

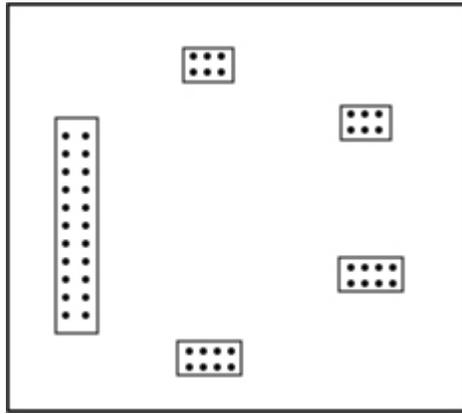
22.615, MHD Theory of Fusion Systems
Prof. Freidberg
Lecture 13: PF Design II – The Coil Solver

Introduction

1. Let us assume that we have successfully solved the Grad–Shafranov equation for a fixed boundary plasma equilibrium.
2. The second important task is to calculate the location of the PF coils and the current flowing in each coil to generate the desired equilibrium.
3. This is an overdetermined problem. A finite number of coils cannot in general match the normal and tangential magnetic field on an infinite number of points on the plasma surface.
4. Thus, the problem reduces to one of obtaining a best fit with the existing coils. In the fitting procedure, we obtain an accurate estimate of the error, thereby providing insight as to the plausibility of the original equilibrium.
5. We shall see that in designing the PF system, it is far simpler computationally to vary the coil currents (linear algebra) as opposed to changing their location (nonlinear problem).
6. There are two common classes of problems involved in the design of the PF system
 - a. Fixed, flat top design points
 - b. Dynamic, time dependent operational analysis
7. For the fixed design points there are two issues worth noting
 - a. During flat top operation, the eddy currents in the vacuum chamber can be assumed to have decayed to zero. This is a great simplification.
 - b. The OH transformer is slightly subtle. It is not clear what value of the flux swing corresponds to the point of interest in the flat top operation. Recall that even though $I_p = \text{const}$, I_Ω is varying in time and its value will have some influence on the remaining EF currents. For a tightly coupled transformer this effect is not too large, as very little of the transformer flux returns through the plasma.
8. The dynamic mode is far more complicated to calculate as a model must be introduced to account for the vacuum chamber eddy currents. Also, circuit equations must be introduced and we must be careful to distinguish between active and passive coils.
9. We now describe how the flat top problem can be solved by fast, efficient, and robust computational techniques

Flat Top Operation

1. From the fixed boundary Grad–Shafranov solver assume we know
 - a. The shape of the surface: $R = R(\mu), Z = Z(\mu)$
 - b. The poloidal flux in the plasma: ψ_b
 - c. The poloidal field around the surface: $\partial\psi/\partial n$
2. Assume we know the location of the EF coils and the OH transformer. Model each coil as a group of filaments.



3. There is no unique way to obtain a best fit for the EF currents. The method described below is fast and robust and requires no iteration.
4. There are two steps to the procedure.
 - a. Assign an arbitrary current I_j to each PF coil. Treat the plasma as a rigid perfect conductor. Calculate the poloidal field on the surface of the plasma as a function of the I_j : \hat{B}_p is linearly related to the I_j .
 - b. Determine the I_j by minimizing the difference between \hat{B}_p and the poloidal field calculated from the fixed boundary equilibria B_p . This is the “best fit” procedure.

The Vacuum Field

1. The magnetic field surrounding the plasma can be written as

$$\hat{B} = B_c + B_\Omega + B_p + \nabla\Phi$$

2. B_Ω is the field due to ohmic transformer, whose current is assumed to be specified: I_Ω is a known quantity

$$B_{\Omega} = \frac{\mu_0 I_{\Omega}}{K_{\Omega}} \sum_{k=1}^{K_{\Omega}} \frac{\nabla \Psi_k \times \mathbf{e}_{\phi}}{R} = \mu_0 I_{\Omega} \frac{\nabla \Psi_k \times \mathbf{e}_{\phi}}{R}$$

