22.812 Spring 2004 Problem Set 2 Solutions
PSB 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21

2.15 John Hamilton is going to buy a car worth $10,000 from a local dealer. He is told that the add-on interest rate is only 1.25% per month, and his monthly payment is computed as follows:

Installment period = 30 Months
Interest = 30(0.0125)($10,000) = $3,750
Credit Check, life insurance, and processing fee = $50
Total amount owed = $10,000 + $3,750 + $50 = $13,800
Monthly Payment = $13,800/30 = $460 per month

What is the effective rate that John is paying for his auto financing?

a. Effective Interest rate per month?
b. Effective annual interest rate?
c. Suppose that John bought the car and made 15 such payments ($460). Now he decides to pay off the remaining debt with one lump payment at the time of the sixteenth payment. What should the size of the payment be?

Solutions

a. To calculate the effective interest rate, we must note the present value of the car and services that John is receiving.

   PV = $10,000 + $50 = $10,050

   Since we already know the size of the payments that John is making, we can use the P/A factor to relate the present value, the payments, the payment period and the interest rate

$$\frac{P}{A} = \frac{(1+i)^N - 1}{i(1+i)^N}$$

   This formula cannot be solved algebraically; therefore, the numerical solution was computed to give the effective monthly interest rate, of the many possible solutions, only one is positive and real. The number of periods is 30, and the present value is $10,050 and the annuity is $460.

   $i = 0.02181376562 \approx 2.181\%$

b. To calculate the effective annual interest rate we must take into account the monthly compounding of this effective interest rate

$$i_a = (1+i)^{12} - 1 = 29.55\%$$

c. To calculate this lump sum payment, we must use the P/A factor to find the present worth of the payments that John has not made at the 15 month point. This will include 15 payments that have not been made. The present value at that time is the amount that John must pay.

$$A(\frac{P}{A}, i, N-15) = A\frac{(1+i)^{15}-1}{i(1+i)^{15}} = \$460\frac{(1+2.181\%)^{15}-1}{2.181\%(1+2.181\%)^{15}} = \$5{,}831.40$$

2.16 A pipeline was built three years ago to last six years. It develops leaks according to the relation log N = 0.07 T – 2.42, for all T greater than 30. N is the total number of leaks from installation, and T is the time in months from installation. It costs $500 to repair a leak. If money is worth 8% per year, and without considering any tax effect, how much can be spent now for a cathodic system that will reduce leaks by 75%?

This problem was solved using a conversion to a discrete cash flow, based on the fact that the given relation models shocks on a system, and thus a continuous system does not adequately represent the phenomena involved.

The first step required is to compute the total payout for each period, based on the function. This is done by solving the given relation for N:

$$\log N = 0.07T - 2.42$$
$$N = 10^{0.07T-2.42}$$

Then, the number of leaks must be multiplied by $500, since that is the cost required to repair each leak. The resulting multiplication allows us to keep a 'running total' of present dollars paid to repair leaks. We must subtract from the running total the amount that has been previously paid to obtain the monthly payout. Then, we can find the present value of each month's payout, using an effective monthly rate of inflation, adapted from the given yearly rate, assuming that the yearly rate is an effective annual rate of 8%:

$$i_m = (1+0.08)^{1/12} - 1 = 0.006434$$

The monthly payout is then discounted using this rate and the formula (P/F, i_m, T). The results are shown in the 'Untreated Plan' section of the table.

In order to compare and determine the willingness to pay for the cathodic treatment, we must repeat the above procedure, but multiply N by 0.25. The WTP is the difference in the two present values of the totals of repair payouts. We must keep in mind, however, that one leak has already occurred (see table).

The sum of discounted monthly payouts for the untreated plan is $171,363.65 and the sum of the discounted monthly payouts for the treated plan is $43,299.09. The difference between these two values is $128,064.56, which is the WTP for the treatment.

