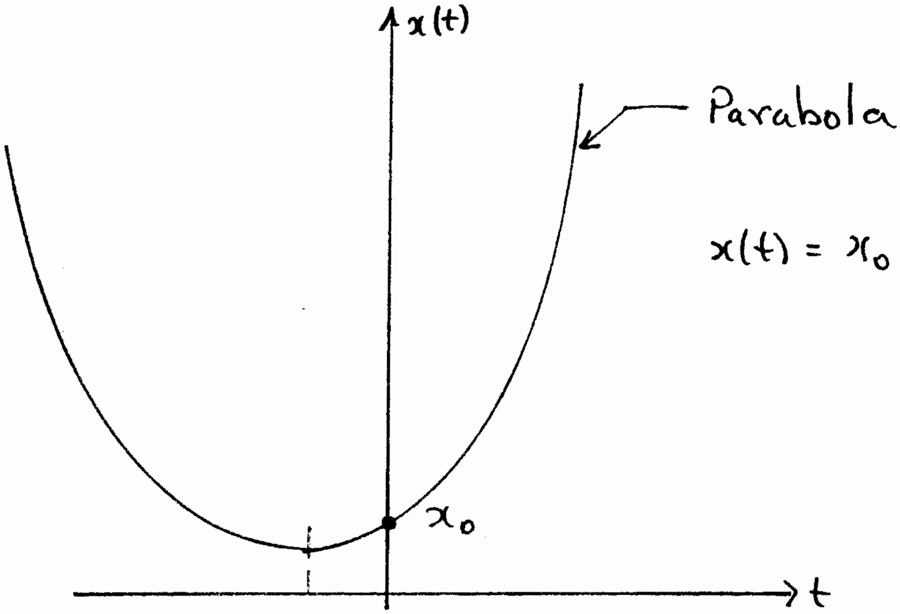
$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$v(t) = \frac{dx}{dt} = v_0 + at$$

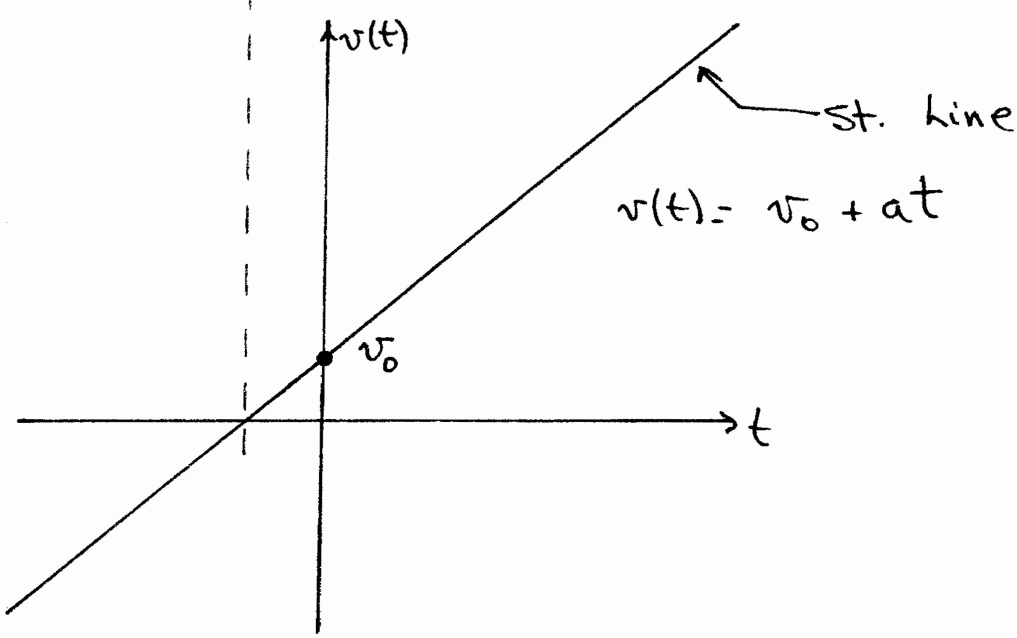
$$a(t) = \frac{dv}{dt} = a \qquad [a \text{ is a constant}]$$

[If $a = 0$, uniform st. line motion]

