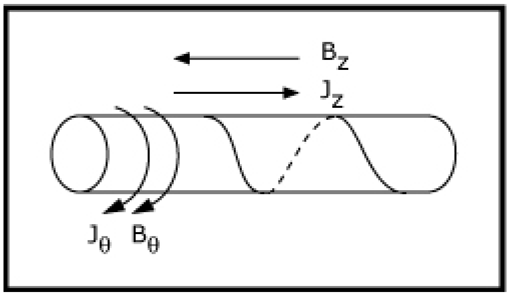
Lecture 5: The Screw Pinch and the Grad-Shafranov Equation

Screw Pinch Equilibria

1. A hybrid combination of Z pinch and θ pinch



2. This combination of fields allows the flexibility to optimize configurations with respect to toroidal force balance and stability.

3. Field components: $\mathrm{B} = B_\theta(r)e_\theta + B_z(r)e_z$, $p = p(r)$

   a. $\nabla \cdot \mathrm{B} = 0$

   $$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0 \quad \text{automatically satisfied}$$

   b. $\mu_0 \mathrm{J} = \nabla \times \mathrm{B} = -\mathrm{B}_z' e_\theta + \left[(rB_\theta)'/r\right]e_z$

   c. $\mathrm{J} \times \mathrm{B} = \nabla p$

   $$\nabla p = \mathrm{p}' e_r$$

   $$\mathrm{J} \times \mathrm{B} = \left(J_\theta B_z - J_z B_\theta\right)e_r$$

   $$= -\frac{1}{\mu_0}\left[\frac{B_\theta}{r}(rB_\theta)' + B_z B_z'\right]$$

   Even though the equations are nonlinear, the forces superpose because of symmetry

   $$\frac{d}{dr}\left(p + \frac{B_z^2 + B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0 \quad \text{screw pinch pressure balance relation}$$

4. The screw pinch has many properties of more realistic, multidimensional toroidal configurations

a. There are two free functions, say $B_\theta(r), B_z(r)$. (The θ pinch, Z pinch are special degenerate cases)

b. The constant pressure contours *p(r)* = constant are circles *r* = constant. The flux surfaces are closed, nested, concentric circles.

c. β can be varied over a wide range if $B_z(0) \neq 0$

d. The magnetic lines wrap around the plasma giving a nonzero rotational transform

General Equilibrium Relation for 1-D Configurations

1. This is a useful relation for defining β

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0 > \int 2\pi r^2 dr$$

2. $$T_1 = 2\pi\int dr\ r^2\left(p + \frac{B_z^2}{2\mu_0}\right)' = -4\pi\int rdr\left(p + \frac{B_z^2}{2\mu_0}\right) + 2\pi r^2\left(p + \frac{B_z^2}{2\mu_0}\right)\Bigg|_0^a$$

$$= 2\pi a^2\left(\frac{B_0^2}{2\mu_0}\right) = -4\pi\int rdr\left(p + \frac{B_z^2}{2\mu_0}\right) = -4\pi\int rdrp + 4\pi\int rdr\,\frac{\left(B_0^2 - B_z^2\right)}{2\mu_0}$$

3. $$T_2 = \frac{2\pi}{\mu_0}\int dr\ r^2\,\frac{B_\theta}{r}\left(rB_\theta\right)' = \frac{2\pi}{\mu_0}\int dr\left[\frac{\left(r^2 B_\theta^2\right)}{2}\right]' = \frac{\pi r^2 B_\theta^2}{\mu_0}\Bigg|_0^a$$

$$= \frac{\pi a^2}{\mu_0}\left(\frac{\mu_0 I}{2\pi a}\right)^2 = \frac{\mu_0 I^2}{4\pi}$$

4. The general equilibrium relation is given by

$$2\pi\int prdr = \frac{\mu_0 I^2}{8\pi} + 2\pi\int\frac{B_0^2 - B_z^2}{2\mu_0}r\ dr$$

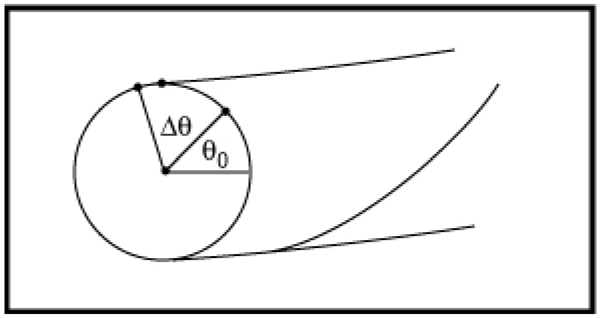
plasma energy line density   poloidal tension   toroidal diamagnetism

