Lecture 14: Formulation of the Stability Problem

Hierarchy of Formulations of the MHD Stability Problem for Arbitrary 3-D Systems

1. Linearized equations of motion
2. Normal mode $\rightarrow$ eigenvalue approach
3. Variational approach
4. Energy principle
5. Extended Energy Principle

Linearized Equations of Motion

Assume we have an equilibrium satisfying

static

$\underline{J}_0 \times \underline{B}_0 = \nabla p_0 \quad \underline{V}_0 = E_0 = 0$

$\nabla \times \underline{B}_0 = u_0 \underline{J}_0 \quad \rho_0$ arb.

$\nabla \times \underline{B}_0 = 0$

$Q_0 = Q_0(\underline{x}) \rightarrow 3D$ in general

Linearize the Equation $\quad Q(\underline{x}, t) = Q_0(\underline{x}) + \tilde{Q}_1(\underline{x}, t)$

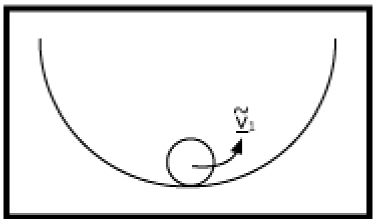
Mass:	$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0$	$\frac{\partial \tilde{\rho}_1}{\partial t} + \nabla \cdot \rho_0 \tilde{\underline{v}}_1 = 0$
Energy:	$\frac{dp}{dt} + rp\nabla \cdot \underline{v} = 0$	$\frac{\partial \tilde{p}_1}{\partial t} + \tilde{\underline{v}}_1 \cdot \nabla p_0 + rp_0 \nabla \cdot \tilde{\underline{v}}_1 = 0$
amp L:	$\mu \cdot \underline{J} = \nabla \times \underline{B}$	$\mu_0 \tilde{\underline{J}}_1 = \nabla \times \tilde{\underline{B}}_1$
$\nabla \cdot \underline{B}$:	$\nabla \cdot \underline{B} = 0$	$\nabla \cdot \tilde{\underline{B}}_1 = 0$
Faraday:	$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} = \nabla \times \underline{v} \times \underline{B}$	$\frac{\partial \tilde{\underline{B}}_1}{\partial t} = \nabla \times \left(\tilde{\underline{v}}_1 \times \underline{B}_0\right)$
Momentum:	$\rho \frac{dv}{dt} = \underline{J} \times \underline{B} - \nabla p$	$\rho_0 \frac{\partial \tilde{\underline{v}}_1}{\partial t} = \tilde{\underline{J}}_1 \times \underline{B}_0 + \underline{J}_0 \times \tilde{\underline{B}}_1 - \nabla \tilde{p}_1$

Simplify the PDE's by introducing the displacement sector $\tilde{\underline{\xi}}$ and appropriate initial conditions

a. $\tilde{\underline{v}}_1 = \dfrac{\partial \tilde{\underline{\xi}}}{\partial t}$

b. $\tilde{\underline{\xi}}$ is the plasma displacement away from equilibrium

Initial Conditions: Assume the plasma is in its equilibrium position moving away with a small velocity



$$\tilde{\underline{\xi}}(\underline{x},0) = \tilde{\underline{B}}_1(\underline{x},0) = \tilde{\rho}_1(\underline{x},0) = \tilde{p}_1(\underline{x},0) = 0$$

$$\tilde{\underline{v}}_1(\underline{x},0) = \frac{\partial \tilde{\underline{\xi}}}{\partial t}(\underline{x},0) \neq 0$$

Simplify the equations $\left(\nabla \cdot \tilde{\underline{B}}_1 = 0 \text{ not needed, redundant}\right)$

