22.615, MHD Theory of Fusion Systems
Prof. Freidberg
Lecture 22

Ballooning mode equation

$$\delta W = \frac{\pi}{\mu_0}\int d\psi\, W(\psi)$$

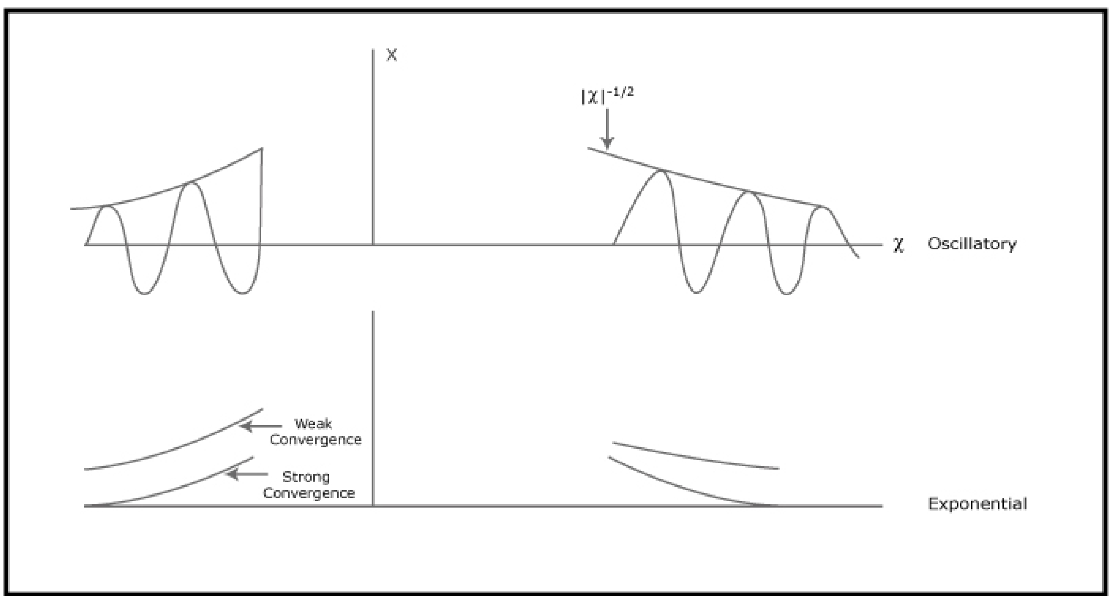
$$W(\psi) = \int_{-\infty}^{\infty} J d\chi \left[\left(k_n^2 + k_t^2\right)\left(\frac{1}{JB}\frac{\partial X}{\partial \chi}\right)^2 - \frac{2\mu_0 R B_p}{B^2}\frac{dp}{d\psi}\left(k_t^2\kappa_n - k_t k_n \kappa_t\right)X^2\right]$$

Mercier Criterion

1. In using the quasimode representation we have had to assume the solution X_φ converges sufficiently rapidly as $\chi \to \pm\infty$.

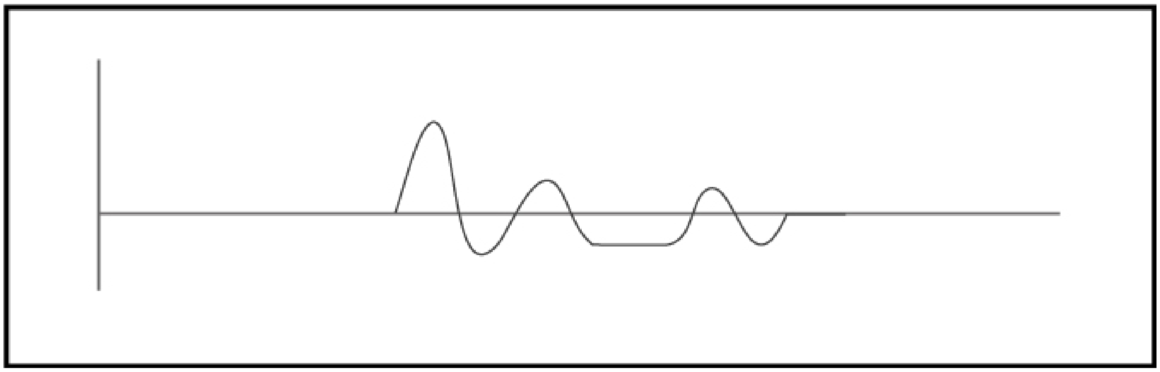
2. Whether or not convergence is acceptable depends upon equilibrium profiles and parameters.