5. This suggests the following cylindrical definitions

$$\beta_p = \frac{16\pi^2\int pr\ dr}{\mu_0 I^2}\qquad \text{poloidal } \beta$$

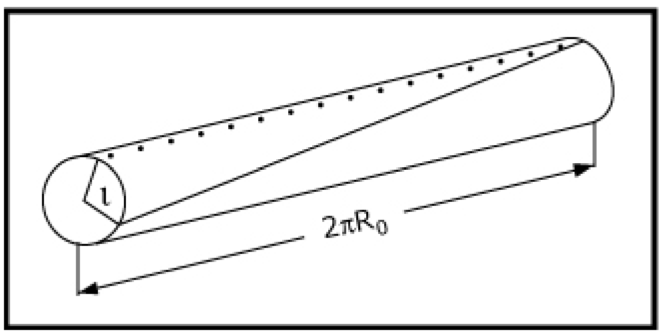
$$\beta_T = \frac{4\mu_0 \int pr\ dr}{a^2 B_0^2} \qquad \text{toroidal } \beta$$

Calculate the Rotational Transform

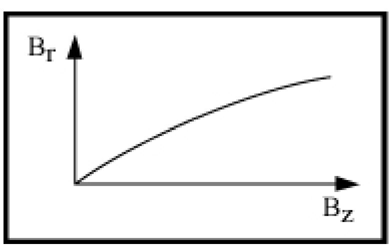


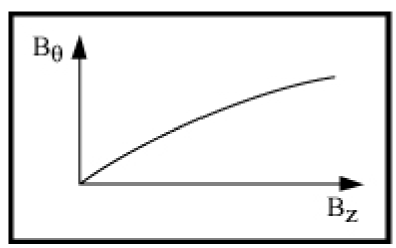
1. Note that $\Delta\theta$ is independent of θ_0 because of cylindrical symmetry.

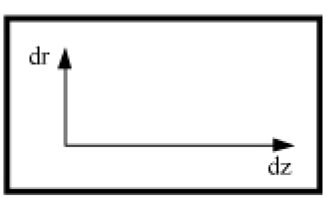
2. The average value of $\Delta\theta$ is just the change θ as the magnetic line moves one length along the torus

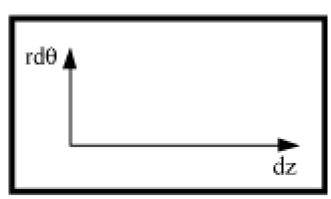


3. Calculate the field line trajectories









$$\frac{dr}{dz} = \frac{B_r}{B_z} = 0 \qquad (1)$$

$$\frac{d\theta}{dz} = \frac{B_\theta}{rB_z} \qquad (2)$$

4. Equation (1) implies that *r(z)* = const. The magnetic lines lie on circles. This is not surprising since the *p(r)* = const. surfaces are circles.

5. Solve Eq. (2)

r = const.