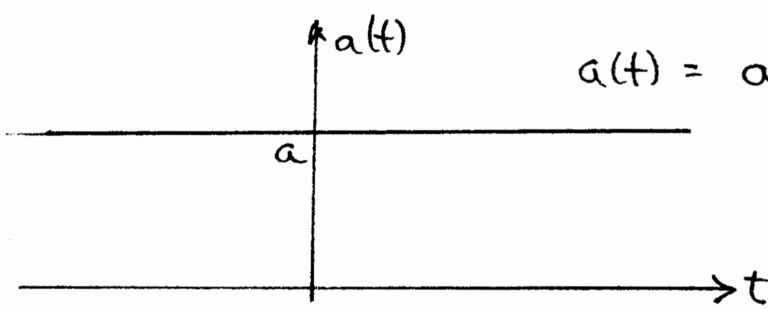
Uniformly Accelerated Motion



$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$



$$v(t) = v_0 + at \qquad = \frac{dx}{dt}$$



$$a(t) = a \text{ [constant] } = \frac{d^2x}{dt^2} = \frac{dv}{dt}$$

Example

How long does it take a car to travel 30m if it accelerates from rest at a rate of $2.0 m/s^2$?

Known	Wanted
$x_0 = 0$	
$v_0 = 0$	
$a = 2.0 m/s^2$	
$x = 30 m$ $\longrightarrow$	$t = ?$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$30 = 0 + (0) t + \frac{1}{2} \times 2 t^2$$

$$\therefore t = \sqrt{30} = \underline{\underline{5.5 \, s}}$$

Example

Particle is at the coordinate position $x_0 = 5m$ at $t=0$ and moving with a velocity $v_0 = 20m/s$. The particle then starts to decelerate (i.e. acceleration opposite to v). At $t = 10s$ the particle has a velocity $v = 2m/s$.

a) what is the acceleration?
b) What is the position function?
c) How long is it before the particle returns to $x = 5m$.

$$\left.\begin{array}{l} x_0 = 5m \\ v_0 = 20m/s \end{array}\right\} t=0 \qquad a = ?$$

$$x(t) = ?$$

$v(10) = 2m/s \quad t = 10s.$

$$v = v_0 + at \qquad (1)$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad (2)$$

$$v^2 - v_0^2 = 2a(x - x_0) \qquad (3)$$

From (1) $\quad a = \dfrac{v - v_0}{t} = \dfrac{2 - 20}{10} = -1.8\, m/s^2$

$$\therefore \boxed{x = 5 + 20t - \frac{1.8}{2}t^2} \quad \text{Position Function}$$

Use position function to determine when particle returns to $x = 5m$.

