

22.615, MHD Theory of Fusion Systems
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Lecture 12: PF Design I - The Plasma

Introduction

1. The PF system in a tokamak is one of the most critical design components, as it is responsible for centering the plasma, shaping the plasma, and, in the case of an ohmic plasma, establishing the pulse length.
2. The task is in general quite complicated since the plasma properties vary with time. Also, there are significant transient eddy currents in the vacuum chamber early in the discharge.
3. The task is further complicated by the need for high fields, leading to critical problems in magnet stress and cooling.
4. Finally, we in general desire elongated plasmas. However, such configurations are usually MHD unstable to vertical-like displacements. The PF system also has the responsibility of providing feedback control against such modes.
5. One of the great successes of the theory program is development of MHD equilibrium analysis to address these issues.

The Classic, Well Posed Approach to PF Design

1. Intuitively, we can see that the following approach represents a well posed procedure for analyzing the PF system.
2. Imagine a PF system consisting of an OH transformer and a series of EF coils for positioning and shaping.
3. Assume that the coil currents are specified as a function of time as well as the plasma free functions $\rho(\psi, t)$, $F(\psi, t)$.
4. This should and does determine the evolution of the system.
5. The solution to this problem would give the profiles (as a function of R, Z) in the plasma at each instant of time.
6. It would also give the plasma shape as a function of time as this formulation represents a free boundary problem. (Across the plasma surface, two conditions, not one, must be satisfied: $\mathbf{n} \cdot \mathbf{B} = 0$ and $[\rho + B^2/2\mu_0] = 0$.)
7. A final output is the plasma current $I_p(t)$ driven by the OH transformer.
8. A number of numerical codes have been developed to self-consistently address this problem. TSC is one such notable code.

9. The classic approach has the advantage of being able to specify the coil currents (or voltages) as a function of time, an important engineering consideration.
10. Also, to within numerical accuracy it generates the self-consistent solution to the problem.
11. There are a number of practical disadvantages to the classic approach.
 - a. The codes take far too long to run, making parametric studies difficult to carry out.
 - b. It is much harder to calculate a free boundary than a fixed boundary equilibrium. More iterations are required, as the plasma shape is an additional unknown function.
 - c. The plasma current that comes out may not be the one you want. This requires further iteration with the OH programming.
 - d. The plasma shape evolution that comes out may not be the one you want. This too requires further, sometimes quite unintuitive iteration on the multi-coil EF system.
 - e. Unless precautions are taken, the numerical solutions may be dominated by vertical instabilities.
12. Codes of this type would appear to be more useful once the coil currents have by one means or another, been predetermined.

An Efficient Inverse Approach to PF Design

1. All of the problems described above can be resolved by adopting an alternate inverse approach to PF design.
2. The end result is again a numerical code, but one which runs far faster than the classic codes and allows the engineer/physicist to specify more meaningful physical quantities.
3. Specifically, the user specifies a desired time evolution of plasma current, plasma shape, and p , F profiles. The code quickly calculates a best fit to the desired OH current and EF currents, including if desired, the effects of vacuum chamber eddy currents.
4. In this and the next lecture we describe how the procedure can be implemented.
5. There are two basic components to the formulation.
 - a. A fast fixed boundary Grad-Shafranov solver.
 - b. A fast coil current solver.

The Grad-Shafranov Solver

1. A fast Grad-Shafranov solution is difficult to compute for several reasons.
 - a. The equation is a nonlinear PDE.
 - b. The natural mathematical constants are not the same as the desired physical constants. This makes information counting somewhat subtle and ultimately leads to additional iteration.
2. Consider first the practical mathematical formulation that addresses problem 1b.
3. The issues become apparent once appropriate normalized variables are introduced.
4. Keep in mind that by specifying plasma shape evolution as an input, we are concerned with fixed boundary equilibria. As stated, this is much simpler than the general free boundary problem.
5. Recall that the exact Grad-Shafranov equation is given by

$$\Delta^* \psi = -\mu_0 R^2 \frac{\partial p}{\partial \psi} - F \frac{\partial F}{\partial \psi}$$

where $\Delta^* \psi = R^2 \nabla \cdot (\nabla \psi / R^2)$ and $p(\psi, t)$, $F(\psi, t)$ are free functions of ψ and t , specified as inputs either by intuitive guessing or by separately solving the transport evolution equations. The goal is to find $\psi(R, Z, t)$.

