22.615 Midterm

Spring 2007

Overview

The purpose of this problem is to investigate the potential for ITER to achieve steady state operation by means of a combination of lower hybrid current drive and the bootstrap current. There are two basic parts to the problem.

The first part is analytical and consists of formulating the equilibrium problem for reversed shear, advanced tokamak (AT) operation. The second part involves writing a code to determine the equilibrium and performance properties that one would require to achieve steady state operation with a bootstrap fraction of 75%. The end goal is to calculate the bootstrap fraction as a function of plasma β and to see if the required β for 75% bootstrap current is compatible with MHD stability limits. The formulation is similar to that given in lecture 12.

Part (1)

A. Derive a normalized form of the Grad-Shafranov equation suitable for studying steady state MHD equilibria for AT operation. Specifically, introduce the following normalizations:

$$\Psi(R,Z) = \Psi_b \psi(x,y) \qquad 0 \le \psi \le 1$$
$$R = R_0 + ax \qquad -1 \le x \le 1$$
$$Z = ay \qquad -\kappa \le y \le \kappa$$

Define the free pressure function as follows

$$p(\Psi) = p_0 h(\psi) \qquad 1 \ge h \ge 0$$

Define the toroidal field free function as $F^2(\Psi) = R_0^2\left[B_0^2 - 2\mu_0 p(\Psi) + 2B_0 B_2(\Psi)\right]$ where the new function B_2 has the form

$$B_2(\Psi) = k_0 B_0 f(\psi) \qquad\qquad 1 \geq f \geq 0$$

(Note that this definition is a little different from the one given in the notes.) Do not make the large aspect ratio expansion; that is, treat $\varepsilon \equiv a / R_0$ as finite. To make the analogy with the notes in class introduce the parameters

$$\beta_0 = \frac{8\pi^2 a^2 p_0}{\mu_0 I^2}$$
$$l_0 = \frac{4\pi\Psi_b}{\mu_0 R_0 I}$$
$$C_0 = \frac{k_0 B_0^2 a^2 R_0^2}{2\Psi_b^2}$$

B. Introduce a set of inverse coordinates defined by $x = x(\rho,\mu)$ and $y = y(\rho,\mu)$ where $\psi = \rho^2$. Assume the shape of the plasma is given by

$$x = \sigma_0\left(1-\rho^2\right) + \rho\cos\left(\mu + D\sin\mu\right)$$
$$y = \left[\kappa_0 + \left(\kappa_a - \kappa_0\right)\rho^2\right]\rho\sin\mu$$
$$D(\rho) = \left(\sin^{-1}\delta_a\right)\left[\delta_0 + \left(1-\delta_0\right)\rho^2\right]\rho^2$$

where κ_a, δ_a are the surface elongation and triangularity. The quantities $\sigma_0, \kappa_0, \delta_0$ are fitting parameters to be determined from the analysis. Note that for ITER, $\varepsilon = 0.32$, $\kappa_a = 1.7$, and $\delta_a = 0.4$. Choose the free functions as follows

$$h = (1-\psi)^2$$
$$f = \frac{(1-\psi)^2(1+3\alpha+2\psi)}{1+3\alpha}$$

Here, α is a profile parameter which for ITER can be taken to be $\alpha = 0.07$. Note that these choices lead to a monotonically decreasing pressure profile and a flux surface averaged toroidal current profile (approximately proportionate to $df / d\psi$) which is peaked off-axis. These are the usual profile goals of AT operation.