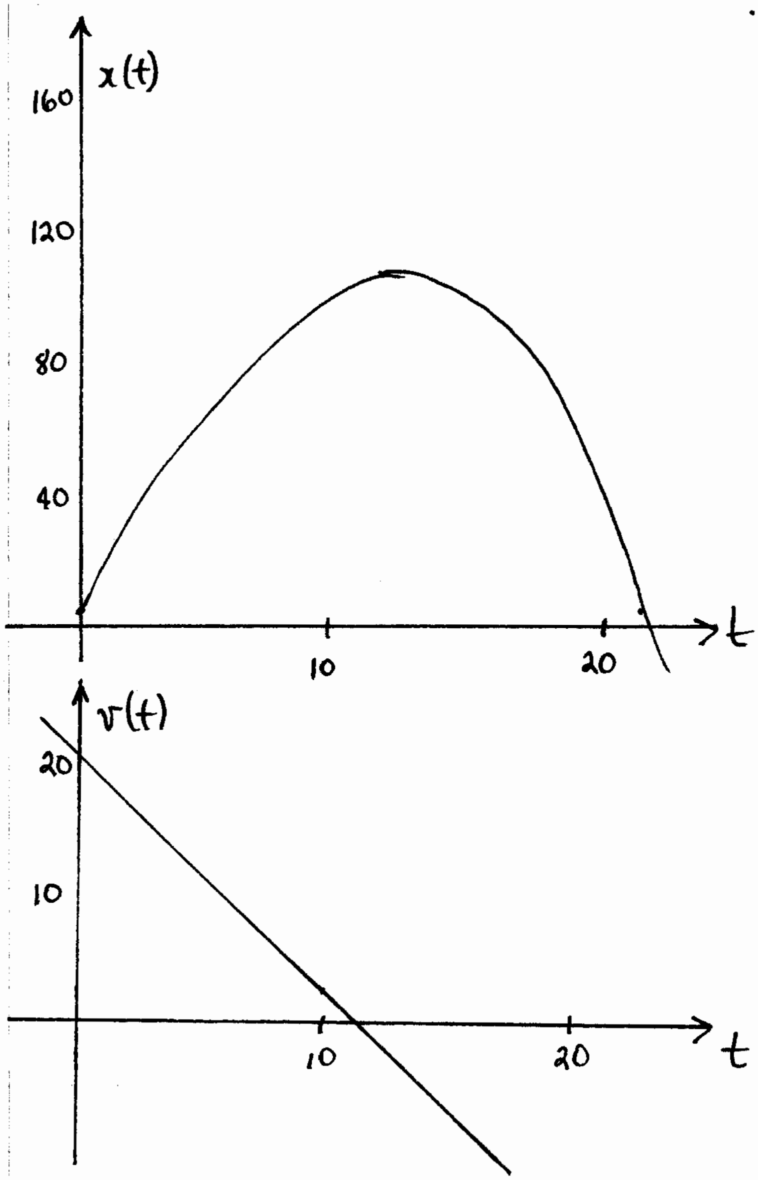
$x = 5m$

$$\therefore \quad 5 = 5 + 20t - 0.9t^2$$

$$0.9t^2 - 20t = 0$$

$$(0.9t - 20)\,t = 0$$

$$t = 0 \quad \text{or} \quad t = \frac{20}{.9} = 22.22\,s.$$



Acceleration of Gravity

Important class of constant acceleration problems involves gravit
Body released near the surface of the earth is accelerated downwards under the influence of gravity.

"Free Fall" – downward motion proceeds with constant accel.

Greeks : Aristotle (384 - 322) BC

- Heavier bodies fall faster
  - ↳ philosophical truths from logical deduction

Galileo (1564 - 1642)

- careful experiments and observations
- established mechanics as a science

All objects near the earth accelerate at the same constant rate when other external effects are excluded : wind, etc.

One of the most precisely and rigorously tested laws of nature.

Difference is $< 1 \times 10^{-10}$ for different objects
$< 1 \times 10^{-12}$ for special cases.

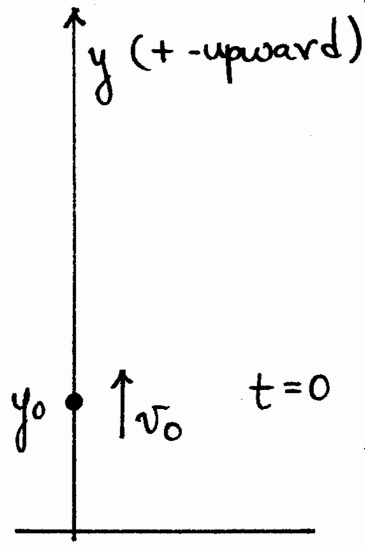
$$g = 9.81 \text{ m/s}^2$$
$$= 32.2 \text{ ft/s}^2$$

[5'th force ??]

- Varies slightly with latitude and longitude
- Will see later how to obtain g – Univ. Law of Grav.

Eq. of Motion $a = -g$

Take a coordinate system with $y > 0$ upward. The equations of motion with constant $\underline{a}$ become



$$a = -g$$

$$v = v_0 - gt$$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2$$

$$v^2 - v_0^2 = -2g(y - y_0)$$

Related to Cons. (KE+PE)

$$\left.\begin{array}{l} y_0 = \text{position} \\ v_0 = \text{velocity} \end{array}\right\} t = 0$$

"Gees"

· Acceleration sometimes measured in units of the acceleration due to gravity.