$$\Psi_{\Omega} = \frac{1}{K_{\Omega}} \sum_{k=1}^{K_{\Omega}} \Psi_k$$

$$\Psi_k(R, Z) = \frac{(RR_k)^{1/2}}{2\pi} \left[\frac{(2 - k_k^2) K(k_k) - 2E(k_k)}{k_k} \right]$$

$$k_k^2 = \frac{4RR_k}{(R + R_k)^2 + (Z - Z_k)^2}$$

The filaments modeling the transformer are located at R_k, Z_k . Alternatively, and more common, is to specify the volt seconds linking the magnetic axis. This is a slightly more complicated constraint than fixing I_{Ω} .

3. B_p is the field due to a filament carrying a current I_p located at the geometric center of the plasma. The necessity for this term will be apparent shortly. I_p is assumed known.

$$B_p = \mu_0 I_p \frac{\nabla \Psi_p \times \mathbf{e}_{\phi}}{R}$$

$$\Psi_p(R, Z) = \frac{(RR_p)^{1/2}}{2\pi} \left[\frac{(2 - k_p^2) K(k_p) - 2E(k_p)}{k_p} \right]$$

$$k_p^2 = \frac{4RR_p}{(R + R_p)^2 + (Z - Z_p)^2}$$

The geometric center of the plasma is located at R_p, Z_p .

4. B_c is the field due to the currents flowing in the EF system. There are J such coils, each carrying a current I_j and modeled by a group of filaments.

$$B_c = \sum_{j=1}^J \frac{\mu_0 I_j}{K_j} \left[\sum_{k=1}^{K_j} \frac{\nabla \Psi_{jk} \times \mathbf{e}_{\phi}}{R} \right] = \sum_{j=1}^J \mu_0 I_j \frac{\nabla \Psi_j \times \mathbf{e}_{\phi}}{R}$$

$$\Psi_j = \frac{1}{K_j} \sum_{k=1}^{K_j} \Psi_{jk}$$

$$\psi_{jk}(R, Z) = \frac{(RR_{jk})^{1/2}}{2\pi} \left[\frac{(2 - k_{jk}^2) K(k_{jk}) - 2E(k_{jk})}{k_{jk}} \right]$$

$$k_{jk}^2 = \frac{4RR_{jk}}{(R + R_{jk})^2 + (Z - Z_{jk})^2}$$

The j 'th EF coil is modeled by K_j filaments located at R_{jk} , Z_{jk} and carrying a current I_k/K_j . The I_j 's are the basic unknowns in the problem.

5. $\nabla \Phi$ is the scalar magnetic potential representing the remainder of the field in the vacuum and satisfies the following equation

$$\nabla^2 \Phi = 0$$

6. Φ has two important properties
 - a. Φ is regular everywhere in the vacuum region, since the singular coil currents have been explicitly subtracted from the complete solution.
 - b. Φ is single valued, since the plasma current has been subtracted from the complete solution. This is why we needed B_ρ .

7. Because of this Φ satisfies the following boundary conditions

$$\Phi(\infty) = 0 \text{ regularity}$$

$$\mathbf{n} \cdot \mathbf{B} \Big|_\rho = \left[\mathbf{n} \cdot (\mathbf{B}_c + \mathbf{B}_\Omega + \mathbf{B}_\rho) + \mathbf{n} \cdot \nabla \Phi \right]_\rho = 0$$

where Φ and $\mathbf{n} \cdot \nabla \Phi$ are single value periodic functions of μ on the plasma surface.

8. Ultimately we need to calculate \hat{B}_ρ on the plasma surface in terms of the I_j . To do this we will need to evaluate $\Phi(\mu)$ on the plasma surface. We already know $\mathbf{n} \cdot \nabla \Phi$ from the boundary condition above.
9. The function $\Phi(\mu)$ on the plasma surface is found from Green's theorem.

