22.615, MHD Theory of Fusion Systems
Prof. Freidberg
Lecture 17: Stability of Simple Function

Memory of the Energy Principle

$\delta W \geq 0$ for all displacements implies stability

The potential energy is given by

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

$$\delta W_F = \frac{1}{2}\int_P d\underline{r}\left[\frac{|\underline{Q}|^2}{\mu_0} - \underline{\xi}_\perp^* \cdot \underline{J}\times\underline{Q} + rp\left|\nabla\cdot\underline{\xi}\right|^2 + \left(\underline{\xi}_\perp\cdot\nabla p\right)\nabla\cdot\underline{\xi}_\perp^*\right]$$

$$\delta W_S = \frac{1}{2}\int dS\left|\underline{n}\cdot\underline{\xi}_\perp\right|^2 \underline{n}\cdot\left[\!\left[\nabla\left(p+\frac{B^2}{2\mu_0}\right)\right]\!\right]$$

$$\delta W_V = \frac{1}{2}\int_V d\underline{r}\frac{\hat{B}_1^2}{\mu_0}, \quad \nabla\times\underline{\hat{B}}_1 = \nabla\cdot\underline{\hat{B}}_1 = 0 \quad \underline{n}\cdot\underline{\hat{B}}_1\Big|_{S_w} = 0$$

$$\underline{n}\cdot\underline{\hat{B}}_1\Big|_{S_p} = \underline{\hat{B}}_1\cdot\nabla\left(\underline{n}\cdot\underline{\xi}_\perp\right) - \left(\underline{n}\cdot\underline{\xi}_\perp\right)\underline{n}\cdot\left(\underline{n}\cdot\nabla\right)\underline{\hat{B}}$$

Only Appearance of $\xi_\parallel$

$$\delta W_1 = \frac{1}{2}\int \gamma p\left|\nabla\cdot\underline{\xi}\right|^2$$

a. Minimizing condition : close $\xi_\parallel$ as $\nabla\cdot\underline{\xi} = 0\left[\underline{B}\cdot\nabla\left(\nabla\cdot\underline{\xi}\right) = 0\right]$

b. Possible if operator $\underline{B}\cdot\nabla$ can be inserted.

c. Not possible : symmetry $\underline{B}\cdot\nabla \equiv 0 \quad \nabla\cdot\underline{\xi} = \nabla\cdot\underline{\xi}_\perp + \underline{B}\cdot\nabla\frac{\xi_\parallel}{B} = \nabla\cdot\underline{\xi}_\perp$

d. Not possible : closed line case: $\nabla\cdot\underline{\xi} = F(p), \nabla\cdot\underline{\xi} = \left\langle\nabla\cdot\underline{\xi}_\perp\right\rangle = \dfrac{\oint\frac{\mu}{B}\nabla\cdot\underline{\xi}_\perp}{\oint\frac{\mu}{B}}$

Final Step: Intuitive Form of δW_F

$$\delta W_F = \frac{1}{2}\int d\underline{r}\left[\frac{|\underline{Q}|^2}{\mu_0} - \underline{\xi}_\perp^*\cdot\left(\underline{J}\times\underline{Q}\right) + rp\left|\nabla\cdot\underline{\xi}\right|^2 + \left(\underline{\xi}_\perp\cdot\nabla p\right)\nabla\cdot\underline{\xi}_\perp^*\right] \qquad \text{standard}$$

a. $|\underline{Q}|^2 = |\underline{Q}_\perp|^2 + |Q_\parallel|^2$

b. $\underline{\xi}_\perp^* \cdot \underline{J} \times \underline{Q} = \left(\underline{\xi}_\perp^* \times \underline{b}\right) \cdot \underline{Q}_\perp J_\parallel + \left(\underline{\xi}_\perp^* \cdot \underline{J}_\perp \times \underline{b}\right) Q_\parallel$

c. $\underline{J}_\perp = \dfrac{b \times \nabla p}{B} \qquad \left(\underline{J} \times \underline{B} = \nabla p\right)$

d. $Q_\parallel = \underline{b} \cdot \nabla \times \left(\underline{\xi}_\perp \times \underline{B}\right)$