$$\underline{a}(\text{gees}) = \left(\frac{a}{g}\right) \qquad \text{(dimensionless)}$$

$$a = g\,\underline{a}(\text{gees}) \qquad g = 9.81 \text{ m/s}^2$$

$$\underline{a} = 1\,\text{gee} \qquad a = g$$

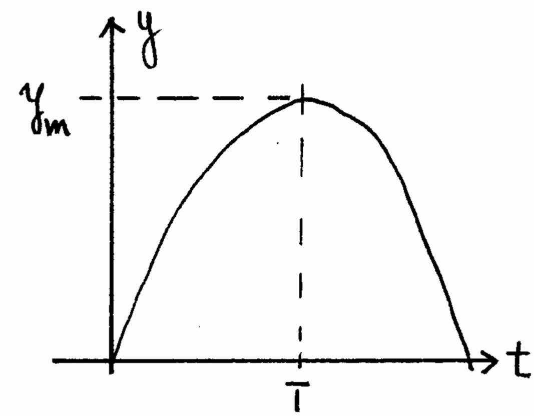
$$\underline{a} = 2\,\text{gees} \qquad a = 2g.$$

Example

A ball is thrown vertically upward from the ground with an initial velocity of 25 m/s.

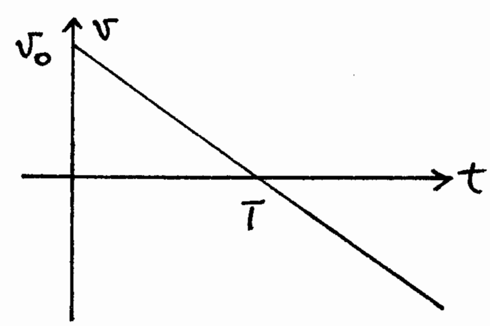
a) How long does it take to reach its maximum height?
b) How high does it rise?
c) What is the velocity when it hits the ground again?
d) What is the time for the total trip?

$$\left.\begin{aligned} y_0 &= 0 \\ v_0 &= 25\,\text{m/s} \\ a &= -g \end{aligned}\right\} t=0$$



$$y(t) = v_0 t - \tfrac{1}{2} g t^2$$

$$v(t) = v_0 - gt$$



What defines maximum height?

At $t = T$ $\quad v(T) = 0$

$$\therefore\ v(T) = 0 = v_0 - gT$$

$$\therefore\ T = \frac{v_0}{g} = \frac{25\,\text{m/s}}{9.8\,\text{m/s}^2} = 2.55\,\text{s}$$

$$v^2 - v_0^2 = -2g(y - y_0)$$

At $v(T) = 0$ $\quad y(T) = y_m$.

$$0 - v_0^2 = -2g(y_m - 0)$$

$$y_m = \frac{v_0^2}{2g} = \frac{(25\,m/s)^2}{2 \times 9.81\,m/s^2} = 31.9\,m.$$

Since $v^2 - v_0^2 = -2g(y - \underset{\parallel\, 0}{y_0})$

$y = 0$ on return.

$$\therefore v^2 = v_0^2$$

$$v = \pm v_0 = 25\,m/s.$$

$$y = v_0 t - \tfrac{1}{2} g t^2$$

$$0 = v_0 t - \tfrac{1}{2} g t^2 \quad \text{[on return]}$$

Solve $\quad t = 0$

$$t = \frac{2v_0}{g} = 2T$$

Example Problem

A man wishes to catch a bus to MIT. The bus is stopped by the curb. The man runs at a rate of 6 m/s towards the bus. When he is 15 m from the bus the bus starts accelerating at the rate of $1 \, m/s^2$.

a) Will he catch the bus?
b) How many seconds?
c) How far will the bus travel?
d) For what accel. of bus a, would he not catch it?

To catch the bus means both must arrive at same position at the same time.