$$\frac{d\theta}{dz} = \frac{B_\theta(r)}{rB_z(r)}$$

$$d\theta = \frac{B_\theta}{rB_z}dz$$

$$\int_0^{\Delta\theta} d\theta = \int_0^{2\pi R_0} \frac{B_\theta}{rB_z}dz$$

6. The angle $\Delta\theta$ is by definition just equal to ι

$$\iota(r) = \frac{2\pi R_0 B_\theta}{rB_z}$$

7. The safety factor is defined as

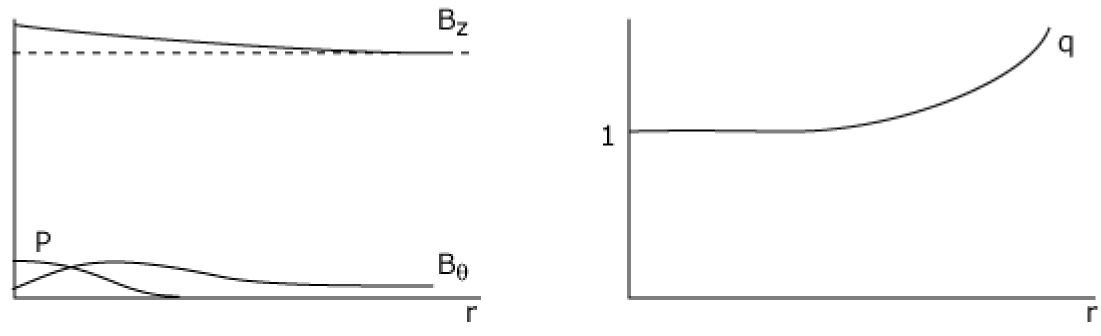
$$q(r) = \frac{2\pi}{\iota(r)}$$

$$q(r) = \frac{rB_z}{R_0 B_\theta}$$

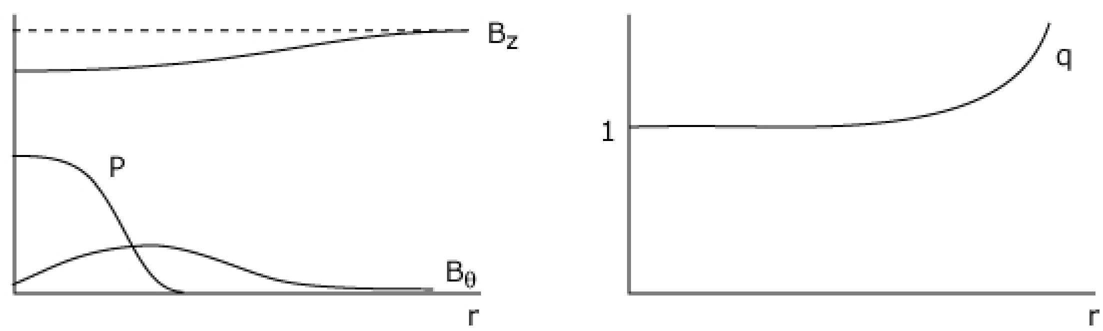
8. Note $\iota(r) = 0$ for a θ pinch and $\iota(r) = \infty$ for a Z pinch

9. The 1-D radial pressure balance relation for a screw pinch accurately describes radial pressure balance in all fusion configurations of interest