$$= \underline{b} \cdot \left(\underline{B} \cdot \nabla \underline{\xi}_\perp - \underline{\xi}_\perp \cdot \nabla B - \underline{B}\, \nabla \cdot \underline{\xi}_\perp\right)$$

$$= -B\left(\nabla \cdot \underline{\xi}_\perp - 2\underline{\xi}_\perp \cdot \underline{\kappa}\right) + \frac{\mu_0}{B}\; \underline{\xi}_\perp \cdot \nabla p \qquad \underline{\kappa} = \underline{b} \cdot \nabla \underline{b}$$

2. Substitute task

$$\delta W_F = \frac{1}{2} \int d\underline{r} \left[\overset{1}{\frac{|\underline{Q}_\perp|^2}{\mu_0}} + \overset{2}{\frac{B^2}{\mu_0}\left|\nabla \cdot \underline{\xi}_\perp + 2\underline{\xi}_\perp \cdot \underline{\kappa}\right|^2} + \overset{3}{\gamma p \left|\nabla \cdot \underline{\xi}\right|^2} \overset{4}{- 2\left(\underline{\xi}_\perp \cdot \nabla p\right)\left(\underline{\kappa} \cdot \underline{\xi}_\perp^*\right)} \overset{5}{- J \times \left(\underline{\xi}_\perp^* \times \underline{b}\right) \cdot \underline{Q}_\perp} \right]$$

1. line bending > 0
2. magnetic compression > 0
3. plasma compression > 0
4. pressure driven modes + or -
5. current driven modes + or -

Classes of MHD Instability

1. Internal or fixed boundary: plasma surface is held fixed during perturbation: $\underline{n} \cdot \underline{\xi}_\perp \big|_{S_p} = 0$, same as a conducting wall
2. External or free boundary: plasma surface is allowed to move: $\underline{n} \cdot \underline{\xi}_\perp \neq 0$. Often the most severe stability criteria
3. Current driven modes: also called kink modes. $J_\parallel$ is the most dominant destabilizing term. Modes driven by parallel current. Important in tokamaks, RFP: (K-S limit, saw tooth oscillations, disruptions). In general modes have long wavelength, low m, n. Cures: tight aspect ratio, low current, packed current profiles, conducting wall.

4. Pressure driven modes: $\kappa\nabla p$ dominant destabilizing term. Special cases interchange or flute, following mode, sausage instability. Kaydoms criterion, mercuir criterion. Important in tokamak, RFP, stillaratio, EBT, mirror. In general long $\parallel$ wavelength, short $\perp$ wavelength. Cures: low β, shear, average formable curvature (min B, magnetic wall)

Applications Today

1. θ pinch
2. Z pinch

Procedure

1. Sine equilibrium $\underline{J}_0, \underline{B}_0, P_0$
2. Test in compressibility condition for $\xi_{\parallel}$
3. Minimize δW with respect to $\underline{\xi_{\perp}}$
4. If $\delta W_{min} > 0$ stable

   $\delta W_{min} < 0$ unstable

   $\delta W_{min} = 0$ marginally stable

θ Pinch

1. Equilibrium: $p(r), B_z(r), J_\theta(r)$

$$\mu_0 J_\theta = -B_z'$$

$$p(r) + \frac{B_z^2(r)}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

2. Stability: $\underline{\xi}(\underline{r}) = \underline{\xi}(r)e^{\iota m\theta + \iota k\xi}$: Fourier analyze analog θ and z

3. Check compressibility

$$\nabla \cdot \underline{\xi} = \nabla \cdot \underline{\xi_{\perp}} + \nabla \cdot \left(\xi_{\parallel}\underline{e}_x\right) = \nabla \cdot \underline{\xi_{\perp}} + \iota k\xi_{\parallel} \qquad \left(\xi_{\parallel} = \xi_z\right)$$

Set $\nabla \cdot \underline{\xi} = 0$: $\quad \xi_{\parallel} = -\frac{\iota}{k}\nabla \cdot \underline{\xi_{\perp}} \qquad$ OK if $k \neq 0$