3. Analysis of Euler-Lagrange equation for X indicates that there are two classes of solutions for large $|\chi|$ depending on profiles

4. 

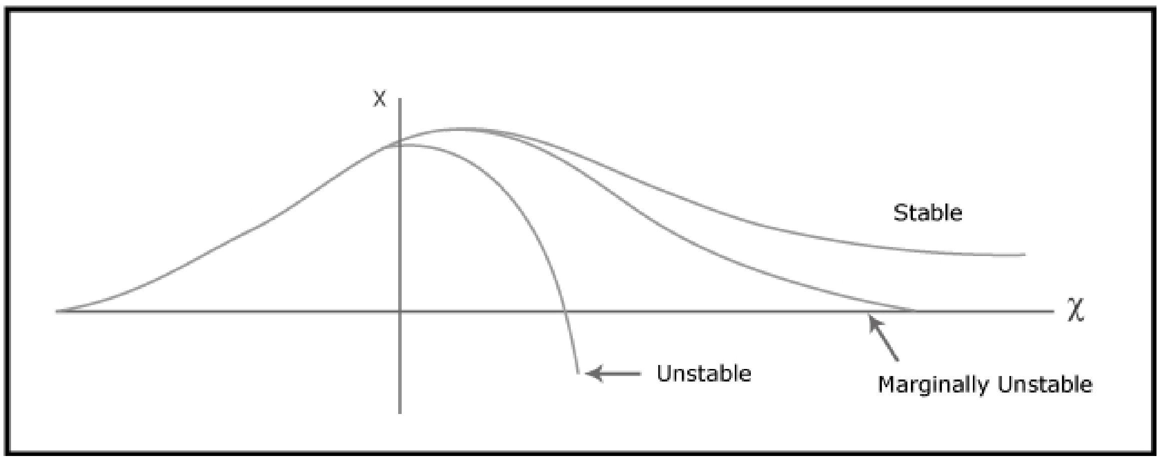
5. Oscillating solutions give rise to unbounded energy: $W \to \infty$. Strong convergence gives rise to bounded energy

6. Oscillatory case implies that ballooning mode formation is not valid as $\chi \to |\infty|$. However, for this case a trial function of the form



leads to $\delta W < 0$ (instability)

7. For exponential solutions, one starts with the strongly converging solution as $\chi = -\infty$ and integrates to the right.



8. The condition of oscillatory solutions is known as the Mercier criterion. When the Mercier criterion is violated, the solutions oscillate for large χ. The ballooning mode equation is not valid but this does not matter as the system is already unstable to interchanges.

9. When the solution's do not oscillate the Mercier criterion is satisfied and the system is stable to interchanges, and the ballooning mode formalism is solid. In this case one integrates the equation and looks to see if there is a zero-crossing. If there is one, the system is unstable to ballooning modes

10. Relation of Mercier-Suydam

Suydam: local behavior as $x \to 0$ near regular surface in space
Mercier: behavior as $\chi \to \infty$ in pseudo angle

Mercier is actually fourier transform of "Suydam like" analysis

$$e^{\iota S} : S \alpha \left(\psi - \psi_0\right) \int_{\chi_0}^{\chi} \frac{\partial}{\partial \psi_0} \left(\frac{JB_\theta}{R}\right) d\chi'$$

$$\approx (\psi - \psi_0)\,\chi\,\frac{dq(\psi)}{d\psi}$$

pseudo angle

radial localization

$$\approx xk$$

transform variable

Forms of the Mercier Criterion

1. Exact form $D_M < 1/4$ for stability

$$D_M = \frac{\mu_0 p'}{q'^2}\left[2\left\langle\frac{RB_p\kappa_n}{B^2}\right\rangle + \left\langle\frac{\Lambda}{B^4}\right\rangle - \left\langle\frac{1}{B^2}\right\rangle\left\langle\frac{\Lambda}{B^2}\right\rangle\right]\Bigg/\frac{R^2B_p^2}{JB^2}$$

$$\Lambda = F\left(\mu_0 p' F - \frac{R^2B_p^2}{J}\frac{\partial\hat{q}}{\partial\psi}\right)$$

$$\langle Q\rangle = \int_0^{2\pi}\frac{QB^2}{R^2B_p^2}\,J\,d\chi\Bigg/\int_0^{2\pi}\frac{B^2}{R^2B_P^2}\,J\,d\chi$$