6. Introduce normalized coordinates as follows.

$$\psi = \psi_b(t) \bar{\psi}(R, Z, t)$$

$$R = R_0 + ax$$

$$Z = ay$$

$$p(\psi, t) = p_0(t) h(\bar{\psi}, t)$$

$$F^2(\psi, t) = R_0^2(t) B_0^2(t) \left[1 + 2k_0(t) f(\bar{\psi}, t) \right]$$

where

- a. $\bar{\psi}(R, Z, t)$ is the new poloidal flux variable normalized so that $\bar{\psi} = 0$ on the magnetic axis and $\bar{\psi} = 1$ on the plasma surface. (One of these is always arbitrary.)
- b. $\psi_b(t)$ is the poloidal flux on the boundary. While it is natural to specify $\psi_b(t)$ mathematically, in practice we wish instead to specify $I_p(t)$. One cannot simultaneously specify ψ_b and I as this overdetermines the system. Thus, $\psi_b(t)$ must be viewed as an unknown, to be determined in terms of $I_p(t)$.
- c. $R_0(t)$ is the geometric center of the plasma.
- d. $a(t)$ is horizontal radius of the plasma. When $y = 0$, the inner and outer surfaces of the plasma are given by $R_0 \pm a$ corresponding to $x = \pm 1$.
- e. $p_0(t)$ is the pressure on axis, assumed to be known from the input data.
- f. $B_0(t)$ is the vacuum toroidal field at $R = R_0$, also known from the input.
- g. $k_0(t)$ is a parameter representing the plasma dia- or paramagnetism. We shall see that $k_0(t)$ is actually an output of the calculation. If we give p and I , the dia- or paramagnetism is a determined quantity. If we give p and k_0 , we cannot specify I_p or ψ_b .
- h. $h(\bar{\psi}, t)$ and $f(\bar{\psi}, t)$ are input profile functions normalized so that
- $$h(0, t) = f(0, t) = 1 \text{ magnetic axis}$$
- $$h(1, t) = f(1, t) = 0 \text{ plasma boundary}$$

7. The Grad-Shafranov equation becomes

$$\Delta^* \bar{\psi} = -2C_0 \frac{\partial f}{\partial \bar{\psi}} - \frac{2\beta_0}{l_0^2} \frac{R^2}{R_0^2} \frac{\partial h}{\partial \bar{\psi}}$$

with

$$\Delta^* \bar{\psi} = \frac{\partial^2 \bar{\psi}}{\partial x^2} - \frac{1}{1 + \epsilon x} \frac{\partial^2 \bar{\psi}}{\partial x} + \frac{\partial^2 \bar{\psi}}{\partial y^2}$$

$$\beta_0(t) = \frac{8\pi^2 a^2 p_0}{\mu_0 I^2}$$

$$l_0(t) = \frac{4\pi\psi_b}{\mu_0 I R_0}$$

$$C_0(t) = \frac{k_0 B_0^2 a^2 R_0^2}{2\psi_b^2}$$

8. Note that β_0 can be assumed as a known input since a , ρ_0 , and I are inputs. In contrast l_0 is an unknown constant since it involves ψ_b . Similarly C_0 is unknown since it involves k_0, ψ_b .
9. The parameter l_0 is a measure of the normalized internal inductance. Its value can be expressed in terms of β_0 and C_0 by assuming that the total current $I(t)$ is a known input quantity.

a. $I(t) = \int J_\phi dS$

- b. Replace RJ_ϕ by the right hand side of the Grad-Shafranov equation, yielding

c.
$$\frac{1}{l_0} = -\frac{C_0}{2\pi} \int dS \frac{R_0}{R} \frac{\partial f}{\partial \psi} - \frac{\beta_0}{2\pi l_0^2} \int dS \frac{\partial h}{\partial \psi}$$