4. Evaluate terms in δW_F

a. $\underline{\kappa} = \underline{b} \cdot \nabla \underline{b} = \frac{\partial}{\partial z} \underline{e}_x = 0$ no pressure driven terms

b. $J_{\|} = \underline{J} \cdot \underline{b} = J_0 \underline{e}_\theta \cdot \underline{e}_x = 0$ no current driven terms

5. Conclusion:

$\delta W_F \geq 0$ sum of positive terms

$\delta W_s = 0$ no surface currents

$\delta W_v \geq 0$ positive term

θ pinch is stable at any value of β worst case: $\delta W \to 0$ as $k \to 0$

Z Pinch

1. Equilibrium: $\rho(r), B_\theta(r), J_z(r)$

$$\mu_0 J_z = \frac{(rB_\theta)'}{r}$$

$$\rho' + \frac{B_\theta}{\mu_0 r}(rB_\theta)' = 0$$

2. Stability: $\underline{\xi}(\underline{r}) = \underline{\xi}(r)^{\iota m\theta + \iota kz}$

3. Check incompressibility: $\| \to \theta$

$$\nabla \cdot \underline{\xi} = \nabla \cdot \underline{\xi_\perp} + \nabla \cdot \xi_{\|} \underline{e}_\theta = \nabla \cdot \underline{\xi_\perp} + \frac{\iota m \xi_{\|}}{r} \qquad \xi_{\|} = \xi_\theta$$

But $\nabla \cdot \underline{\xi} = 0$ $\quad \xi_{\|} r - \frac{\iota}{m} r \nabla \cdot \underline{\xi_\perp}$ ok if $m \neq 0$

4. Evaluate terms in δW_F

a. $J_{\|} = \underline{J} \cdot \underline{b} = J_z \underline{e}_z \cdot \underline{e}_\theta$ no current driven terms

b. $\underline{\kappa} = \underline{b} \cdot \nabla \underline{b} = +\underline{e}_\theta \cdot \nabla \underline{e}_\theta = \frac{1}{r}\frac{\partial}{\partial \theta} \underline{e}_\theta = -\frac{\underline{e}_r}{r}$

$$-2\left(\underline{\xi_\perp} \cdot \nabla p\right)\left(\underline{\xi_\perp^*} \cdot \underline{\kappa}\right) = \left(-2\xi_r p'\right)\left(-\frac{\xi_r^*}{r}\right) = \frac{2p'}{r}\left|\xi_r\right|^2 \quad < 0 \text{ if } p' < 0$$

destabilizing term

c. $\underline{B}_1 = \nabla \times \left(\underline{\xi}_\perp \times \underline{B}\right) = \underline{B} \cdot \nabla \underline{\xi}_\perp - \underline{\xi}_\perp \cdot \nabla \underline{B} - \underline{B} \nabla \cdot \underline{\xi}_\perp$

$$B_{1r} = \frac{\iota m B_\theta}{r}\xi r - \cancel{\frac{B_\theta \xi_\theta}{r}} + \cancel{\frac{B_\theta \xi_\theta}{r}} = \frac{\iota m B_\theta}{r}\xi r$$

$$B_{1z} = \frac{\iota m B_\theta}{r}\xi_z$$

$$\left.\begin{matrix} B_{1r} \\ B_{1z} \end{matrix}\right\} \quad \underline{B_{1\perp}} = \frac{\iota m B_\theta}{r}\underline{\xi_\perp}$$

$$\left|\underline{B}_{1\perp}\right|^2 = \frac{m^2 B_\theta^2}{r^2}\left[\left|\xi r\right|^2 - \left|\xi_z\right|^2\right]$$

d. $\nabla \cdot \underline{\xi_\perp} + 2\underline{\xi_\perp} \cdot \underline{\kappa} = \frac{1}{r}\left(r\xi_r\right)' + \iota k \xi_z - \frac{2\xi_r}{r} = r\left(\frac{\xi_r}{r}\right)' + \iota k \xi_z$