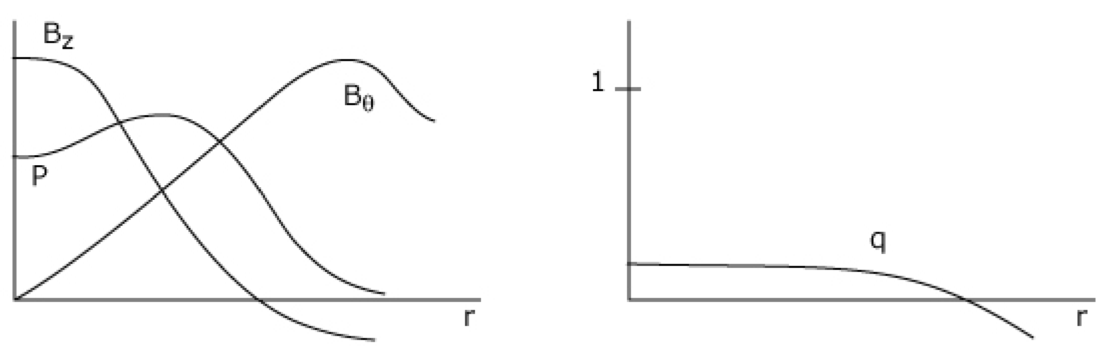
Examples



Ohmic tokamak



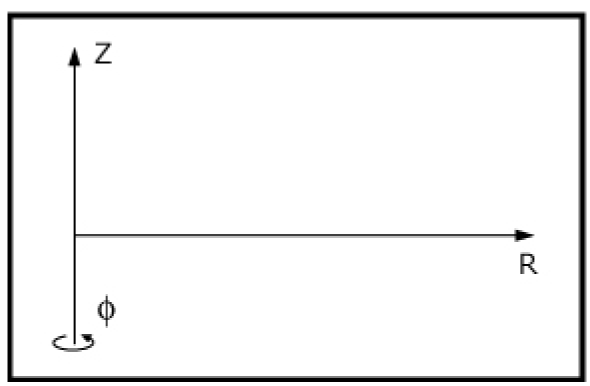
High β tokamak



High β tokamak

Toroidal Force Balance

1. Consider the axisymmetric torus, the simplest, multi-dimensional configuration
2. We shall derive the Grand–Shafranov equation for axisymmetric equilibria
3. This provides a complete description of toroidal equilibrium
   a. radial pressure balance
   b. toroidal force balance
   c. β limits
   d. q profiles
   e. magnetic well, etc….
4. It applies to the following configurations
   a. RFP
   b. ohmic tokamak (circular and noncircular)
   c. high β tokamak (circular and noncircular)
   d. flux conserving tokamak (circular and noncircular)
   e. spherical tokamak (circular and noncircular)
   f. spheromak (circular and noncircular)
   g. toroidal multipole (circular and noncircular)
5. Grad–Shafranov equation
   a. exact (no expansion)
   b. axisymmetric $\partial/\partial\phi = 0$
   c. 2-D
   d. nonlinear
   e. partial differential equation
   f. elliptic characteristics



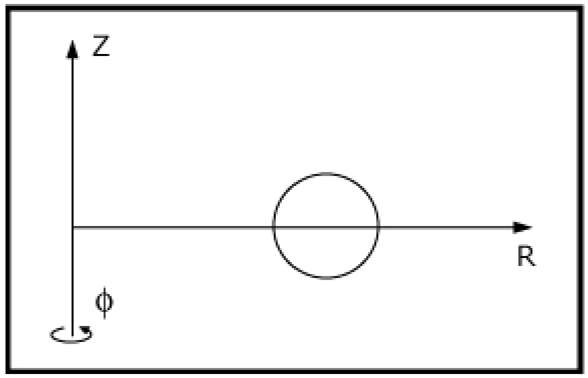
6. Plan of action
   a. Derive the exact Grad–Shafranov equation

b. Solve by means of an asymptotic expansion in a/R

c. Zero order: $(a/R)^0 \rightarrow$ 1-D screw pinch radial pressure balance

d. First order: $(a/R)^1 \rightarrow$ toroidal force balance

Derivation



1. Geometry

2. Axisymmetry

$$\frac{\partial}{\partial \phi} = 0$$

$$Q(R, \phi, Z) \rightarrow Q(R, Z)$$

3. We solve in this order

$$\nabla \cdot \mathrm{B} = 0$$

$$\mu_0 \mathrm{J} = \nabla \times \mathrm{B}$$

$$\mathrm{J} \times \mathrm{B} = \nabla p$$

4. $\nabla \cdot \mathrm{B} = 0$

a. $$\frac{1}{R}\frac{\partial}{\partial R} R B_R + \underbrace{\frac{1}{R}\frac{\partial B\phi}{\partial \phi}}_{=\,0} + \frac{\partial B_Z}{\partial Z} = 0$$

b. B_ϕ is arbitrary as of now

c. $$\left.\begin{aligned} B_Z &= \frac{1}{R}\frac{\partial \psi}{\partial R} \\ B_R &= -\frac{1}{R}\frac{\partial \psi}{\partial Z} \end{aligned}\right\} \quad \text{introduce “flux” function } \psi$$

d. These results can be summarized as follows

$$\mathrm{B} = B_\phi e_\phi + \mathrm{B}_p$$

$$B_p = \frac{1}{R}\nabla\psi\times e_\phi$$

$$\psi = \psi(R, Z)$$

5. Why is ψ the "flux" function?

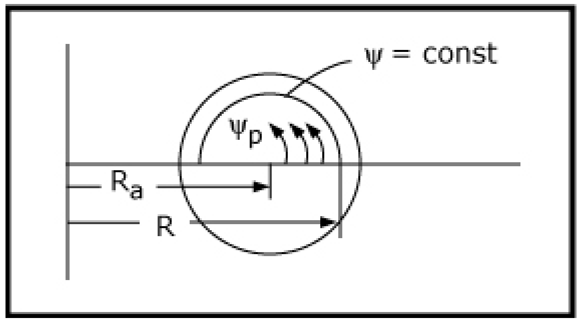
a. $B_p = \nabla\times A = \nabla\times\left(A_\phi e_\phi\right) = \frac{1}{R}\frac{\partial}{\partial R}RA_\phi e_Z - \frac{\partial A_\phi}{\partial Z}e_R$