10. The value of C_0 is more subtle. It is related to the value of k_0 and is ultimately determined by the fact that $\bar{\psi}$ is a normalized quantity varying between $0 < \bar{\psi} < 1$. In particular requiring (1) regularity, (2) $\bar{\psi} = 0$ on axis, and (3) $\bar{\psi} = 1$ on the surface overdetermines the system and solutions are possible only for a special value of C_0 .
11. To see this explicitly consider a large aspect ratio tokamak with $h = f = 1 - \bar{\psi}$. Let $x = \rho \cos \theta, y = \rho \sin \theta$. The Grad-Shafranov equation becomes

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \bar{\psi}}{\partial \rho} \right) = 2 \left(C_0 + \frac{\beta_0}{l_0^2} \right)$$

12. The general solution is

$$\bar{\psi} = C_1 + C_2 \ln \rho + \left(C_0 + \frac{\beta_0}{l_0^2} \right) \frac{\rho^2}{2}$$

13. Regularity and the condition $\bar{\psi}(0) = 0$ imply that $C_1 = C_2 = 0$

14. In order for $\bar{\psi}(1) = 1, C_0$ must satisfy

$$C_0 + \frac{\beta_0}{I_0^2} = 2$$

15. The flux-current constraint reduces to

$$\frac{1}{I_0} = \frac{1}{2} \left(C_0 + \frac{\beta_0}{I_0^2} \right)$$

16. This leads to

$$I_0 = 1$$

$$C_0 = 2 - \beta_0$$

17. The general case is completely analogous.

18. A simple way to find C_0 in the general case is by taking a moment of the Grad-Shafranov equation. Actually any moment will suffice if the exact solution is known. For an approximate solution, the energy moment is quite accurate. Multiply the Grad-Shafranov equation by $(1 - \bar{\psi})/R^2$ and integrate over the plasma volume.

$$C_0 \int (1 - \bar{\psi}) \frac{R_0^2}{R^2} \frac{\partial f}{\partial \bar{\psi}} d\tau + \frac{\beta_0}{I_0^2} \int (1 - \bar{\psi}) \frac{\partial h}{\partial \bar{\psi}} d\tau = -\frac{1}{2} \int \left(\frac{R_0}{R} \nabla \bar{\psi} \right)^2 d\tau$$

19. The fact that $\bar{\psi}$ must by definition lie between 0 and 1 determines the appropriate normalization condition to evaluate C_0 .

Summary of the Normalized Equations

1. Inputs: $\rho_0(t), I(t), R_0(t), a(t), B_0(t)$, other shape parameters $\kappa(t), \delta(t), f(\bar{\psi}, t), h(\bar{\psi}, t)$

2. Solve

$$\Delta^* \bar{\psi} = -2C_0 \frac{\partial f}{\partial \bar{\psi}} - \frac{2\beta_0}{I_0^2} \frac{R^2}{R_0^2} \frac{\partial h}{\partial \bar{\psi}}$$

$$\frac{1}{I_0} = -\frac{C_0}{2\pi} \int \frac{R_0}{R} \frac{\partial f}{\partial \bar{\psi}} dS - \frac{\beta_0}{2\pi I_0^2} \int \frac{R}{R_0} \frac{\partial h}{\partial \bar{\psi}} dS$$

$$C_0 \int (1 - \bar{\psi}) \frac{R_0^2}{R^2} \frac{\partial f}{\partial \bar{\psi}} d\tau + \frac{\beta_0}{I_0^2} \int (1 - \bar{\psi}) \frac{\partial h}{\partial \bar{\psi}} d\tau = -\frac{1}{2} \int \left(\frac{R_0}{R} \nabla \bar{\psi} \right)^2 d\tau$$