		Untreated Plan				Treated Plan			
T	Months from Present	N	Discretized Total Payout	Monthly Payout	Discounted Monthly Payout	N'	Discretized Total Payout	Monthly Payout	Discounted Monthly Payout
30		0.479							
31		0.562							
32		0.661							
33		0.776							
34		0.912							
35	-1	1.072	$ 500	$ 500	$ 503.22	0.268	$ 500	$ 500	$ 503.22
36	0	1.259	$ 500	$ -	$ -	0.315	$ 500	$ -	$ -
37	1	1.479	$ 500	$ -	$ -	0.370	$ 500	$ -	$ -
38	2	1.738	$ 500	$ -	$ -	0.434	$ 500	$ -	$ -
39	3	2.042	$ 1,000	$ 500	$ 490.47	0.510	$ 500	$ -	$ -
40	4	2.399	$ 1,000	$ -	$ -	0.600	$ 500	$ -	$ -
41	5	2.818	$ 1,000	$ -	$ -	0.705	$ 500	$ -	$ -
42	6	3.311	$ 1,500	$ 500	$ 481.13	0.828	$ 500	$ -	$ -
43	7	3.890	$ 1,500	$ -	$ -	0.973	$ 500	$ -	$ -
44	8	4.571	$ 2,000	$ 500	$ 474.99	1.143	$ 1,000	$ 500	$ 474.99
45	9	5.370	$ 2,500	$ 500	$ 471.96	1.343	$ 1,000	$ -	$ -
46	10	6.310	$ 3,000	$ 500	$ 468.94	1.577	$ 1,000	$ -	$ -
47	11	7.413	$ 3,500	$ 500	$ 465.94	1.853	$ 1,000	$ -	$ -
48	12	8.710	$ 4,000	$ 500	$ 462.96	2.177	$ 1,500	$ 500	$ 462.96
49	13	10.233	$ 5,000	$ 1,000	$ 920.01	2.558	$ 1,500	$ -	$ -
50	14	12.023	$ 6,000	$ 1,000	$ 914.13	3.006	$ 2,000	$ 500	$ 457.06
51	15	14.125	$ 7,000	$ 1,000	$ 908.28	3.531	$ 2,000	$ -	$ -
52	16	16.596	$ 8,000	$ 1,000	$ 902.47	4.149	$ 2,500	$ 500	$ 451.24
53	17	19.498	$ 9,500	$ 1,500	$ 1,345.06	4.875	$ 2,500	$ -	$ -
54	18	22.909	$ 11,000	$ 1,500	$ 1,336.46	5.727	$ 3,000	$ 500	$ 445.49
55	19	26.915	$ 13,000	$ 2,000	$ 1,770.55	6.729	$ 3,500	$ 500	$ 442.64
56	20	31.623	$ 15,500	$ 2,500	$ 2,199.04	7.906	$ 4,000	$ 500	$ 439.81

57	21	37.154	$ 18,500	$ 3,000	$ 2,621.98	9.288	$ 5,000	$ 1,000	$ 873.99
58	22	43.652	$ 21,500	$ 3,000	$ 2,605.22	10.913	$ 5,500	$ 500	$ 434.20
59	23	51.286	$ 25,500	$ 4,000	$ 3,451.42	12.822	$ 6,500	$ 1,000	$ 862.85
60	24	60.256	$ 30,000	$ 4,500	$ 3,858.02	15.064	$ 8,000	$ 1,500	$ 1,286.01
61	25	70.795	$ 35,000	$ 5,000	$ 4,259.29	17.699	$ 9,000	$ 1,000	$ 851.86
62	26	83.176	$ 41,500	$ 6,500	$ 5,501.68	20.794	$ 10,500	$ 1,500	$ 1,269.62
63	27	97.724	$ 48,500	$ 7,000	$ 5,887.01	24.431	$ 12,500	$ 2,000	$ 1,682.00
64	28	114.815	$ 57,000	$ 8,500	$ 7,102.81	28.704	$ 14,500	$ 2,000	$ 1,671.25
65	29	134.896	$ 67,000	$ 10,000	$ 8,302.83	33.724	$ 17,000	$ 2,500	$ 2,075.71
66	30	158.489	$ 79,000	$ 12,000	$ 9,899.70	39.622	$ 20,000	$ 3,000	$ 2,474.92
67	31	186.209	$ 93,000	$ 14,000	$ 11,475.81	46.552	$ 23,500	$ 3,500	$ 2,868.95
68	32	218.776	$ 109,000	$ 16,000	$ 13,031.37	54.694	$ 27,500	$ 4,000	$ 3,257.84
69	33	257.040	$ 128,500	$ 19,500	$ 15,780.45	64.260	$ 32,500	$ 5,000	$ 4,046.27
70	34	301.995	$ 150,500	$ 22,000	$ 17,689.76	75.499	$ 38,000	$ 5,500	$ 4,422.44
71	35	354.813	$ 177,000	$ 26,500	$ 21,171.90	88.703	$ 44,500	$ 6,500	$ 5,193.11
72	36	416.869	$ 208,000	$ 31,000	$ 24,608.80	104.217	$ 52,500	$ 8,000	$ 6,350.66