Application of Green's Theorem

1. Recall Green's theorem

$$\mathbf{\nabla} \cdot (\Phi \nabla \hat{G} - \hat{G} \nabla \Phi) = \Phi \nabla^2 \hat{G} - \hat{G} \nabla^2 \Phi = \Phi \nabla^2 \hat{G} \quad \int dr$$

$$\int \Phi \nabla^2 \hat{G} dr = \sum s_j \int (\Phi \hat{\mathbf{n}} \cdot \nabla \hat{G} - \hat{G} \hat{\mathbf{n}} \cdot \nabla \Phi) dS_j$$

c. Assume $\nabla^2 \hat{G} = \delta(r - r')$. Also there is only one S_j corresponding to the plasma surface since $\Phi(\infty) \rightarrow 0$ by regularity.

d. Thus

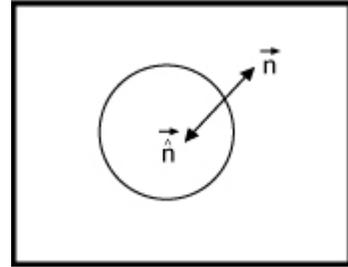
$$\sigma\Phi = \int (\hat{G}n' \cdot \nabla' \Phi - \Phi n' \cdot \nabla' \hat{G}) dS$$

$$n' = -\hat{n}'$$

1 for r in the region

$$\sigma = \frac{1}{2} \text{ for } r \text{ on the surface}$$

0 for r out of the region



e. Choose r to lie on the plasma surface

$$\Phi(S_p) = 2 \int [\hat{G}n' \cdot \nabla' \Phi(S_p') - \Phi(S_p')n' \cdot \nabla' \hat{G}] dS_p'$$

f. Once we know \hat{G} , this is an integral equation for $\Phi(S_p)$ since we know $n \cdot \nabla' \Phi(S_p')$ from the boundary conditions.

2. Solve the integral equation

$$dS_p' = R' dl_p' d\phi' = R'(R_\mu'^2 + Z_\mu'^2)^{1/2} d\mu' d\phi'$$

$$\hat{G} = -\frac{1}{4\pi} \frac{1}{|r - r'|}$$

3. Since the problem is axisymmetric, we can integrate the Green's function over the toroidal angle ϕ'

$$G(\mu, \mu') = \int_0^{2\pi} \hat{G} d\phi'$$

$$= \frac{-kK(k)}{2\pi(RR')^{1/2}}$$

$$k^2(\mu, \mu') = \frac{4RR'}{(R'+R)^2 + (Z'-Z)^2}$$

In these expressions $R = R(\mu)$, $Z(\mu)$, $R' = R(\mu')$, $Z' = Z(\mu')$ corresponding to the plasma surface.

4. The equation for Φ becomes

$$\Phi(\mu) = 2 \int \left(\widehat{G} n' \cdot \nabla' \Phi' - \Phi' n' \cdot \nabla' G \right) J' d\mu'$$

$$J' = J(\mu') = R' \left(R_{\mu'}^2 + Z_{\mu'}^2 \right)^{1/2}$$

5. This equation can be solved by Fourier analysis

$$\Phi = \sum_{-M}^M a_m e^{im\mu}$$

$$J n \cdot \nabla \Phi = \sum_{-M}^M \left(c_m + \sum_{j=1}^J D_{mj} I_j \right) e^{im\mu}$$

6. The coefficients c_m and D_{mj} are known from the boundary condition on $n \cdot \nabla \Phi$, which can be written as

$$J n \cdot \nabla \Phi = \frac{\partial}{\partial \mu} \left(\mu_0 I_\rho \Psi_\rho + \mu_0 I_\Omega \Psi_\Omega + \sum_{j=1}^J \mu_0 I_j \Psi_j \right)$$

