22.615, MHD Theory of Fusion Systems
Prof. Freidberg
Lecture 2: The Moment Equations

Boltzmann-Maxwell Equations

1. Recall that the general coupled Boltzmann-Maxwell equations can be written as

   a. $$\frac{\partial f_j}{\partial t} + \mathrm{v} \cdot \nabla f_j + \frac{q_j}{m_j}\left(\mathrm{E} + \mathrm{v} \times \mathrm{B}\right) \cdot \nabla_v \mathrm{f}_j = \sum_k C_{jk} + s_j$$

   $$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

   $$\nabla \times \mathrm{B} = \mu_0 \mathrm{J} + \frac{1}{c^2}\frac{\partial \mathrm{E}}{\partial t}$$

   $$\nabla \cdot \mathrm{E} = \frac{\sigma}{\epsilon_0}$$

   $$\nabla \cdot \mathrm{B} = 0$$

   b. The particle and electromagnetic equations are coupled by

   $$\sigma = \sum_j q_j n_j = \sum_j q_j \int f_j d\mathrm{v}$$

   $$\mathrm{J} = \sum_j q_j n_j \mathrm{u}_j = \sum_j q_j \int v f_j d\mathrm{v}$$

   c. The collision operators C_{jk} arise from elastic collisions and satisfy a corresponding set of conservation relations.

2. Derivation of fluid equations – take moments as follows:

   a. mass

   $$\int \left[\frac{df_j}{dt} - \sum_k C_{jk} - s_j\right] d\mathrm{v} = 0$$

   b. momentum

   $$\int m_j \mathrm{v}\left[\frac{df_j}{dt} - \sum_k C_{jk} - s_j\right] d\mathrm{v} = 0$$

   c. energy

   $$\int \frac{m_j v^2}{2}\left[\frac{df_j}{dt} - \sum_k C_{jk} - s_j\right] d\mathrm{v} = 0$$

3. Introduce macroscopic quantities n_j (r, t), u_j (r, t), p_j (r, t).

4. The moment equations become a set of coupled, time dependent PDE's relating the various macroscopic quantities.

5. There initial Boltzmann equation is a single scalar equation in f_j: $f_j = f_j$ (r, v, t). There are seven independent variables.

6. The resulting moment equations contain six fluid variables: n_j, u_j, T_j, p_j, all functions of (r, t). There are four independent variables.

7. The fluid equations are far simpler to solve.

8. At a basic mathematics level the moment method appears to be ill conceived. You cannot solve a partial differential equation by integrating over several independent variables and then solving a reduced equation.

9. Example consider $\psi = \psi(r, \theta, t)$ satisfying

   a. $$\frac{\partial \psi}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} rf(r,\theta)\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta} g(r,\theta)\frac{\partial \psi}{\partial \theta} = 0$$

   b. Integrated over θ assuming periodicity: define $\langle G \rangle = (1/2\pi)\int G d\theta$

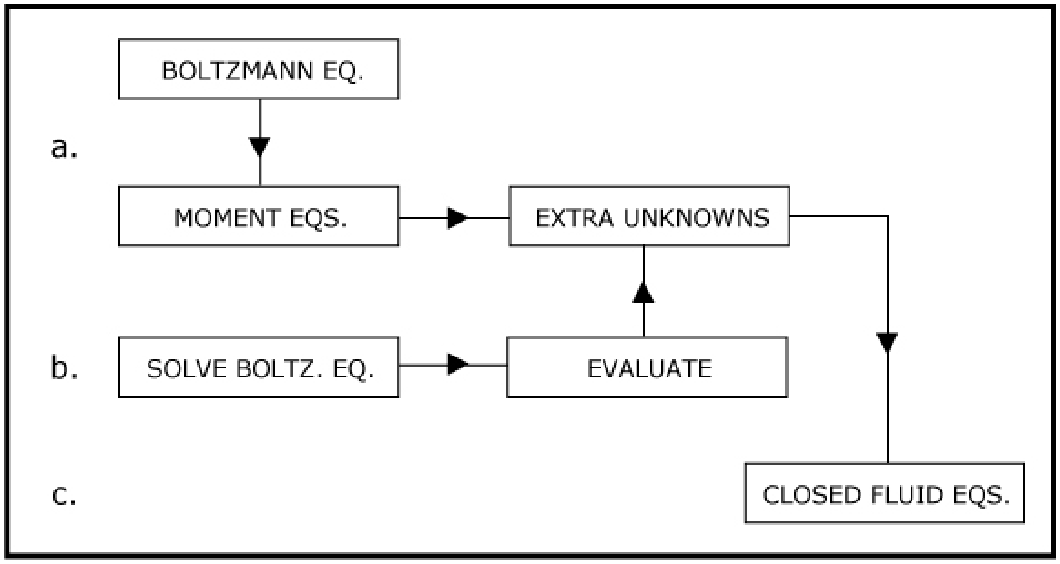
   $$\frac{\partial}{\partial t}\langle \psi \rangle = \frac{1}{r}\frac{\partial}{\partial r} r \left\langle f \frac{\partial \psi}{\partial r} \right\rangle$$