Express all quantities in terms of $\tilde{\underline{\xi}}$

Mass:	$\frac{\partial \tilde{\rho}_1}{\partial t} + \nabla \cdot \rho_0 \tilde{\underline{v}}_1 = 0 \rightarrow \frac{\partial}{\partial t}\left[\tilde{\rho}_1 + \nabla \cdot \rho_0 \tilde{\underline{\xi}}\right] = 0$	$\tilde{\rho}_1 = -\nabla \cdot \left(\rho_0 \tilde{\underline{\xi}}\right)$
Energy:	$\frac{\partial}{\partial t}\left(\tilde{p}_1 + \tilde{\underline{\xi}} \cdot \nabla p_0 + r p_0 \nabla \cdot \tilde{\underline{\xi}}\right) = 0$	$\tilde{p}_1 = -\tilde{\underline{\xi}}_0 \nabla p_0 - r p_0 \nabla \cdot \tilde{\underline{\xi}}$
Faraday:	$\frac{\partial}{\partial t}\left(\tilde{\underline{B}}_1 - \nabla \times \left(\tilde{\underline{\xi}} \times \underline{B}_0\right)\right) = 0$	$\tilde{\underline{B}}_1 = \nabla \times \left(\tilde{\underline{\xi}} \times \underline{B}_0\right)$
Ampere	$\mu_0 \tilde{\underline{J}}_1 = \nabla \times \tilde{\underline{B}}_1$	$\mu_0 \tilde{\underline{J}}_1 = \nabla \times \nabla \times \left(\tilde{\underline{\xi}} \times \underline{B}_0\right)$
Momentum:	$\rho_0 \frac{\partial \tilde{\underline{v}}_1}{\partial t} = \tilde{\underline{J}}_1 \times \underline{B}_0 + \underline{J}_0 \times \tilde{\underline{B}}_1 - \nabla \tilde{\rho}_1$	

a. $\rho_0 \dfrac{\partial^2 \tilde{\underline{\xi}}}{\partial t^2} = \underline{F}\left(\tilde{\underline{\xi}}\right) \qquad C\, \tilde{\underline{\xi}}(\underline{x},0) = 0, \dfrac{\partial \tilde{\underline{\xi}}}{\partial t}(x,0) = \text{given}, + \text{B.C.}$

b. $\underline{F}\left(\tilde{\underline{\xi}}\right) = \dfrac{1}{\mu_0}\left[\nabla \times \nabla \times \left(\tilde{\underline{\xi}} \times \underline{B}_0\right)\right] \times \underline{B}_0 + \dfrac{\left(\nabla \times \underline{B}_0\right)}{\mu_0} \times \left[\nabla \times \left(\tilde{\underline{\xi}} \times \underline{B}_0\right)\right] + \nabla\left(\tilde{\underline{\xi}} \cdot \nabla p_a + r p_0 \nabla \cdot \tilde{\underline{\xi}}\right)$

Initial Value Approach

Solve the linear equations of motion

Advantages: a. directly gives time evolution of the system

b. fastest growing mode automatically appears

c. good stand for nonlinear calculations

Disadvantages: a. more information contained than required to determine stability

b. extra work is required analytically and numerically to determine this information

c. tough to find marginal stability

Normal Mode Approach

A more efficient procedure that treats one mode at a time

1. The initial $\tilde{\underline{\xi}}(\underline{x}, 0)$ can be decomposed into normal modes

2. Each mode is then analyzed separately

3. To do this we fourier analyze in time

   $$\widetilde{Q}(\underline{x}, t) = Q(\underline{x})e^{-i\omega t} \qquad \tilde{\underline{\xi}}(\underline{x}, t) = \tilde{\underline{\xi}}(\underline{x})e^{-i\omega t}$$

4. Why is this legitimate?

5. Note: The equations for $\tilde{\rho}_1, \tilde{\underline{B}}_1, \tilde{\underline{J}}_1, \tilde{p}_1$ do not have any time durations. Hence, no explicit ω's appear. We find (drop 0 subscript)

   $$\rho_1 = -\nabla \cdot (\rho\underline{\xi})$$

   $$p_1 = -\underline{\xi} \cdot \nabla p - rp\nabla \cdot \underline{\xi}$$

   $$\underline{B}_1 = \nabla \times (\underline{\xi} \times \underline{B})$$

   $$\mu_0 \underline{J}_1 = \nabla \times \nabla (\underline{\xi} \times \underline{B})$$