2.17 A market survey indicates that the price of a 10-oz jar of instant coffee has fluctuated over the last few years as follows:

Period	-4	-3	-2	-1	0	1
Price ($)	2.83	3.13	3.47	4.67	5.83	?

a. Assuming that the base period (price index = 100) is period -4 (four periods ago), compute the average price index for this instant coffee.
b. Estimate the price at the time period 1, if the current price trend is expected to continue.

Solutions

a. The price index is easily computed by taking the ratio of the price in a period to that of the price in a base period and normalizing to 100. The average is then computed by taking the mean of the indices over the 5 periods in the table.

Period	-4	-3	-2	-1	0	1
Price ($)	2.83	3.13	3.47	4.67	5.83	?
Index	100	110.6007	122.6148	165.0177	206.0071	?

The average index is 140.85

b. The average trend is given by the average inflation rate in the price over the period. This can be established based on the base period price and the current price.

$$PI_0 = (1+\bar{f})^5 PI_{-4}$$

$$\bar{f} = \left(\frac{PI_0}{PI_{-4}}\right)^{1/5} - 1 = 0.1556$$

The average inflation rate can then be used to find the expected value of the price index of coffee in the next period.

$$PI_1 = (1+\bar{f})PI_0$$

$$PI_1 = (1+0.1556)206 = 238.04$$

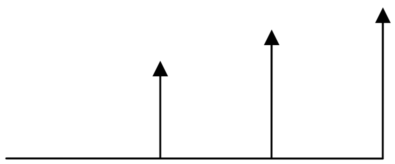
The price is the price index, normalized to unity, times the price of the coffee in the base period.

$$P_1 = \left[\frac{238.04}{100}\right]\$2.83 = \$6.74$$

2.18 The annual operating costs of a small electrical generating unit are expected to remain the same ($200,000) if the effects of inflation are not considered. The best estimates indicate that the annual inflation-free rate of interest will be 5% and the annual inflation rate 6%. If the generator is to be used 3 more years, what is the present equivalent of its operating costs using actual dollar analysis?

Solution

First construct the Cash Flow Diagram



Every year, because of inflation, the operating payments will increase. However, the trend is predictable. We can calculate the cost in each of the three years as follows

$$A_1 = (1+f)^1 A' = (1.06)^1 (\$200{,}000)$$
$$A_2 = (1+f)^2 A' = (1.06)^2 (\$200{,}000)$$
$$A_3 = (1+f)^3 A' = (1.06)^3 (\$200{,}000)$$

It is convenient to work algebraically and solve for the present value of the payments symbolically. We must include the discounting due to the real interest rate as well. We calculate the real interest rate as follows

$$i = i' + f + i'f = 11.3\%$$

To calculate the present value, we use the P/F factor for each payment that the generator makes.

$$P = \left(\frac{P}{F}, i, 1\right) A_1 + \left(\frac{P}{F}, i, 2\right) A_2 + \left(\frac{P}{F}, i, 3\right) A_3$$
$$P = A'\left[\left(\frac{1+f}{1+i}\right) + \left(\frac{1+f}{1+i}\right)^2 + \left(\frac{1+f}{1+i}\right)^3\right] = 2.72A' = \$544{,}000$$

One can check the result by calculating the P/A factor using the real interest rate, the result is 2.72. This confirms the approach using actual dollar analysis.