2. For tokamaks, pressure is low: $\beta \sim \in, \in^2$. As with Suydam criterion, Mercier criterion is satisfied over most of the discharge because of low β. Only problem is near the origin.

   a. For a $\beta_p \sim 1$ tokamak with circular cross section, Mercier becomes

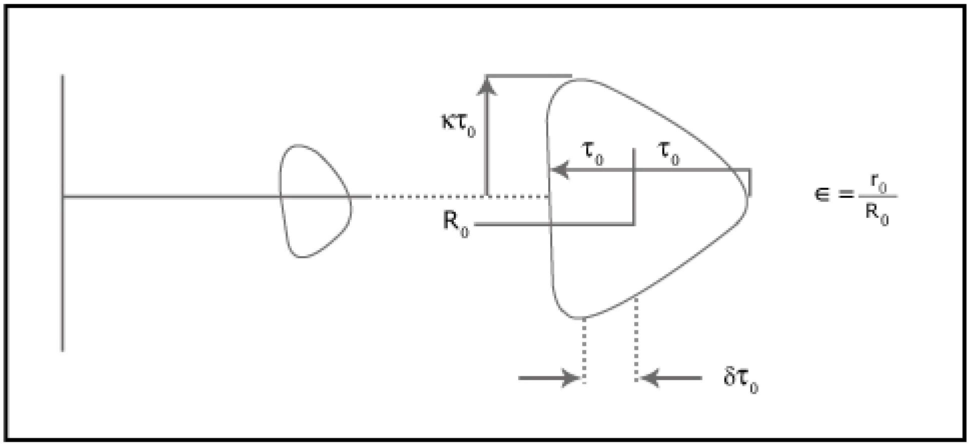
$$\frac{r^2 q'^2}{q^2} + 4r\beta'\left(1 - q^2\right) > 0$$

   Near $r=0$ q' is very small and we require

$$\boxed{q_0 > 1}$$

   b. Near the origin for non-circular tokamaks, the criterion becomes

$$1 < q_0^2\left\{1 - \frac{4}{1+3\kappa^2}\left[\frac{3}{4}\frac{\kappa^2-1}{\kappa^2+1}\left(\kappa^2 - \frac{2\delta}{\in}\right) + \frac{(\kappa-1)^2\beta_{p0}}{\kappa(\kappa+1)}\right]\right\}$$



for $\kappa = 1$, triangularity and β_p have no effect.

for $\kappa > 1$, β_p is destabilizing, $+\delta$ stabilizing

good Mercier: elongation and outward triangularity, moderate β_p and $q_0 \gtrsim 1$

c. Why do toroidal effects introduce such big changes since they are of order $\in$. Compute $\kappa_n = \underline{n} \cdot \left(\underline{b} \cdot \nabla \underline{b}\right)$

Circle: $\kappa_n \approx -\dfrac{B_\theta^2}{rB_\theta^2}$ $\qquad \underline{b} \approx \underline{e}_z + \dfrac{B_\theta}{B_0}\underline{e}_\theta$

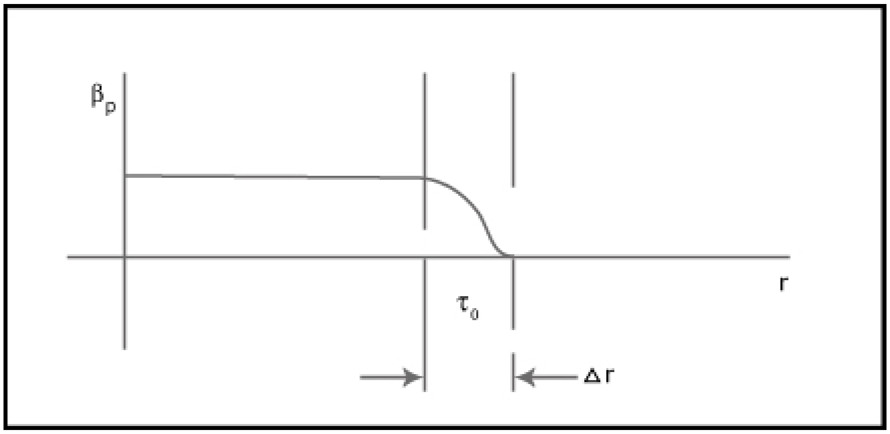
Torus: $\kappa_n \approx -\overset{\in^2}{\dfrac{B_\theta^2}{rB_0^2}} - \overset{\in}{\dfrac{\underline{e}_R}{R}}$ $\qquad \underline{b} \approx \underline{e}_\theta + \dfrac{B_\theta}{B_0}\underline{e}_\theta$