   c. This equation is a correct expression, but not a useful one since two different averages appear: $\langle \psi \rangle, \langle f \partial \psi / \partial r \rangle$. Integrating over θ leads to a single, simpler reduced equation, but with two unknowns.

10. This is a general property of moment equations. When we take moments of the Boltzmann equation, we will obtain a set of correct relations, but there will be more unknowns than equations.

11. How do we resolve this problem? We will close the set of equations by solving the Boltzmann equation and then evaluating some of the higher order, additional unknowns.

12. This would seem to make the entire procedure circular. If we are going to solve the Boltzmann equation anyway why bother with the moment equations?



13. There is method to this madness.

    a. First, even if we know the solution to the Boltzmann equation the moments represent more useful information in that they describe the measurable physical quantities in the system.

    b. Second, and equally important, we are not just going to "simply solve" the Boltzmann equation for the extra unknowns. The full equation is enormously complicated to solve.

    c. Instead, we shall solve the Boltzmann equation by means of various expansions $\left(\text{e.g. } m_e/m_i,\ r_L/a,\ \omega/\Omega,\ \text{etc}\right)$.

    d. Each order in the expansion is exponentially more painful to calculate than the previous order.

    e. By having a carefully defined set of moment equations, we can determine beforehand exactly how many terms are needed in the expansions. In addition we can rewrite the moment equations in such a way as to further minimize the number of terms required.

    f. These two reasons (physical variables, minimum algebra) are strong motivation for using the moment procedure.

Moment Equations

1. Procedure to be followed:

   a. Calculate exact moments: mass, momentum, energy.

   b. Introduce random velocity $\mathrm{v} = \mathrm{u}_j (\mathrm{r}, t) + \mathrm{w}$.

   c. Define physical variables n_j, u_j, T_j, p_j...

   d. Do some algebraic bookkeeping.

   e. Arrive at a set of moment equations (with more unknowns than equations).

2. Now focus on the momentum equation from which we derive the MHD equilibrium equation. Show that this equation is valid for both MHD stability analysis and transport phenomena. Motivation: almost no knowledge of higher order unknowns is required.

3. Spend half a semester investigating applications of the MHD equilibrium equation.

4. Derive in detail the collision operators C_{jk}.

5. Spend half a semester investigating applications of the transport equations in cylindrical geometry.

6. Consider next the conservation of mass equation: we derive this carefully, carrying out all the steps.

   a. $$\int d\mathrm{v}\left[\frac{\partial f_j}{\partial t} + \mathrm{v}\cdot\nabla f_j + \frac{q_j}{m_j}\left(\mathrm{E} + \mathrm{v}\times B\right)\cdot\nabla_v f_j - \sum_k C_{jk} - s_j\right] = 0$$

   b. Define $n_j = \int f_j d\mathrm{v}$ particle number density

      $n_j u_j = \int \mathrm{v} f_j d\mathrm{v}$ particle flux

   c. $$\int d\mathrm{v}\frac{\partial f_j}{\partial t} = \frac{\partial}{\partial t}\int d\mathrm{v} f_j = \frac{\partial n_j}{\partial t}$$