3. Calculate
 - a. $\bar{\psi}(x, y, t)$
 - b. $I_0(t)$
 - c. $C_0(t)$

Fast Solution using Inverse Coordinates and Variational Techniques

1. The solution to the Grad–Shafranov equation can be found quite accurately and extremely quickly using inverse coordinates and variational techniques.
2. Grad–Shafranov variational procedure

$$L = \int dx dy \left[\frac{R_0}{R} \frac{(\nabla \bar{\psi})^2}{2} - 2C_0 \frac{R_0}{R} f - \frac{2\beta_0}{I_0^2} \frac{R}{R_0} h \right]$$

3. Vary L

$$\delta L = - \int dx dy \frac{R_0}{R} \left[\Delta^* \bar{\psi} + 2C_0 \frac{\partial f}{\partial \psi} + \frac{2\beta_0}{I_0^2} \frac{R^2}{R_0^2} \frac{\partial h}{\partial \psi} \right] \delta \bar{\psi}$$

4. Introduce inverse coordinates, with variational parameters. Let $\bar{\psi} = \rho^2$ and

$$x = x(\rho, \mu) = \sigma_0 (1 - \rho^2) + \rho \cos[\mu + D(\rho) \sin \mu]$$

$$y = y(\rho, \mu) = [\kappa_0 + (\kappa_a - \kappa_0) \rho^2] \rho \sin \mu$$

$$D(\rho) = \rho^2 [\eta_1 + (1 - \eta_1) \rho^2] \sin^{-1} \delta_a$$

where κ_a and δ_a are inputs representing the surface elongation and triangularity. The quantities σ_0 , κ_0 , and η_1 are variational parameters obtained by extremizing L . Only a few parameters are needed, because the interior surfaces in general follow the shape of the plasma surface.

5. The solution to the Grad-Shafranov equation reduces to the solution of five simultaneous algebraic equations, three for the variational parameters and two for the extra constraints.
6. Variational equations: denote $\sigma_0, \kappa_0, \eta_1$, by α_j , $1 \leq j \leq 3$.

$$\frac{\delta L}{\delta \alpha_j} = 0 = \int L_\psi \frac{\delta \bar{\psi}}{\delta \alpha_j} d\rho d\mu$$

$$L_\psi = 2\rho \frac{\partial}{\partial \rho} \left[\frac{\rho R_0}{\Delta R} (x_\mu^2 + y_\mu^2) \right] - 2\rho \frac{\partial}{\partial \mu} \left[\frac{\rho R_0}{\Delta R} x_\mu x_\rho + y_\mu y_\rho \right] + \Delta \left(C_0 \frac{R_0}{R} \frac{\partial f}{\partial \rho} + \frac{\beta_0}{l_0^2} \frac{R}{R_0} \frac{\partial h}{\partial \rho} \right)$$

$$\frac{\delta \bar{\psi}}{\delta \alpha_j} = \frac{1}{\Delta} \left(y_\mu \frac{\partial x}{\partial \alpha_j} - x_\mu \frac{\partial y}{\partial \alpha_j} \right)$$

$$\Delta = x_\rho y_\mu - y_\rho x_\mu$$

7. Constraint equations

$$\frac{1}{l_0} + C_0 S_0 + \frac{\beta_0}{l_0^2} U_0 = 0$$

$$C_0 S_1 + \frac{\beta_0}{l_0^2} U_1 = -\frac{1}{\pi} \int d\rho d\mu \frac{R_0}{\Delta R} (x_\mu^2 + y_\mu^2)$$

where

$$S_n = \frac{1}{2\pi} \int \Delta d\rho d\mu \frac{(1 - \rho^2)^n}{2\rho} \frac{R_0}{R} \frac{\partial f}{\partial \rho}$$

$$U_n = \frac{1}{2\pi} \int \Delta d\rho d\mu \frac{(1 - \rho^2)^n}{2\rho} \frac{R}{R_0} \frac{\partial h}{\partial \rho}$$

8. Solving five nonlinear algebraic equations with good starting guesses is a simple computational task, requiring only .02 seconds on a Cray and about 10-20 seconds on a Mac.