$$\approx -\frac{B_\theta^2}{rB_0^2} - \frac{1}{R_0}\left(1 - \frac{r}{R_0}\cos\theta + \ldots\right)\left(\underline{e}_r \cos\theta - \underline{e}_\theta \sin\theta\right)$$

$$\left\langle \kappa_r \right\rangle \approx -\frac{B_\theta^2}{rB_0^2}\left(1 - \frac{q^2}{2}\right)$$

Ballooning Modes

1. Simple limit ballooning mode equation for $\beta_p \sim 1$, circular cross section plasma with gradient in p'. $r\beta_p' \sim 1/\in$



2. $\chi \to \theta \qquad J \to \dfrac{r}{B_\theta} \qquad \kappa_n \to -\dfrac{\cos\theta}{R_0}$

   $R \to R_0 \quad B \to B_0$

   $B_p \to B_\theta(r) \qquad \dfrac{dp}{d\psi} \to \dfrac{1}{rB_\theta}\dfrac{dp}{dr}$

   and $\hat{\chi} = \int_{\chi_0}^{\chi} \dfrac{\partial}{\partial\psi}\dfrac{JB_\theta}{R}\,d\chi' = \dfrac{1}{R_0 B_\theta}\left[q'(\theta-\theta_0) + \dfrac{2\mu_0 rqp'}{R_0 B_\theta^2}(\sin\theta - \sin\theta_0)\right]$

   $$\frac{\partial}{\partial\chi}\left(\frac{\mu_0 \dfrac{dp}{d\psi}F}{B^2}\right) = -\frac{2\mu_0 r}{R_0 B_0 B_\theta}\frac{dp}{dr}\sin\theta$$

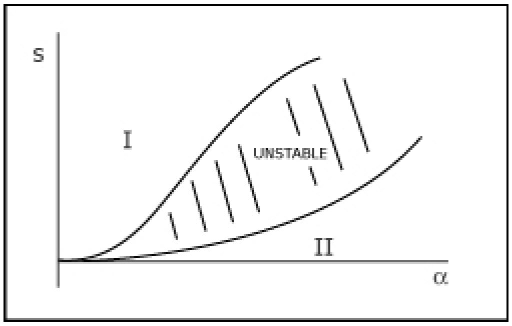
3. This gives for the Euler-Lagrange equation

   $$\frac{\partial}{\partial\theta}\left[\left(1+\Lambda^2\right)\frac{\partial X}{\partial\theta}\right] + \alpha\left[\Lambda\sin\theta + \cos\theta\right]X = 0$$

   $$\Lambda = S(\theta-\theta_0) - \alpha\left(\sin\theta - \sin\theta_0\right)$$

   $$S = \frac{rq'}{q} \qquad \alpha = -\frac{2\mu_0 r^2 p'}{R_0 B_\theta^2} = -q^2 R_0 \beta'$$

4. Solved numerically gives S vs α diagram



I first region of stability (goes unstable at high α)

II second region of stability (eventually becomes stable at high α)