The momentum equation becomes ($\underline{F}$ has no time derivation)

$$-\omega^2\rho\underline{\xi} = \underline{F}\left(\underline{\xi}\right)$$

$$\underline{F}\left(\underline{\xi}\right) = \underline{J}_1 \times \underline{B} + \underline{J} \times \underline{B}_1 - \nabla p_1$$

We only need B.C. $\rightarrow \omega^2$ eigenvalue

This is normal mode approach

Advantages: a. more amenable to analysis

b. directly addresses stability question (examine from ω)

c. more convenient numerically

Disadvantages: a. cannot be generalized for nonlinear calculation

b. still relatively complicated

Properties of $\vec{F}$

1. To proceed further (variational approach, energy principle) we need to understand the properties of the force operator $\underline{F}\left(\underline{\xi}\right)$

2. We show that

   a. $F\left(\underline{\xi}\right)$ is self adjoint

   b. ω^2 is purely real

   c. the normal modes are orthogonal

3. Self adjointness (2 procedures)

   a. subtle but elegant

   b. direct but complicated

The basic self adjoint property is associated with the conservation of energy; there is no dissipation in the system

Self Adjoint Property

a. $\int \underline{\eta} \cdot \underline{F}\left(\underline{\xi}\right) d\underline{r} = \int \underline{\xi} \cdot \underline{F}\left(\underline{\eta}\right) d\underline{r}$

where $\underline{\xi}$ and $\underline{n}$ are any two arbitrary, independent sectors satisfying the boundary conditions

b. Simple self adjoint equations: $\int \underline{\eta} \cdot \underline{\xi} \, d\underline{r}$, $\int \underline{\eta} \cdot \frac{\partial^2 \underline{\xi}}{\partial x^2} d\underline{r} = \int \underline{\xi} \cdot \frac{\partial^2}{\partial x^2} \underline{\eta} \; d\underline{r}$

c. A non self adjoint equation: $\int \underline{\eta} \cdot \frac{\partial \underline{\xi}}{\partial x} \; d\underline{r} = -\int \underline{\xi} \cdot \frac{\partial \underline{\eta}}{\partial x} \; d\underline{r}$

Direct Demonstration: Very Tedious Calculation

1. Assume $\underline{n} \cdot \underline{\xi} = \underline{n} \cdot \underline{\eta} = 0$ as plasma boundary (can be generalized)

2. $$\underline{\eta} \cdot \underline{F}\left(\underline{\xi}\right) d\underline{r} = -\int d\underline{r} \left\{ \frac{1}{\mu_0}\left(\underline{B} \cdot \nabla \underline{\xi_\perp}\right)\left(\underline{B} \cdot \nabla \underline{\eta_\perp}\right) + rp\left(\nabla \cdot \underline{\xi}\right)\left(\nabla \cdot \underline{\eta}\right) \right.$$

$$+ \frac{B^2}{\mu_0}\left(\nabla \cdot \underline{\xi_\perp} + 2\underline{\xi_\perp} \cdot \underline{\kappa}\right)\left(\nabla \cdot \underline{\eta_\perp} + 2\underline{\eta_\perp} \cdot \underline{\kappa}\right)$$

$$\left. - \frac{4B^2}{\mu_0}\left(\underline{\xi_\perp} \cdot \underline{\kappa}\right)\left(\underline{\eta_\perp} \cdot \underline{\kappa}\right) + \left(\underline{\eta_\perp}\underline{\xi_\perp} : \nabla\nabla\right)\left(p + \frac{B^2}{2\mu_0}\right) \right\}$$

$\vec{F}$ is self adjoint by inspection: switch $\underline{\xi}$ and $\underline{\eta}$, get the same result.