Man: $x_m = x_{om} + v_m t$

Bus: $x_B = x_{oB} + v_{oB} t + \frac{1}{2} a t^2$

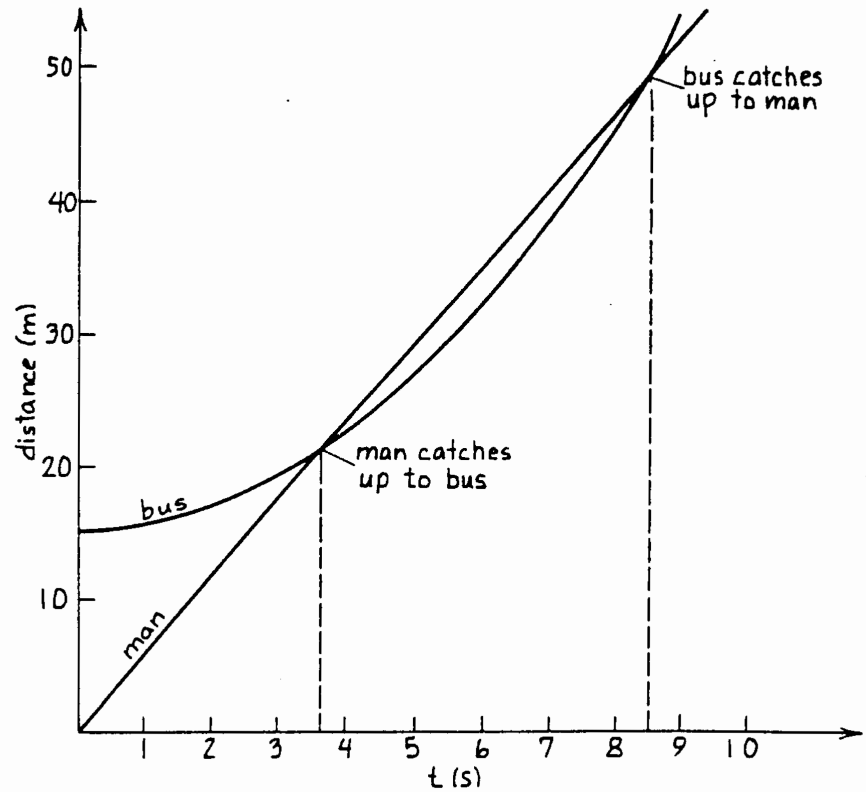
Require $x_m = x_B$

$$\therefore \cancel{x_{om}} + v_m t = x_{oB} + \cancel{v_{oB} t} + \frac{1}{2} a t^2$$

$$t = \frac{v_m}{a}\left[1 \pm \left(1 - \underbrace{\frac{2 x_{oB} a}{v_m^2}}_{\leq 1 \text{ for real solution}}\right)^{1/2}\right]$$

In general 2 correct times.

choose man to be at origin of coord. system at $t=0$.



$x_{om} = 0$

$x_{oB} = 15\,m$

$v_m = 6\,m/s$

$a = 1\,m/s^2$

$v_{oB} = 0$

$$\frac{2 x_{oB} a}{v_m^2} = \frac{2 \times 15 \times 1}{6 \times 6} = 0.83 \quad \text{(OK) [Real Roots]}$$

$$t = \left(\tfrac{6}{1}\right)\left[1 \pm (1 - 0.83)^2\right]$$

$$= 3.5\,s \text{ and } 8.4\,s$$

How far has bus travelled?

$$\underbrace{x_B - x_{oB}}_{\text{dist. travelled}} = v_{oB} t + \frac{1}{2} a t^2$$

$$= 0 + \frac{1}{2} \times 1 \times (3.5)^2$$

$$= 6\,m$$

Problem

stone thrown upward from top of building with an initial velocity of 20 m/s straight upward. The building is 50m high, and the stone just misses the building on the way down.

a) What is the time needed for the stone to reach its maximum height?

$$v = v_0 - gt$$

At max. height $v = 0$.

$$\therefore \; 20\,m/s - 9.8\,\frac{m}{s^2}\,t_1 = 0$$

$$t_1 = 2.04\,s$$

b) What is the maximum height?

$$y = v_0 t - \frac{1}{2} g t^2$$

$$y_{max} = 20 \times 2.04 - \frac{1}{2} \times 9.8 \times (2.04)^2$$

$$= 20.4\,m$$

c) What is the time needed for stone to return to level of thrower?

$$y = v_0 t - \frac{1}{2} g t^2$$

At level of thrower

$y = 0.$

$\therefore \quad 20t - 4.9t^2 = 0$

$t = 0$ and $t = 4.08s$

$\uparrow$ initial $\qquad \uparrow$ required time

d) The velocity of the stone at this instant?

$v = v_0 - gt$

$= 20 - 9.8 \times 4.08$

$= -20.0 \text{ m/s}.$

[same in magnitude as initial velocity]

e) what is velocity and position at $t = 5s$?

$v = v_0 - gt$

$= 20 - 9.8 \times 5 = -29.0s.$

$y = v_0 t - \frac{1}{2} g t^2$

$= 20 \times 5 - \frac{1}{2} \times 9.8 \times 5^2 = -22.5s.$

f) What is velocity and time when stone hits ground? $\quad v = -37.1 \text{ m/s} \quad t = 5.83s$