$$B^2\left|\nabla \cdot \underline{\xi_\perp} + 2\underline{\xi_\perp} \cdot \underline{\kappa}\right|^2 = B_\theta^2\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2\left|\xi_z\right|^2 + \iota k r \xi_z\left(\frac{\xi_r^*}{r}\right)' - \iota k r \xi_z^*\left(\frac{\xi_r}{r}\right)'\right]$$

e. $\nabla \cdot \underline{\xi} = 0 \quad (\text{for } m \neq 0)$

$$= \frac{\left(r\xi_r\right)'}{r} + \iota k \xi_z \quad (\text{for } m = 0)$$

$$\gamma p\left|\nabla \cdot \underline{\xi}\right|^2 = \gamma p\left[\left|\frac{\left(r\xi_r\right)'}{r}\right|^2 + k^2\left|\xi_z\right|^2 + \frac{\iota k \xi_z}{r}\left(r\xi_r^*\right)' - \frac{\iota k \xi_z^*}{r}\left(r\xi_r\right)'\right] \quad (m = 0)$$

Examine $m \neq 0$

$$\delta W_F = \frac{1}{2}\int d\underline{r}\left\{\frac{m^2 B_\theta^2}{\mu_0 r^2}\left[\left|\xi_r\right|^2 + \left|\xi_r\right|^2\right] + \frac{2p'}{r}\left|\xi_r\right|^2 + \frac{B_\theta^2}{\mu_0}\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2\left|\xi_z\right|^2 + \iota k r \xi_z\left(\frac{\xi_r^*}{r}\right)' - \iota k r \xi_z^*\left(\frac{\xi_r}{r}\right)'\right]\right\}$$

Minimize δW_F

1. Note that ξ_z only appears algebraically: complete squares

   ξ_z terms:

   a. $$\frac{B_\theta^2}{\mu_0}\left[\overbrace{\left(\frac{m^2}{r^2}+k^2\right)}^{k_0^2(r)}|\xi_z|^2+\imath kr\xi_z\left(\frac{\xi_r^*}{r}\right)-\imath kr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right]$$

   b. $$\frac{B_\theta^2 k_0^2}{\mu_0}\left[\left|\xi_z\frac{\imath kr}{k_0^2}\left(\frac{\xi_r}{r}\right)'\right|^2-\frac{k^2r^2}{k_0^4}\left|\left(\frac{\xi_r}{r}\right)'\right|^2\right]$$

   c. Choose $\xi_z=\dfrac{ikr}{k_0^2}\left(\dfrac{\xi_r}{r}\right)'$ minimizing condition

2. Then $(\xi_r\equiv\xi)$

$$\delta W_F=\frac{1}{2\mu_0}\int d\underline{r}\left[|\xi|^2\left(\frac{m^2B_\theta^2}{r^2}+\frac{2\mu_0p'}{r}\right)+B_\theta^2\left|r\left(\frac{\xi}{r}\right)'\right|^2\underbrace{\left(1-\frac{k^2}{k_0^2}\right)}_{\frac{m^2}{m^2+k^2r^2}}\right]$$

$$\delta W_F=\frac{1}{2\mu_0}\int d\underline{r}\left[|\xi|^2\left(\frac{m^2B_\theta^2}{r^2}+\frac{2\mu_0p'}{r}\right)+\frac{m^2B_\theta^2}{m^2+k^2r^2}\left|r\left(\frac{\xi}{r}\right)'\right|^2\right]$$

only appearance of k

3. k appears only in a satisfying term this term is minimized by choosing

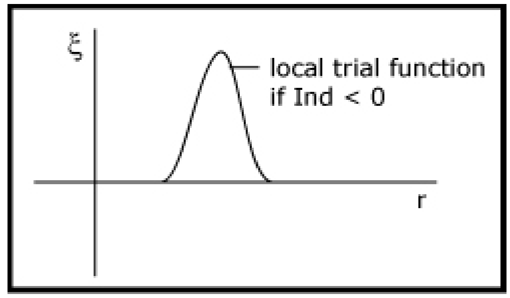
   $\boxed{k^2\to\infty}$

4. Then

$$\delta W_F=\frac{1}{2\mu_0}\int d\underline{r}|\xi|^2\left(\frac{m^2B_\theta^2}{r^2}+\frac{2\mu_0p'}{r}\right)$$