7. Thus

$$c_m = \frac{im}{2\pi} \int_0^{2\pi} \left(\mu_0 I_\rho \Psi_\rho + \mu_0 I_\Omega \Psi_\Omega \right) e^{-im\mu} d\mu$$

$$D_{mj} = \frac{im}{2\pi} \int_0^{2\pi} \mu_0 \Psi_j e^{-im\mu} d\mu$$

8. Substituting into the integral equation yields a linear algebra problem for the unknown coefficients a_m

$$(\bar{I} + \bar{A}) \cdot a = \bar{C} \cdot (c + \bar{D} \cdot i)$$

$$A_{mm'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} (J' n' \cdot \nabla' G) e^{im'\mu' - im\mu} d\mu d\mu'$$

$$C_{mm'} = -\frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} G e^{im'\mu' - im\mu} d\mu d\mu'$$

$$i = (I_1, I_2, I_3, \dots, I_j) \text{ unknown currents}$$

9. Solve for a

$$a = d + \bar{E} \cdot i$$

$$d = (\bar{I} + \bar{A})^{-1} \cdot \bar{C} \cdot c$$

$$\bar{E} = (\bar{I} + \bar{A})^{-1} \cdot \bar{C} \cdot \bar{D}$$

10. Knowing a it is straightforward to evaluate \hat{B}_p on the surface, which requires a knowledge of $d\Phi(\mu)/d\mu$

$$\hat{B}_p(\mu) = b^{(f)}(\mu) + b^{(c)}(\mu) \cdot i$$

$$b^{(f)}(\mu) = \mu_0 I_p \frac{n \cdot \nabla \Psi_p}{R} + \mu_0 I_\Omega \frac{n \cdot \nabla \Psi_\Omega}{R} + \frac{R}{J} \sum_m imd_m e^{im\mu}$$

$$b_j^{(c)}(\mu) = \mu_0 \frac{n \cdot \nabla \Psi_j}{R} + \frac{R}{J} \sum_m imE_{mj} e^{im\mu}$$

11. The quantities $b^{(f)}$ and $b^{(c)}$ are required in order to calculate and minimize the error.

Determination of i

1. The vector i is determined by minimizing the difference between $\hat{B}_p(\mu)$ calculated above and $B_p(\mu)$ as calculated from the fixed boundary Grad-Shafranov code.

2. Define the error ϵ as follows

$$\epsilon = \int (\hat{B}_p - B_p)^2 dS_p / \int B_p^2 dS_p$$

3. A straightforward calculations yields

$$\epsilon = i \cdot \bar{M} \cdot i - 2n \cdot i + p$$

$$M_{jj'} = \frac{\int_0^{2\pi} b_j^{(c)} b_{j'}^{(c)} J d\mu}{\int_0^{2\pi} B_p^2 J d\mu}$$

$$n_j = \frac{\int_0^{2\pi} b_j^{(c)} (B_p - b^{(f)}) J d\mu}{\int_0^{2\pi} B_p^2 J d\mu}$$

$$\rho = \frac{\int_0^{2\pi} (B_p - b^{(f)})^2 J d\mu}{\int_0^{2\pi} B_p^2 J d\mu}$$

4. The best fit is obtained by minimizing ϵ with respect to the individual I_j . This leads to a set of linear algebraic equation for the I_j

a. Set

$$\frac{\partial \epsilon}{\partial I_j} = 0$$

b. The result is

$$\bar{M} \cdot i = n$$

c. This is a standard numerical problem.

5. Once i is found, then ϵ can be calculated.
6. For sufficiently small ϵ , the equivalent free boundary problem would give a comparable shape as the fixed boundary.
7. For sufficiently large ϵ , no choice of EF currents can reproduce the desired shape.
8. The procedure is robust, giving an answer even for very complicated (e.g. clover leaf) shape plasma with one EF coil. Obviously the error would be large. However, there is no need to hunt around aimlessly with free boundary codes, not knowing whether a satisfactory answer exists or not.
9. Finally, note that the quantities \bar{M} and n depend only upon the shape of the plasma and the location of the PF coils. Once these are set, the plasma parameters can be varied independently without recomputing \bar{M} and n .