Show that ω^2 is real

1. $-\omega^2 \rho \underline{\xi} = \underline{F}\left(\underline{\xi}\right) \quad > \int d\underline{r} \, \underline{\xi}^*$

2. $-\omega^2 \int \rho \left|\underline{\xi}\right|^2 d\underline{r} = \int \underline{\xi}^* \cdot \underline{F}\left(\underline{\xi}\right) d\underline{r}$

3. Similarly $-\omega^{2*} \rho \underline{\xi}^* = \underline{F}\left(\underline{\xi}^*\right) \qquad > \int d\underline{r} \, \underline{\xi}$

   ($\underline{F}$: real operator)

4. $-\omega^{2*} \int \rho \left|\underline{\xi}\right|^2 d\underline{r} = \int \underline{\xi}^* \cdot \underline{F}\left(\underline{\xi}^*\right) d\underline{r}$

5. Subtract the equations

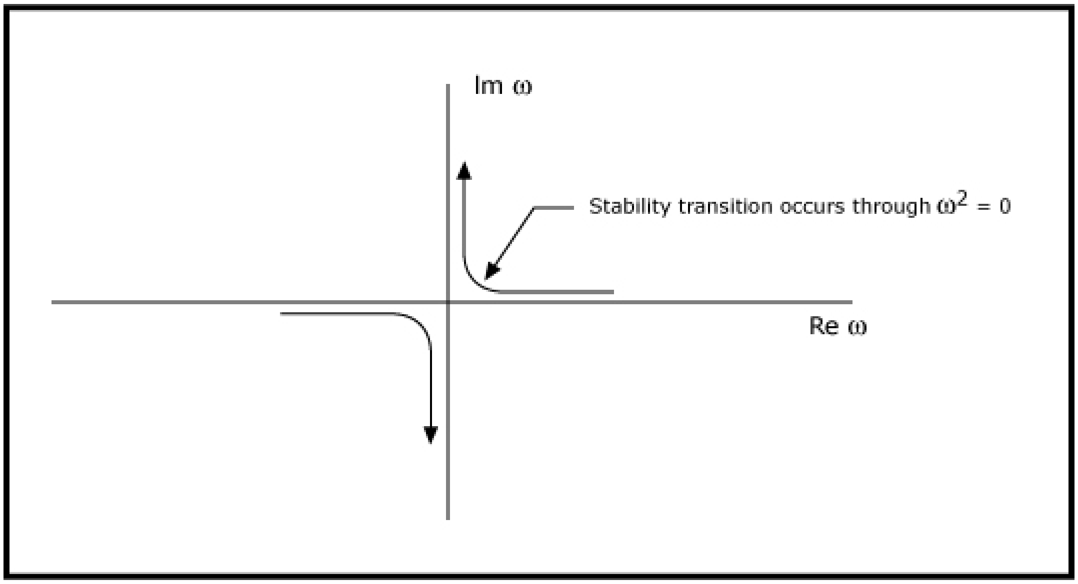
$$-\left(\omega^2 - \omega^{2*}\right) \int \rho \left|\underline{\xi}\right|^2 d\underline{r} = \int d\underline{r}\left[\underline{\xi}^* \cdot \underline{F}\left(\underline{\xi}\right) - \underline{\xi} \cdot \underline{F}\left(\underline{\xi}^*\right)\right]$$

=0 because of the self adjoint property

6. Therefore

   - $\omega^2 = \omega^{2*}$
   - ω^2 is real

This has important consequences



1. At marginal stability we note that by definition $\omega_i = 0$
2. In ideal MHD $\omega_r = 0$ also!! This is a big help. There is no need to find ω_r
3. $-\omega^2 \rho \underline{\xi} = \underline{F}(\underline{\xi})$

   $\underline{F}$ real, ω^2 real $\rightarrow$ $\underline{\xi}$ real

4. That $\underline{\xi}$ is real is not initially obvious $\tilde{\underline{\xi}}(\underline{r}, t) = \tilde{\underline{\xi}}(r)\, e^{-i\omega t}$
5. We continue to allow complex $\underline{\xi}$ to simplify fourier analysis in space, later on.