5. Stability condition

$$\boxed{-rp' < \frac{m^2 B_\theta^2}{2\mu_0}}$$



6. Simplify using equil relation $\mu_0 p' + \frac{B_\theta}{r}(rB_\theta)' = 0$

$$B_\theta (rB_\theta)' = B_\theta \left(r^2 \frac{B_\theta}{r}\right)' = r^2 B_\theta \left(\frac{B_\theta}{r}\right)' + 2B_\theta^2$$

or $$= r\left(\frac{B_\theta^2}{2}\right)' + B_\theta^2 = \left(\frac{rB_\theta^2}{2}\right)' + \frac{B_\theta^2}{2}$$

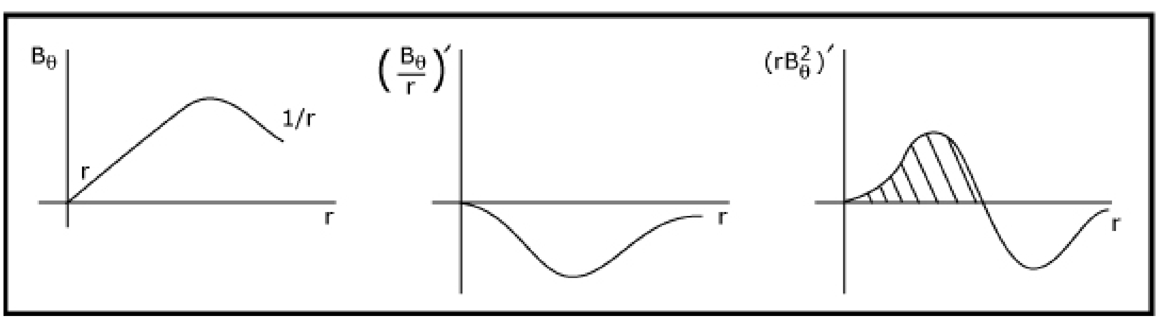
7. Then

$$\frac{r^2}{B_\theta}\left(\frac{B_\theta}{r}\right)' < \frac{1}{2}\left(m^2 - 4\right) \qquad (1)$$

or

$$\frac{\left(rB_\theta^2\right)'}{B_\theta^2} < \frac{1}{2}\left(m^2 - 1\right) \qquad (2)$$

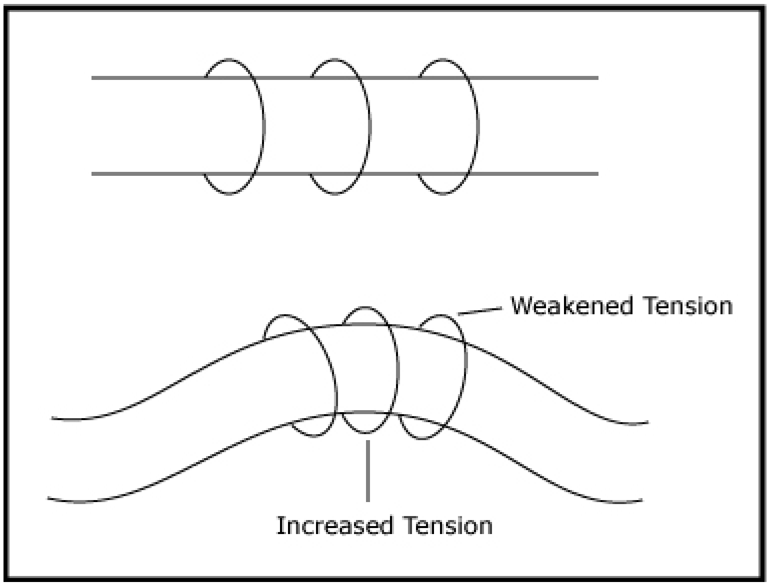
8. Typical profile



From (1) $\boxed{\text{stability for } m \ge 2}$

From (2) $\boxed{\text{instability for } m=1}$ near small r, $k \longrightarrow \infty$