5. Region I maximum β. Set $\alpha \approx \cdot 6\, S$ to determine critical profile for maximum β. For $q_a \gg 1, q_0 = 1$

$$\beta_t \le \cdot 3 \frac{\in}{q_a} \qquad \text{(circle)}$$

6. Numerical studies by Sykes, Yamazaki

$$\beta_t \lesssim \cdot 22 \frac{\in \kappa}{q_*} \qquad q_* = \frac{2B_0 \overbrace{A}^{\text{cross sectional area}}}{\mu_0 R_0 I}$$

$$= 0.44 \frac{I_0}{aB_0}$$

For optimized profiles with elongation and outer triangularity

7. For $\kappa = 2, q_0 = 1.5, \in = 1/3 \rightarrow \beta_t \approx 10\%$

Second Stability

1. Why does such a region exist

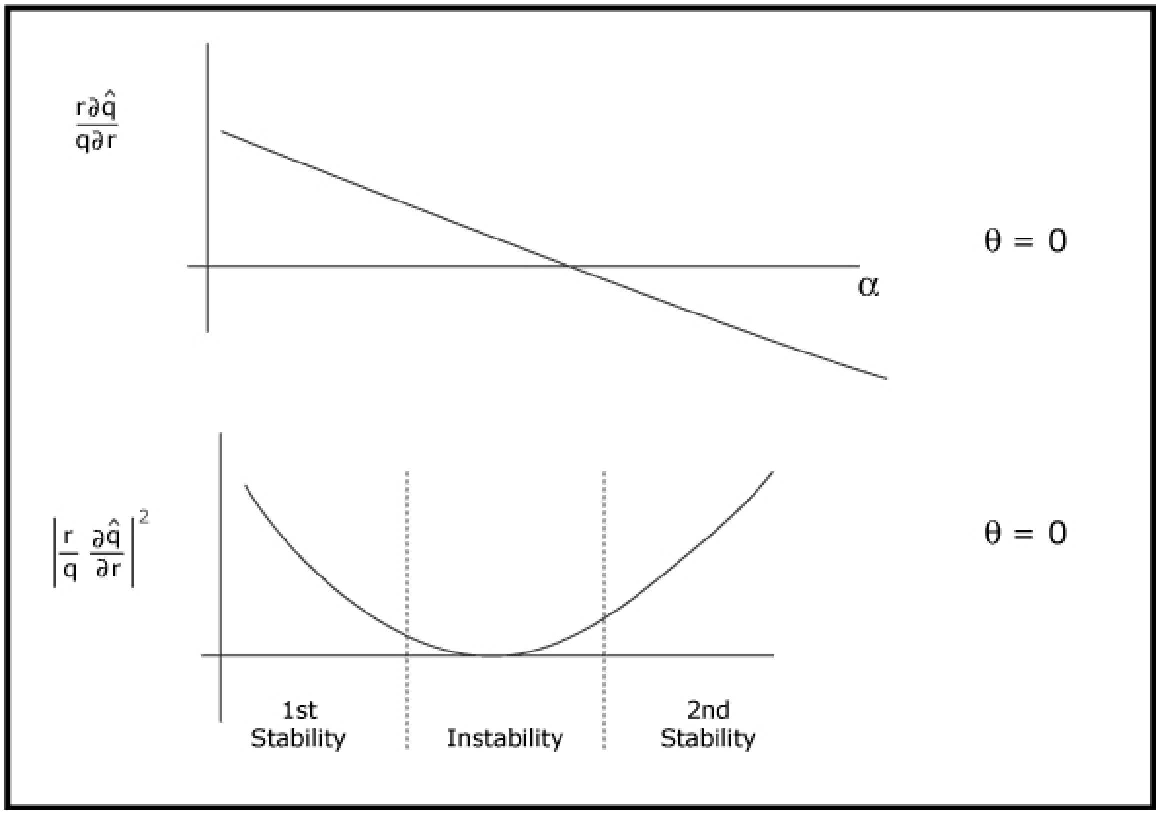
2. Examine local shear

$$q(\psi) = \frac{1}{2\pi}\int_0^{2\pi} \hat{q}(\psi, \chi)\, d\chi \qquad \hat{q} = \frac{JB_\theta}{R}$$

local shear

$$\frac{r}{q}\frac{\partial \hat{q}}{\partial r} = \overbrace{S}^{\text{shear}} - \underbrace{\alpha \cos\theta}_{\text{pressure driven modulation}}$$

3. Note: bad curvature occurs as $\theta = 0$ due to toroidal field. Stabilizing term due to shear $\alpha\left(\frac{rq'}{q}\right)^2$