   $\underline{\xi}(\underline{r}) = \underline{\xi}(r)\, e^{im\theta + ikz}$ for example

Show that the Normal Modes are Orthogonal

1. Consider two normal modes (assume real $\underline{\xi}$ now)

$$-\omega_m^2 \rho \, \underline{\xi_m} = \underline{F}\left(\underline{\xi_m}\right) \qquad > \int \cdot \underline{\xi_n} \, d\underline{r}$$

$$-\omega_m^2 \rho \, \underline{\xi_n} = \underline{F}\left(\underline{\xi_n}\right) \qquad > \int \cdot \underline{\xi_m} \, d\underline{r}$$

2. Subtract

$$\left(\omega_n^2 - \omega_m^2\right) \int \rho \, \underline{\xi_m} \cdot \underline{\xi_n} \, d\underline{r} = \int \left[\underline{\xi_n} \cdot \underline{F}\left(\underline{\xi_m}\right) - \underline{\xi_m} \cdot \underline{F}\left(\underline{\xi_n}\right) \right] d\underline{r}$$

=0 by the self adjoint property

3. For $n \neq m, \omega_n^2 \neq \omega_m^2$

$$\int \rho \, \underline{\xi_n} \cdot \underline{\xi_m} \, d\underline{r} = 0 \text{ orthogonal property}$$

4. For n=m choose

$$\int \rho \, \underline{\xi_m^2} \, d\underline{r} = 1 \text{ orthonormal}$$

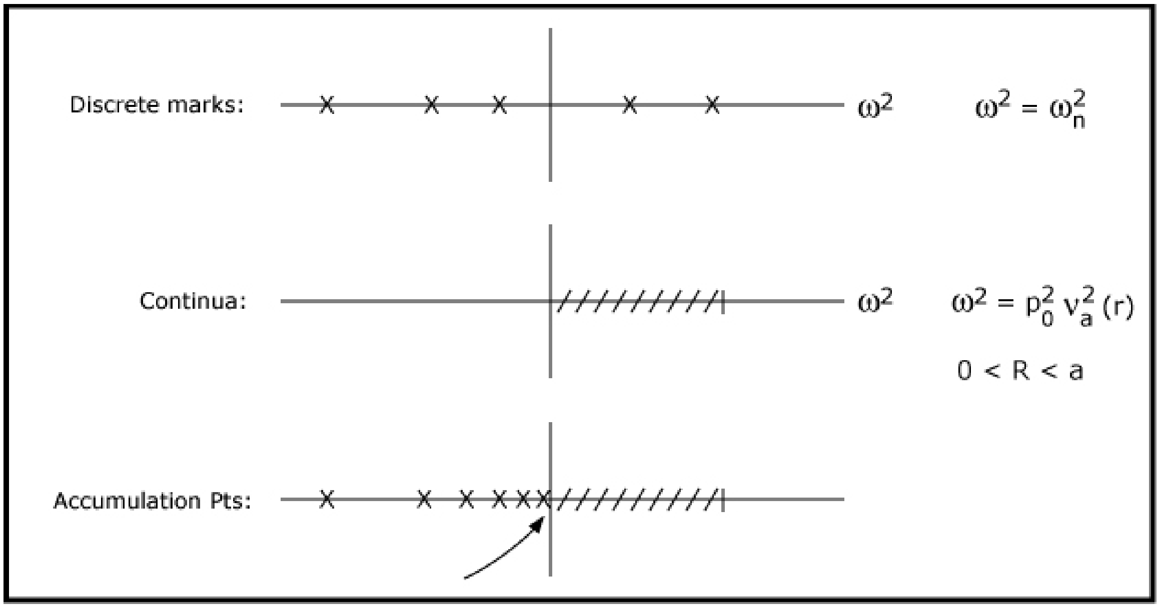
Spectrum of E

In general $\underline{F}$ exhibits both discrete eigenvalues and continua

Spectrum: $-\omega^2 \rho \, \underline{\xi} = \underline{F}\left(\underline{\xi}\right) \rightarrow \tilde{\underline{\xi}} = \left(\omega^2 + \frac{\underline{F}}{\rho}\right)^{-1}$ [initial conditions]

The points where $\left(\omega^2 + \frac{\underline{F}}{\rho}\right)^{-1}$ do not exist define the spectrum of $\underline{F}$

Examples



Continua significantly complicate MHD analysis for general initial value problems. They require more than just picking up the pole contributions from the displace transform.

However the continua lie on stable side of the spectrum and thus do not affect stability.

Accumulation points: these provide a simple necessary condition for stability.