9. Physical Mechanism



Examine m=0

1. $$\delta W_F = \frac{1}{2\mu_0}\int d\underline{r}\left\{B_\theta^2\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2\left|\xi_z\right|^2 + \iota k r\xi_z\left(\frac{\xi_r^*}{r}\right)' - \iota k r\xi_z^*\left(\frac{\xi_r}{r}\right)'\right]\right.$$
$$\left. + \mu_0\gamma p\left[\left|\frac{(r\xi_r)'}{r}\right|^2 + k^2\left|\xi_z\right|^2 + \frac{\iota k\xi_z}{r}\left(r\xi_r^*\right)' - \frac{\iota k\xi_r^*}{r}\left(r\xi_r\right)'\right] + \frac{2\mu_0 p'}{r}\left|\xi_r\right|^2\right\}$$

Note again that ξ_z only appears algebraically: complete squares.

2. ξ_z terms:

a. $$k^2\left(B_\theta^2 + \mu_0\gamma p\right)\left|\xi_z\right|^2 + \left[B_\theta^2 r\left(\frac{\xi_r^*}{r}\right)' + \mu_0\gamma p\frac{\left(r\xi_r^*\right)}{r}\right]\left(\iota k\xi_z\right) + c.c.$$

b. $$\left(B_\theta^2 + \mu_0\gamma p\right)\left|k\xi_z - \frac{\iota\left[B_\theta^2 r\left(\frac{\xi_r}{r}\right)' + \mu_0\gamma p\frac{\left(r\xi_r\right)'}{r}\right]}{B_\theta^2 + \mu_0\gamma p}\right|^2 - \frac{1}{B_\theta^2 + \mu_0\gamma p}\left|B_\theta^2 r\left(\frac{\xi_r}{r}\right)' + \mu_0\gamma p\frac{\left(r\xi_r\right)'}{r}\right|^2$$

3. Only appearance of ξ_z is in a stabilizing term. δW is minimized by choosing

$$\delta_z = \frac{\iota}{k\left(B_\theta^2 + \mu_0\gamma p\right)}\left[B_\theta^2 r\left(\frac{\xi_r}{r}\right)' + \mu_0\gamma p\frac{\left(r\xi_r\right)'}{r}\right]$$

4. δW_F becomes $\left(\xi_i \equiv \xi\right)$

$$\delta W_F = \frac{1}{2\mu_0}\int d\underline{r}\left\{\frac{2\mu_0 p'}{r}|\xi|^2 + B_\theta^2\left[\left|\xi'\right|^2 + \frac{|\xi|^2}{r^2} - \frac{\left(\xi'\xi^* + \xi^{*'}\xi\right)}{r}\right]\right.$$

$$+\gamma\mu_0 p\left[\left|\xi'\right|^2 + \frac{|\xi|^2}{r^2} - \frac{\left(\xi'\xi^* + \xi^{*'}\xi\right)}{r}\right]$$

$$\left. - \frac{1}{B_\theta^2 + \mu_0\gamma p}\left|\left(B_\theta^2 + \mu_0\gamma p\right)\xi' - \left(B_\theta^2 - \mu_0\gamma p\right)\frac{\xi}{r}\right|^2\right\}$$

$$= \frac{1}{2\mu_0}\int d\underline{r}\left\{\frac{2\mu_0 p'}{r}|\xi|^2 + \frac{|\xi|^2}{r^2}\left[B_\theta^2 + \mu_0\gamma p - \frac{\left(B_\theta^2 - \mu_0\gamma p\right)^2}{B_\theta^2 + \mu_0\gamma p}\right]\right.$$

$$+\frac{\xi'\xi^* + \xi^{*'}\xi}{r}\left[\mu_0\gamma p - B_\theta^2 + \left(B_\theta^2 - \mu_0\gamma p\right)\right]$$

$$\left. + \left|\xi'\right|^2\left[B_\theta^2 + \gamma\mu_0 p - \left(B_\theta^2 - \mu_0\gamma p\right)\right]\right\}$$

5. Thus:

$$\delta W_F = \frac{1}{2\mu_0}\int d\underline{r}\frac{|\xi|^2}{r^2}\left[2\mu_0 r p' + \frac{4\mu_0\gamma p B_\theta^2}{B_\theta^2 + \mu_0\gamma p}\right]$$