2.19 You want to know how much money to set aside now to pay for 1,000 gallons of home heating oil each year for 10 years. The current price of heating oil is $1.00 per gallon, and the price is expected to increase at a 10% compound price change each year for the next 10 years. The money to pay for the fuel will be set aside now in a bank savings account that pays 6% annual interest. How much money do you have to place in the savings account now, if the payment for the fuel is made by end-of-year withdrawals?

Solution

The cash flow diagram is much akin to that shown in the previous question, the payments are increasing in value because of the compound price change. Analogously we can write the solution by extrapolation of the result shown in 2.18

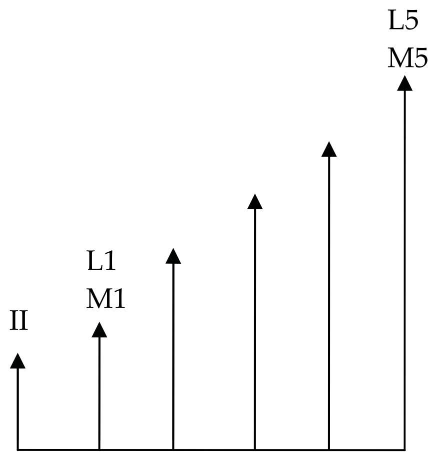
A′ denotes the cost of the oil without the compound price change, and r is the price change rate and i denotes the savings account annual interest rate. The present value of the future payments is the lump sum that must be set aside, and since the oil is purchased at the end of the year, the price compounding and interest accrual are on the same time scale.

$$P = A'\sum_{n=1}^{10}\left(\frac{1+r}{1+i}\right)^n = 12.33A' = \$12{,}330$$

2.20 An investment of $100,000 is required to expand a certain production facility in a manufacturing company. The firm estimates that the labor costs will be $150,000 for the first year but will increase at the rate of 8% over the previous year's expenditure. Material costs, on the other hand, will be $400,000 for the first year but will increase at the rate of 10% per year due to inflation. If the firm's inflation-free interest rate is 10% and the average general inflation rate is expected to be 5% over the next 5 years, determine the total present equivalent operating expenses (with no tax consideration) for the project.

Solution

Construct the Cash Flow Diagram



We know that the initial investment (II) is $100,000. The first year labor cost (L1) is $150,000 and the first year material cost (M1) is $400,000.

The labor and material costs in the nth year can be predicted by the following relationships

$$L_n = (1+g_l)^{n-1} L_1$$
$$M_n = (1+g_m)^{n-1} M_1$$
$$C_n = L_n + M_n$$

g denotes the yearly increase in the costs, for labor g is 8% and for materials g is 10%.

We must also calculate the interest rate given the real interest rate and the average rate of inflation

$$i = i' + f + i'f = 0.155$$

Similar to the procedures in 2.18 and 2.19, we must present value the costs and sum the present values together. The formula is shown below for the total present value

$$TPV = II + \sum_{n=1}^{5} C_n \left(\frac{P}{F}, i, n \right)$$

Substituting the formulae from above in the expression for the TPV, we find a similar result, though somewhat different than that of problem 2.19

$$TPV = II + L_1 \left[\frac{1}{1+i} + \frac{1+g_l}{(1+i)^2} + \frac{(1+g_l)^2}{(1+i)^3} + \dots \right] + M_1 \left[\frac{1}{1+i} + \frac{1+g_m}{(1+i)^2} + \frac{(1+g_m)^2}{(1+i)^3} + \dots \right]$$

calculating the coefficients yields

$$TPV = II + 3.802L_1 + 3.9359M_1 = \$2{,}244{,}683.07 \approx \$2{,}245{,}000$$

2.21 A couple with a 7-year old daughter want to save for their child's college expenses in advance. Assuming that the child enters college at age 18, they estimate than an amount of $20,000 per year in terms of today's dollars will be required to support the child's college expenses for 4 years. The future inflation rate is estimated to be 6% per year and they can invest their savings at 8% compounded quarterly.

a. Determine the equal quarterly amounts the couple must save until they send their child to college.
b. If the couple has decided to save only $500 each quarter, how much will the child have to borrow each year to support her college education.