   d. $$\int d\mathrm{v}\ \mathrm{v}.\nabla f_j = \int d\mathrm{v}\left[\nabla\cdot\left(\mathrm{v}f_j\right) - \underbrace{f_j\nabla\cdot\mathrm{v}}_{=0}\right]$$

      $$= \nabla\cdot\int d\mathrm{v}\ \mathrm{v}f_j = \nabla\cdot\left(n_j\mathrm{u}_j\right)$$

e. $\int d\text{v} \frac{q_j}{m_j} \text{E} \cdot \nabla_v f_j = \frac{q_j}{m_j} \text{E} \cdot \int d\text{v} \left[\text{e}_x \frac{\partial f}{\partial v_x} + \text{e}_y \frac{\partial f_j}{\partial v_y} + \text{e}_z \frac{\partial f}{\partial v_z} \right] = 0$

f. $\int d\text{v} \frac{q_j}{m_j} \text{v} \times \text{B} \cdot \nabla_v f_j = \frac{q_j}{m_j} \int d\text{v} \left[\left(v_y B_z - v_z B_y \right) \frac{\partial f_j}{\partial v_x} + \cdots \right] = 0$

g. $-\int d\text{v} \sum_k C_{jk} = -\sum_k \int d\text{v} C_{jk} = 0$ (conservation of particles)

h. $-\int d\text{v} s_j \equiv -S_{nj}$ (source of density)

7. Combine terms: note one equation, four unknowns n_j, u_j

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \left(n_j u_j \right) = S_{nj}$$

8. Similar procedure for the momentum equation yields

$$\frac{\partial}{\partial t} \left(n_j m_j \text{u}_j \right) + \nabla \cdot \left(n_j m_j \langle \text{vv} \rangle \right) - q_j n_j \left(\text{E} + \text{u}_j \times \text{B} \right) = \sum\nolimits_k' \int m_j \text{v} C_{jk} d\text{v} + \text{S}_{pj}$$

9. a. Here $\sum\nolimits_k'$ denotes $k \neq j$ (due to conservation of momentum in like particle collisions)

b. $\text{S}_{pj} \equiv \int d\text{v} m_j \text{v} s_j = 0$ (source of momentum, zero for practical applications)

c. $\langle Q \rangle = \int Q f_j \, d\text{v} / n_j$

10. Similar procedure for energy equation yields

$$\frac{\partial}{\partial t} \frac{1}{2} m_j n_j \left\langle v^2 \right\rangle + \nabla \cdot \frac{1}{2} m_j n_j \left\langle v^2 \text{v} \right\rangle - q_j n_j u_j \cdot \text{E} = \sum\nolimits_k' \int \frac{m_j v^2}{2} C_{jk} d\text{v} + S_{Ej}$$

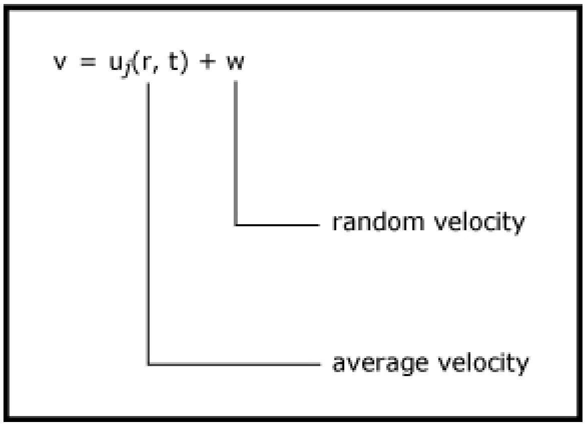
11. a. Here $S_{Ej} \equiv \int d\text{v} \left(m_j v^2 / 2 \right) s_j$ (sources of energy, say due to rf).

b. Also, $k=j$ vanishes from collision term because of conservation of energy in like particle collisions

Plasma Bookkeeping

1. The moment equations can be written in more physical terms by introducing the random velocity and defining various physical quantities in addition to n_j and u_j.