$$\psi = RA_\phi$$

b. $\psi_p = \int B_p \cdot dA$

$$= \int_{R_e}^{R} dR \int B_z\left(R, Z = 0\right) R d\phi$$

$$= \int_{R_e}^{R} 2\pi R \frac{1}{R}\frac{\partial\psi}{\partial R}dR$$



c. $\psi_p = 2\pi\left[\psi(R,0) - \psi(R_a,0)\right]$ $\psi(R_a,0)$ is arbitary $\rightarrow$ set to zero

$$\psi_p = 2\pi\psi$$

d. We usually label the flux surfaces with ψ values rather than p values

6. $\mu_0 J = \nabla\times B$

a. $\mu_0 J = \nabla\times\left[RB_\phi\frac{e_\phi}{R} + \frac{1}{R}\nabla\psi\times e_\phi\right]$

$$= \nabla\left(RB_\phi\right)\times\frac{e_\phi}{R} + RB_\phi\nabla\times\overset{=0}{\frac{e_\phi}{R}} + \frac{e_\phi}{R}\cdot\nabla\left(\nabla\psi\right) - \nabla\psi\cdot\nabla\frac{e_\phi}{R} - \frac{e_\phi}{R}\nabla\cdot\nabla\psi + \nabla\psi\nabla\cdot\overset{=0}{\frac{e_\phi}{R}}$$

$$= \nabla\left(RB_\phi\right)\times\frac{e_\phi}{R} - \frac{e_\phi}{R}\nabla^2\psi + \frac{1}{R^2}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial R}e_R + \frac{\partial\psi}{\partial Z}\overset{=0}{e_z}\right) \rightarrow \frac{1}{R^2}\frac{\partial\psi}{\partial R}e_\phi$$

$$-\left(\frac{\partial\psi}{\partial R}\frac{\partial}{\partial R} + \frac{\partial\psi}{\partial Z}\frac{\partial}{\partial Z}\right)\overset{=0}{\frac{e_\phi}{R}} \rightarrow \frac{1}{R^2}\frac{\partial\psi}{\partial R}e_\phi$$

$$= \nabla RB_\phi\times\frac{e_\phi}{R} - \frac{e_\phi}{R}\left[\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} - \frac{2}{R}\frac{\partial\psi}{\partial R}\right]$$

$$\mu_0 \mathrm{J} = \nabla RB_\phi \times \frac{e_\phi}{R} - \frac{e_\phi}{R}\left[R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2}\right]$$

b. These results can be summarized as follows

$$\mu_0 \mathrm{J} = \mu_0 \mathrm{J}_\phi \mathrm{e}_\phi + \mu_0 \mathrm{J}_p$$

$$\mu_0 \mathrm{J}_p = \frac{1}{R}\nabla RB_\phi \times \mathrm{e}_\phi$$

$$\mu_0 \mathrm{J}_\phi = -\frac{1}{R}\Delta^*\psi$$

$$\Delta^*\psi \equiv R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$$

7. Force balance $\mathrm{J}\times\mathrm{B} = \nabla p$

   Decompose this relation into three components along B, J, $\nabla\psi$

8. B component

   a. $\mathrm{B}\cdot\nabla p = 0$

   b. $\dfrac{B_\phi}{R}\dfrac{\partial p}{\partial\phi} + \dfrac{\nabla\psi\times e_\phi}{R}\cdot\nabla p = 0$

   c. $e_\phi\cdot\nabla\psi\times\nabla p = 0$

   d. $p = p(\psi)$ p is an arbitrary free function of ψ .

   e. There is no way to determine $p(\psi)$ from ideal MHD. We need transport theory or some other simple physical model.

   f. Note: $p(\psi)$ is more of a constraint than $p(r,\theta)$

9. J component

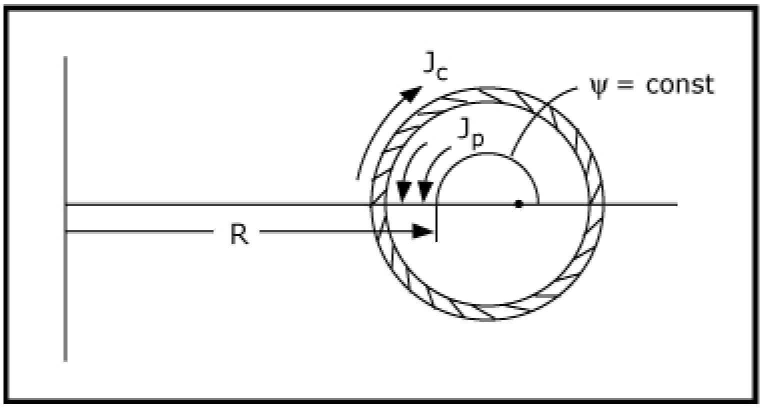
   a. $\mathrm{J}\cdot\nabla p = 0$

   b. $\dfrac{J_\phi}{R}\dfrac{\partial p}{\partial\phi} + \mathrm{J}_p\cdot\nabla p = 0 \rightarrow \dfrac{1}{R}\nabla RB_\phi : \times e_\phi\cdot\nabla\psi\dfrac{dp}{d\psi} = 0$