6. Stability condition

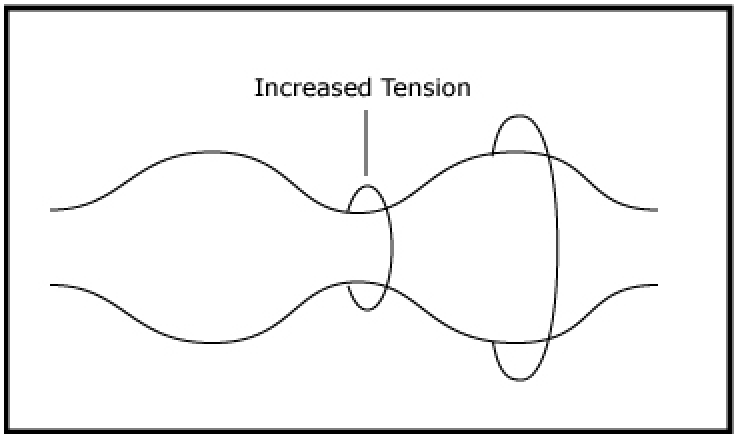
$$\boxed{-\frac{rp'}{p} < \frac{2\gamma B_\theta^2}{\mu_0\gamma p + B_\theta^2}}$$

Instability criterion usually violated in experiments. For Benneth profiles we require $\gamma > 2$ for stability

Instability:

a. competition between increased magnetic pressure and increased particle pressure

b. sausage instability



plasma pushes back less (3 degrees of freedom) than magnetic

pressure compresses plasma (2 degrees of freedom)

c. Stability boundary is independent of k

d. Mode is catastrophic experimentally.

e. Can be stable theoretically if p′ is weak enough. However, reliance on "γ" is suspicious. Not easily stabilized experimentally

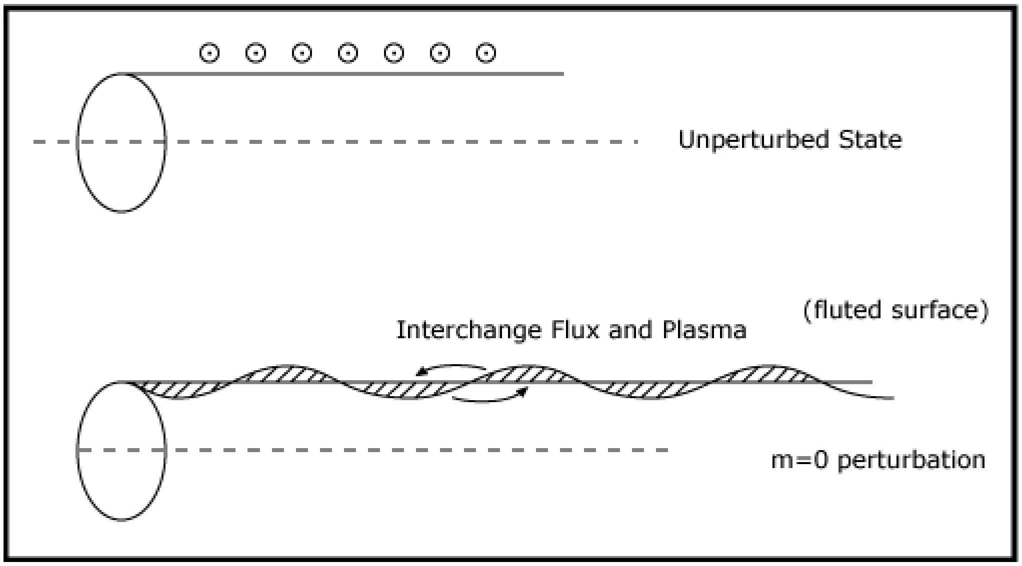
Conclusion

θ pinch stable

z pinch unstable

Single Particle Picture why does curvature enter?