2. The random velocity w: this is a change of independent variables from v to w defined by

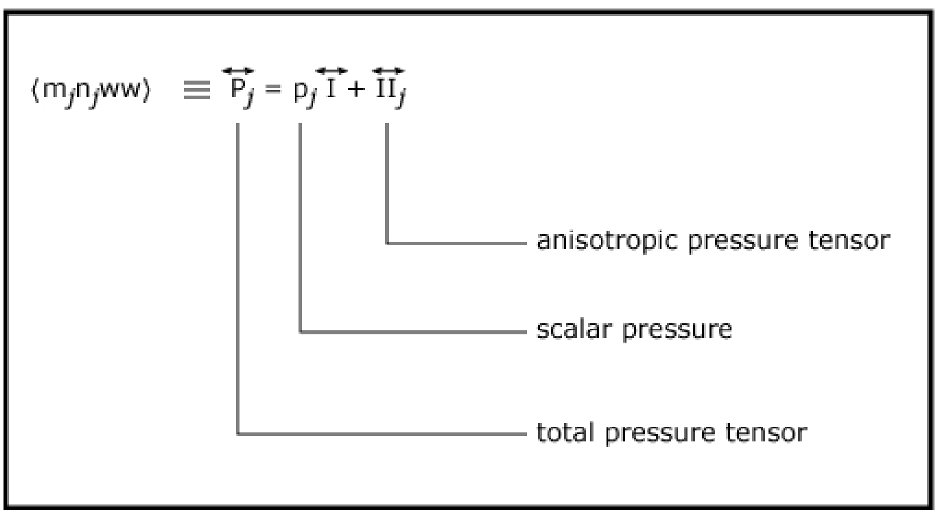


3. By definition dv=dw and $\langle w \rangle = 0$

4. Then

$$\langle vv \rangle = \langle u_j u_j + \underbrace{u_j w}_{=0} + \underbrace{w u_j}_{=0} + ww \rangle = \langle ww \rangle + u_j u_j$$

5. Define



where

$$p_j = \frac{1}{3} n_j m_j \langle \omega^2 \rangle$$

$$\overleftrightarrow{\Pi}_j = n_j m_j \left\langle \mathrm{ww} - \frac{1}{3}\omega^2 \overleftrightarrow{\mathrm{I}} \right\rangle$$

6. Similarly

$$\left\langle v^2 \right\rangle = \left\langle u_j^2 + \underbrace{2\mathrm{w}\cdot \mathrm{u}_j}_{=0} + \omega^2 \right\rangle = u_j^2 + \frac{3p_j}{n_j m_j}$$

$$\left\langle v^2 \mathrm{v} \right\rangle = \left\langle \left(u_j^2 + \underbrace{2\mathrm{w}\cdot \mathrm{u}_j}_{=0} + \omega^2 \right) \mathrm{u}_j + \left(\underbrace{\mathrm{u}_j^2}_{=0} + 2\mathrm{w}\cdot \mathrm{u}_j + \omega^2 \right) \mathrm{w} \right\rangle$$

$$= u_j^2 \mathrm{u}_j + \frac{3p_j u_j}{n_j m_j} + 2\frac{\mathrm{u}_j \cdot \overleftrightarrow{\mathrm{P}}_j}{n_j m_j} + \frac{2\mathrm{h}_j}{m_j n_j}$$

where

$$\mathrm{h}_j \equiv \frac{1}{2} n_j m_j \left\langle \omega^2 \mathrm{w} \right\rangle$$

is the heat flux, the flux of heat due to random motion.

7. Now define

$$\int m_j \left(\underbrace{\mathrm{u}_j}_{=0} + \mathrm{w}\right) C_{jk} d\mathrm{v} = \int m_j \mathrm{w} C_{jk} d\mathrm{w} \equiv \mathrm{R}_{jk}$$

$$\int \frac{m_j}{2}\left(\underbrace{u_j^2}_{=0} + 2\mathrm{u}_j \cdot \mathrm{w} + \omega^2\right) C_{jk} d\mathrm{v} = \mathrm{u}_j \cdot \mathrm{R}_{jk} + \int \frac{m_j \omega^2}{2} C_{jk} d\mathrm{w}$$

$$\equiv \mathrm{u}_j \cdot \mathrm{R}_{jk} + Q_{jk}$$

where R_{jk} is the average momentum transferred due to unlike collisions and Q_{jk} is the heat generated due to unlike collisions.