Solution

a. The first step is to determine the present value of the college education, this is done by using P/F factors for the four yearly college payments that occur in the years 12, 13, 14, and 15. Since we know the real dollar value of these payments, combined with the real interest rate, we can easily calculate the present value as follows:

$$PV = \left(\frac{P}{F}, i', 12\right)A' + \left(\frac{P}{F}, i', 13\right)A' + \left(\frac{P}{F}, i', 14\right)A' + \left(\frac{P}{F}, i', 15\right)A'$$
$$PV = \frac{1}{(1+i')^{12}}A' + \frac{1}{(1+i')^{13}}A' + \frac{1}{(1+i')^{14}}A' + \frac{1}{(1+i')^{15}}A'$$

We must calculate the real interest rate based on the annual interest rate and the inflation rate.

$$i_a = (1+0.08/4)^4 - 1 = 8.24\%$$
$$i' = \frac{i_a - f}{1+f} = \frac{8.24-6}{100+6} = 2.113\%$$

Combined with the present value formula shown above, the present value of the college education using real dollar analysis is: $60,340.

We must now use actual dollar analysis to calculate the quarterly savings amount the couple must put into the savings account, such that the present value of these payments is the same as the present value of the college education.

The second step is to use an A/P factor with the nominal interest rate for 11 years (44 quarters)

$$A = PV\left(\frac{A}{P}, i, 44\right) = \left[\frac{0.02(1+0.02)^{44}}{(1+0.02)^{44}-1}\right] PV = \$2{,}074.96$$

The couple must pay $2,075 into their savings account every quarter to cover the college expenses.

b. Now we must work backwards from the known savings account payments. The present value of the savings can be calculated with the P/A factor.

$$PV = \frac{A}{\left(\frac{A}{P}, i, 44\right)} = \frac{\$500}{\left[\frac{0.02(1+0.02)^{44}}{(1+0.02)^{44}-1}\right]} = \$14{,}539.98$$

No we can convert this present value into an equivalent real set of four payments in the years 12, 13, 14, and 15. We assume that the college payments are made at the end of the year. We choose to have the same real payment from the account so that the child does not have to borrow additional buying power in each subsequent year to pay for college. Let the real payment in present day dollars be A′ and the real cost per year is $20,000. We know the present value of the four real payments, and can therefore calculate the real value of A′.

Alternatively, you could set this up as follows:

A′ = PV (F/P,i′,12) (P/A, i′,4)

Where the first factor calculates the future worth of your PV at the beginning of the college years, and the second factor converts that amount into an equivalent annuity during the 4 college years.

$$PV = \left(\frac{P}{F}, i', 12\right) A' + \left(\frac{P}{F}, i', 13\right) A' + \left(\frac{P}{F}, i', 14\right) A' + \left(\frac{P}{F}, i', 15\right) A'$$

$$PV = \left(\frac{1}{(1+i')^{12}} + \frac{1}{(1+i')^{13}} + \frac{1}{(1+i')^{14}} + \frac{1}{(1+i')^{15}}\right) A'$$

$$A' = \frac{PV}{\left(\frac{1}{(1+i')^{12}} + \frac{1}{(1+i')^{13}} + \frac{1}{(1+i')^{14}} + \frac{1}{(1+i')^{15}}\right)} = \$4{,}819.38$$

From the child’s perspective, what is of real import is the difference between the real cost of college and the real contribution from the parents’ savings account. The real deficit will be denoted with D′ and the actual deficit will be denoted with D. D′ is constant, whereas D depends on the year because of inflation.

$$D' = \$20{,}000 - \$4819.38 = \$15{,}180.62$$

$$D_1 = \$15{,}180.62(1+f)^{12} = \$30{,}546.39$$

$$D_2 = \$15{,}180.62(1+f)^{13} = \$32{,}379.18$$

$$D_3 = \$15{,}180.62(1+f)^{14} = \$34{,}321.93$$

$$D_4 = \$15{,}180.62(1+f)^{15} = \$36{,}381.24$$