   c. $\dfrac{1}{R}\dfrac{dp}{d\psi}\left[e_\phi\cdot\nabla\psi\times RB_\phi\right] = 0$

d. $RB_\phi = F(\psi)$ F is an arbitrary free function

10. Interpretation of $F(\psi)$



a. $I_p = \int \mathrm{J}_p \cdot d\mathrm{A} = \int_0^R dR \int_0^{2\pi} R d\phi \left(\nabla RB_\phi \times \frac{e_\phi}{R} \right) z$

$$= 2\pi \int_0^R dR \frac{\partial F}{\partial R} = 2\pi \left[F(R,0) - F(0,0) \right]$$

$$= RB_\phi \big|_0 = 0$$

b. $I_p = 2\pi F(\psi)$

c. $I_p(\psi)$ is the total poloidal current passing through the circle ψ $(R, 0) =$ const.

11. $\nabla\psi$ component

$$\nabla\psi \cdot (\mathrm{J} \times \mathrm{B} - \nabla p) = 0$$

a. $T_1 = -\nabla\psi \cdot \nabla p = -\frac{dp}{d\psi} (\nabla\psi)^2$

b. $T_2 = \left(\mathrm{J}_p + J_\phi e_\phi \right) \times \left(\mathrm{B}_p + B_\phi e_\phi \right) \cdot \nabla\psi$

$$= \left[\mathrm{J}_p \times \mathrm{B}_p + \mathrm{J}_p \times e_\phi B_\phi + e_\phi \times \mathrm{B}_p J_\phi + \left(e_\phi \times e_\phi \right) J_\phi B_\phi \right] \cdot \nabla\psi$$

c. $T_a = \frac{1}{\mu_0} \left[\frac{1}{R} \nabla F \times e_\phi \times \frac{1}{R} \nabla\psi \times e_\phi \right] \cdot \nabla\psi$

$$= \frac{1}{\mu_0 R^2}\frac{dF}{d\psi}\left[\left(\nabla\psi \times e_\phi\right) \times \left(\nabla\psi \times e_\phi\right)\right] \cdot \nabla\psi = 0$$

d. $T_b = \frac{1}{\mu_0}\left[\left(\frac{1}{R}\nabla F \times e_\phi\right) \times e_\phi\right] \cdot \nabla\psi B_\phi$

$$= \frac{F}{\mu_0 R^2}\frac{dF}{d\psi}\left(\nabla\psi \times e_\phi\right) \times e_\phi \cdot \nabla\psi$$

$$= -\frac{F}{\mu_0 R^2}\frac{dF}{d\psi}\left(\nabla\psi\right)^2$$

e. $T_c = e_\phi \times \frac{\nabla\psi \times e_\phi}{R}\left(-\frac{1}{\mu_0 R}\Delta^*\psi\right) \cdot \nabla\psi$

$$= -\frac{1}{\mu_0 R^2}\Delta^*\psi\left(\nabla\psi\right)^2$$

f. Combine terms

$$\left(\nabla\psi\right)^2\left[-\frac{dp}{d\psi} - \frac{1}{\mu_0 R^2}\frac{d}{d\psi}\frac{F^2}{2} - \frac{1}{\mu_0 R^2}\Delta * \psi\right] = 0$$

g. The Grad–Shafranov equation is given by

$$\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

where

$$p = p\left(\psi\right)$$

free functions

$$F = F\left(\psi\right)$$

$$\mathrm{B} = \frac{1}{R}\nabla\psi \times e_\phi + \frac{F}{R}e_\phi$$

$$\mu_0 \mathrm{J} = \frac{1}{R}\frac{dF}{d\psi}\nabla\psi \times e_\phi - \frac{1}{R}\Delta * \psi e_\phi$$

and $\psi_p = 2\pi\psi,\ I_p = 2\pi F$