Consider m=0 mode



1. Calculate drifts

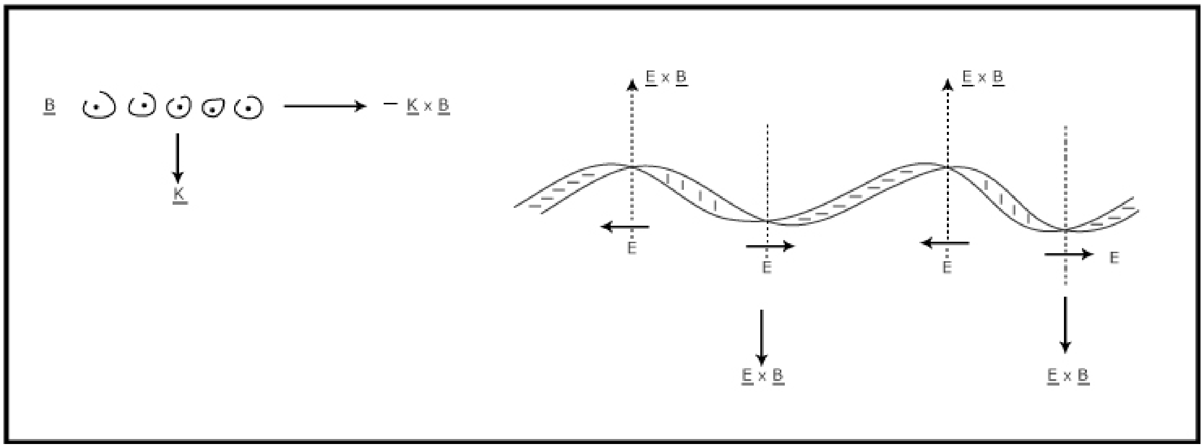
$$V_{\nabla B} = \frac{v_\perp^2}{2\omega_c}\frac{\underline{B}\times\nabla B}{B^2} = -\frac{mv_\perp^2}{2e}\frac{1}{B_\theta^3}\cdot B_\theta\frac{2B_\theta}{2r}\underline{e}_z = -\frac{mv_\perp^2}{2e}\frac{B_\theta'}{B_\theta^2}\underline{e}_z$$

$$V_\kappa = -\frac{v_\perp^2}{\omega_c}\frac{\underline{\kappa}\times B}{B} = \frac{mv_\parallel^2}{er}\frac{\underline{e}_r\times e_z}{B_\theta} = \frac{mv_\parallel^2}{e_r B_\theta}\underline{e}_z$$

2. Assume isotropic plasma $v_\parallel^2 = \frac{v_\perp^2}{2} = v^2$

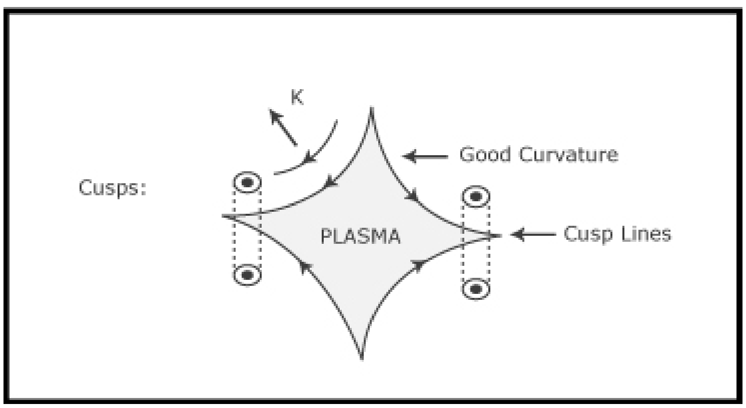
$$v_0 = \frac{mv^2}{eB_\theta^2}\left(\frac{B_\theta^2}{r} - B_\theta'\right) = -\frac{mv^2}{eB_\theta^2}\, r\left(\frac{B_\theta}{r}\right)'$$

In most profiles $\left(\frac{B_\theta}{r}\right)' < 0$: v_D is in same direction as curvature drift.



Curvature drift creates $E \times B$ drift which enhances perturbation

If the curvature drift is in the opposite direction, $\underline{E} \times \underline{B}$ drift would oppose the perturbation $\rightarrow$ stability



Summary