8. As they now stand, the moment equations can be written as

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \left(n_j \mathrm{u}_j \right) = s_{nj}$$

$$\frac{\partial}{\partial t}\left(n_j m_j \mathrm{u}_j \right) + \nabla \cdot \left(n_j m_j \mathrm{u}_j \mathrm{u}_j \right) + \nabla \cdot \overleftrightarrow{\mathrm{P}}_j - q_j n_j \left(\mathrm{E} + \mathrm{u}_j \times \mathrm{B} \right) = \sum\nolimits_k^{\prime} R_{jk}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(n_j m_j u_j^2\right)+\frac{3}{2}\frac{\partial p_j}{\partial t}+\nabla\cdot\left[\frac{1}{2}n_j m_j u_j^2 \mathbf{u}_j+\frac{3}{2}p_j\mathbf{u}_j+\mathbf{u}_j\cdot\overleftrightarrow{\mathbf{P}_j}+\mathbf{h}_j\right]-q_j n_j \mathbf{u}_j\cdot\mathbf{E}=\sum\nolimits_k'\left(\mathbf{u}_j\cdot\mathbf{R}_{jk}+\mathbf{Q}_{jk}\right)+s_{Ej}$$

Algebraic simplification

1. Define the convective derivative, moving with the fluid

$$\frac{dQ_j}{dt}=\frac{\partial \mathbf{Q}_j}{\partial t}+\mathbf{u}_j\cdot\nabla\mathbf{Q}_j$$

2. Define the temperature

$$T_j=p_j/n_j$$

3. The mass equation is OK as is

4. Momentum equation simplifications:

$$\frac{\partial}{\partial t}\left(n_j m_j \mathbf{u}_j\right)+\nabla\cdot\left(n_j m_j \mathbf{u}_j\mathbf{u}_j\right)$$

$$=m_j n_j\frac{\partial \mathbf{u}_j}{\partial t}+m_j\mathbf{u}_j\frac{\partial n_j}{\partial t}+m_j\mathbf{u}_j\nabla\cdot\left(n_j\mathbf{u}_j\right)+m_j n_j\mathbf{u}_j\cdot\nabla\mathbf{u}_j$$

$$=m_j n_j\frac{d\mathbf{u}_j}{dt}+m_j\mathbf{u}_j S_{nj}$$

5. The momentum equation becomes

$$m_j n_j\frac{d\mathbf{u}_j}{dt}+\nabla\cdot\overleftrightarrow{\mathbf{P}}_j-q_j n_j\left(\mathbf{E}+\mathbf{u}_j\times\mathbf{B}\right)=\sum\nolimits_k' R_{jk}-m_j\mathbf{u}_j S_{nj}$$

6. Energy equation simplifications

a. $$\frac{\partial}{\partial t}\frac{n_j m_j u_j^2}{2}+\nabla\cdot\frac{n_j m_j u_j^2}{2}\mathbf{u}_j=\frac{m_j n_j}{2}\frac{\partial}{\partial t}u_j^2+\frac{m_j u_j^2}{2}\frac{\partial n_j}{\partial t}+\frac{m_j u_j^2}{2}\nabla\cdot n_j u_j+\frac{m_j n_j}{2}\mathbf{u}_j\cdot\nabla u_j^2$$

$$=\frac{m_j n_j}{2}\frac{d}{dt}u_j^2+\frac{m_j u_j^2}{2}S_{nj}$$

b. $$\frac{3}{2}\left(\frac{\partial p_j}{\partial t}+\nabla\cdot p_j u_j\right)=\frac{3}{2}\left(n_j\frac{\partial T_j}{\partial t}+T_j\frac{\partial n_j}{\partial t}+T_j\nabla\cdot n_j u_j+n_j u_j\cdot\nabla T_j\right)$$