External Kinks

1. Consider surface current model. p=const, circular cross section

$$\delta W = \delta W_p + \delta W_s + \delta W_v$$

$$\delta W_p = \int_p \frac{B_1^2}{2\mu_0} d\underline{r} \qquad > 0$$

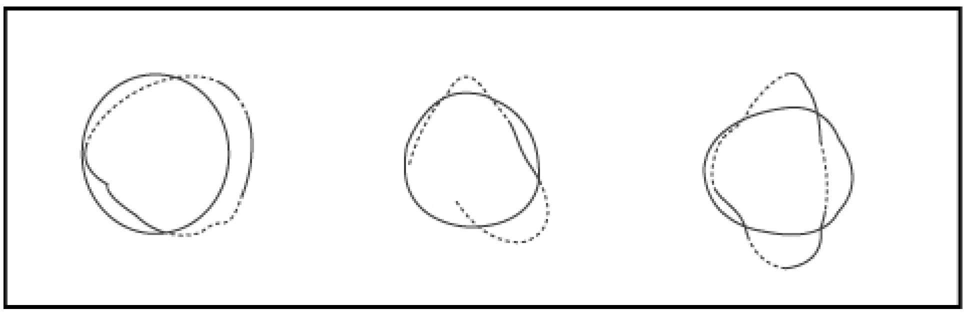
$$\delta W_v = \int_v \frac{\hat{B}_1^2}{2\mu_0} d\underline{r} \qquad > 0$$

$$\delta W_s = \frac{1}{2}\int_s ds \left|\underline{n}\cdot\underline{\xi}\right|^2 n\cdot\left[\!\left[\nabla\left(p + \frac{B^2}{2\mu_0}\right)\right]\!\right]$$

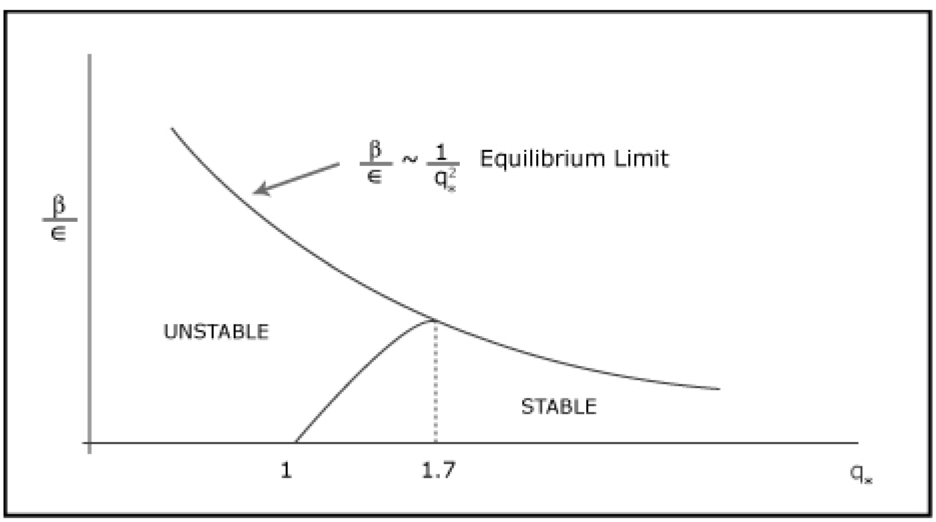
$$= \frac{1}{2}\int dS\left|\underline{n}\cdot\underline{\xi}\right|^2\left[\underbrace{2p\left\langle\kappa_n\right\rangle}_{\text{curvature term}}+\frac{\left\langle B^2\right\rangle}{2\mu_0}\underbrace{\left(\hat{\kappa}_n-\kappa_n\right)}_{\text{kink term}}\right] \qquad \left\langle\ \right\rangle = \frac{Q+\hat{Q}}{2}$$

$$\frac{\delta W_s}{2\pi \in^2 B_0^2 R_0/\mu_0} = -\frac{1}{2\pi}\int_0^{2\pi} d\theta\left|\xi_n\right|^2\left[\underbrace{\left(\frac{B_\theta}{\in B_0}\right)^2}_{\text{kink term}}+\underbrace{\left(\frac{\beta_t}{\in}\right)}_{\text{high }\beta\text{ ballooning effect}}\underbrace{\cos\theta}_{\text{toroidal curvature}}\right]$$

2. Modes have the structure of pressure driven kinks



3. Stability diagram (sharp boundary model)



4. Full numerical studies using optimized profiles and cross-sections

$$\beta_t = \cdot 14 \frac{\in \kappa}{q_*} = \cdot 028 \frac{I}{aB}$$

and $\quad q_* > q_{*\,min}$

Note, no dried $q_{*\,min}$ for following modes.