$$=\frac{3}{2}\left(n_j\frac{dT_j}{dt}+T_j S_{nj}\right)$$

c. Energy equation becomes

$$\frac{m_j n_j}{2}\frac{d}{dt}u_j^2 + \frac{3}{2}n_j\frac{dT_j}{dt} + \nabla\cdot\left(\mathrm{u}_j\cdot\vec{\vec{\mathrm{P}}}_j + h_j\right) - q_j n_j \mathrm{u_j}\cdot\mathrm{E} =$$

$$\sum\nolimits_k^l\left(\mathrm{u}_j\cdot\mathrm{R}_{jk} + Q_{jk}\right) + S_{Ej} - \left(\frac{m_j}{2}u_j^2 + \frac{3}{2}T_j\right)S_{nj}$$

d. Note that $\mathrm{u}_j\cdot$(momentum equation) is equal to

$$\frac{m_j n_j}{2}\frac{d}{dt}u_j^2 + \mathrm{u}_j\cdot\nabla\cdot\vec{\vec{\mathrm{P}}}_j - q_j n_j \mathrm{u}_j\cdot\mathrm{E} = \sum\nolimits_k^l \mathrm{u}_j\cdot\mathrm{R}_{jk} - m_j u_j^2 S_{nj}$$

e. Now subtract (d) from (e)

$$\frac{3}{2}n_j\frac{dT_j}{dt} + \nabla\cdot\left(\mathrm{u}_j\cdot\vec{\vec{\mathrm{P}}}_j + h_j\right) - \mathrm{u}_j\cdot\nabla\cdot\vec{\vec{\mathrm{P}}}_j = \sum\nolimits_k^l Q_{jk} + S_{Ej} + \left(\frac{m_j u_j^2}{2} - \frac{3}{2}T_j\right)S_{nj}$$

f. Note the following identity (recalling that by definition $P_{ij} = P_{ji}$)

$$\nabla\cdot\left(\mathrm{u}\cdot\vec{\vec{\mathrm{P}}}\right) - \mathrm{u}\cdot\nabla\cdot\vec{\vec{\mathrm{P}}} = \frac{\partial}{\partial x_j}\left(u_i P_{ij}\right) - u_i\frac{\partial}{\partial x_j}P_{ij}$$

$$= P_{ij}\frac{\partial}{\partial x_j}u_i$$

$$= \vec{\vec{\mathrm{P}}} : \nabla\mathrm{u}$$

g. The energy equation becomes

$$\frac{3}{2}n_j\frac{dT_j}{dt} + \vec{\vec{\mathrm{P}}}_j : \nabla\mathrm{u}_j + \nabla\cdot h_j = \sum\nolimits_k^l Q_{jk} + S_{Ej} + \left(\frac{m_j u_j^2}{2} - \frac{3}{2}T_j\right)S_{nj}$$

7. Summary of fluid moments

$$\frac{dn_j}{dt} + n_j\nabla\cdot\mathrm{u}_j = S_{nj}$$

$$m_j n_j\frac{d\mathrm{u}_j}{dt} + \nabla\cdot\vec{\vec{\mathrm{P}}}_j - q_j n_j\left(\mathrm{E} + \mathrm{u}_j\times\mathrm{B}\right) = \sum\nolimits_k^l \mathrm{R}_{jk} - m_j\mathrm{u}_j S_{nj}$$

$$\frac{3}{2}n_j\frac{dT_j}{dt} + \vec{\vec{\mathrm{P}}}_j : \nabla\mathrm{u}_j + \nabla\cdot\mathrm{h}_j = \sum\nolimits_k^l Q_{jk} + S_{Ej} + \left(\frac{m_j u_j^2}{2} - \frac{3}{2}T_j\right)S_{nj}$$

8. Assuming the collisional terms are known, the fluid unknowns are n_j, $\mathbf{u}_j$, $\overleftrightarrow{\mathrm{P}}_j$, T_j, and $\mathbf{h}_j$. (17 unknowns, 5 equations)

9. Even with a scalar pressure, there are still 9 unknowns.

10. The moment equations above are exact, if not particularly useful. They do, however, accurately describe both MHD and transport phenomena.