5. For $\kappa = 1, q_* = 1.7 \qquad \beta_t = .08 \in$

   For optimized Troyon $q_{*\,min} = 1,5 \ \ , \kappa = 1.6$

$$\beta_t = .15 \in$$

   Set $\in = 1/3 \rightarrow \quad \boxed{\beta_t = 5\%}$

   Near the regime of reactor interest!!

Requirements on β

1. There are two basic fusion energy requirements where β enters
2. Ignition and Wall Loading
3. Ignition $\left(T_e = T_\iota = T\right)$

   a. $P_f = \frac{n^2 \sigma v}{4} Q_f = Q_f \frac{n^2 T^2}{4} \frac{\sigma v}{T^2}$

   $$\beta = \frac{4\mu_0 nT}{B^2}$$

   b. $P_f = \frac{Q_f}{4} \frac{\sigma v}{T^2} \frac{\beta^2 B^4}{16\mu_0^2}$

   as 10 keV, $\sigma v = 10^{-22}\ m^3/sec$

   c. $P_f = .5 \times 10^6 \beta^2 B^4 \quad \omega/m^3$

   d. $P_f = P_e$

   e. $P_e = \frac{3nT}{\tau} = \frac{3}{\tau} \frac{\beta B^2}{4\mu_0} = .6 \frac{\beta B^2}{\tau} \omega/m^3$

   $$\therefore \ .5 \times 10^6 \beta^2 B^4 = .6 \frac{\beta B^2}{\tau}$$

$$\boxed{\beta\tau = \frac{1.2}{B^2}}$$

4. Wall Loading

.4

a. $P_E = \eta P_f = \eta P_f V = \eta .5 \times 10^6\ \beta^2 B^4\ 2\pi R_0 \pi a^2$

3

b. $P_E = 4 \times 10^6\ \beta^2\ B^4\ R_0 a^2 \approx 4 \times 10^6\ \beta^2\ B^4\ \frac{R_0}{a} a^3 \approx 1.2 \times 10^6\ \beta^2\ B^4 a^3$

$$\boxed{P_E = 1.2 \times 10^6\ \beta^2\ B^4 a^3}$$

c. $P_f = P_W A$

$$P_f\ 2\pi^2\ R_0 a^2 = P_W\ 4\pi^2\ aR_0$$

$$P_f = \frac{2P_W}{a}$$

$$\boxed{.5 \times 10^6\ \beta^2\ B^4 = \frac{2P_W}{a}}$$

d. Eliminate a from step b.

$$a = \frac{P_E^{1/3}}{\beta^{2/3} B^{4/3}} \frac{1}{\left(1.2 \times 10^6\right)^{1/3}} = 9 \times 10^{-3} \left(\frac{P_E}{\beta^2 B^4}\right)^{1/3}$$

$$\boxed{a = 9 \times 10^{-3} \left(\frac{P_E}{\beta^2 B^4}\right)^{1/3}}$$

e. Substitute a into (C)

$$.5 \times 10^6\ \beta^2\ B^4 = \frac{2P_W}{9 \times 10^{-3}} \left(\frac{\beta^2 B^4}{P_E}\right)^{1/3}$$

$$\boxed{\beta B^2 = 3 \times 10^{-3} \frac{P_W^{3/4}}{P_E^{1/4}}}$$

f. For $P_W = 4 \times 10^6\ \omega/m^2$ and $P_E = 10^9$ watts

$$\boxed{\beta B^2 \le 1.5}$$

g. For B=5T at $R = R_0$ then

$$\boxed{\beta < 6\%}$$

5. The Troyon limit

$$\beta < .03\frac{I}{aB}$$

$$\beta(\%) < \beta_N \frac{I_{MD}}{aB} \qquad \beta_N = 3$$

or $$\boxed{\beta < \cdot 15\frac{\in \kappa}{q_*} \approx .15 \times \frac{1}{3} \times \frac{1.8}{1.5} = 6\%}$$