At this point to solve the Grad-Shafranov equation we need to give values for ε, β_0 and determine the values of $C_0, l_0, \sigma_0, \kappa_0, \delta_0$. Rather than use the variational method discussed in class, try and determine these five coefficients using a method of moments. The procedure is as follows. First, derive the relation between flux and current by switching to flux coordinates and setting

$$\oint B_p dl_p = \mu_0 I$$

Next, denote the Grad-Shafranov equation by the abbreviation GS

$$GS \equiv R^2 \nabla \cdot \left(\frac{\nabla \Psi}{R^2} \right) + 2\mu_0 R^2 \frac{dp}{d\Psi} + F \frac{dF}{d\Psi} = 0$$

and derive the following four lowest order moment equations.

$$\int \frac{W_n}{R^2} (GS) \, d\mathbf{r} = 0$$

where

$$\begin{aligned} W_0 &= 1 \\ W_1 &= R - R_0 \\ W_2 &= (R - R_0)^2 \\ W_3 &= Z^2 \end{aligned}$$

Again, you will need to carry out this derivation in flux coordinates. You should now have five nonlinear algebraic equations for the five unknown constants.

Part (2)

A. Write a numerical code based on the above procedure to evaluate $l_0, C_0, \sigma_0, \kappa_0, \delta_0$ as a function of β_0 for the ITER parameters given above. Test your code over a plausible range of β_0. For a typical value of β_0 plot a set of flux surfaces $\psi(x,y) = const.$ using the parameters that you have calculated.

B. Now comes the interesting part. Choose a value for β_0. Set $B_0 = 5.3\,T$, $a = 2\,m$, $R_0 = 6.2\,m$. Derive and evaluate an expression for the safety factor $q(\psi)$. Search for the off-axis minimum value of $q_{\min} = q(\psi_{\min})$ and choose the value of I so that $q_{\min} = 2$. If $q_{\min} < 2$ then it can be shown that hollow current profiles are MHD unstable even at low values of β.

Next, recall that the density profile for H mode operation tends to be relatively flat. Therefore, as a simple model assume that the density profile is given by

$$\frac{n(\rho)}{n_0} = \left[\frac{p(\rho)}{p_0}\right]^{1/3}$$

You should now be in a position to derive and numerically evaluate expressions for the volume averaged beta $\overline{\beta}$, the kink safety factor q_*, the flux surface averaged bootstrap current profile $\langle J_B \rangle$, the total bootstrap current I_B, and the bootstrap fraction f_B. These are defined as follows.

$$\bar{\beta} = \frac{2\mu_0 \bar{p}}{B_0^2} = \frac{2\mu_0}{B_0^2}\left[\frac{1}{Vol}\int p d\mathbf{r}\right]$$

$$q_* = \frac{2\pi a^2 \kappa_a B_0}{\mu_0 R_0 I}$$

$$\langle \mathbf{J}_B \rangle \approx \langle J_B \rangle \mathbf{e}_\phi = -4.7 \frac{R_0^{1/2}}{a^{3/2} B_0} \frac{q(\rho) T(\rho)}{\rho^{1/2}} \frac{dn(\rho)}{d\rho} \mathbf{e}_\phi$$

$$I_B = \int J_B dS$$

$$f_B = \frac{I_B}{I}$$

Here and below the notation $\langle Q \rangle$ denotes

$$\langle Q \rangle = \frac{\oint Q \frac{dl_p}{B_p}}{\oint \frac{dl_p}{B_p}}$$

Carry out the solution procedure for a range of β_0. This should enable you to make a plot of f_B vs $\bar{\beta}$. Generate this plot and calculate the value of $\bar{\beta}$ required to make $f_B = 0.75$.

C. Compare this value of $\bar{\beta}$ with the maximum allowable stable value given by the Troyon limit $\bar{\beta} \le 0.14 \varepsilon \kappa_a / q_*$. Are we safe?

D. An approximate formula for current drive efficiency is given by

$$I_{CD} \approx 1.2 \frac{P_{CD}}{\mathrm{n}_\parallel^2 R_0 \bar{n}_{20}}$$

Here, the units are $I_{CD}(MA), P_{CD}(MW), n_{20}(10^{20} m^{-3})$. For ITER choose the parallel index of refraction $\mathrm{n}_\parallel = 2$ and the average density $\bar{n}_{20} = 1.0$. Ultimately ITER will have $P_{CD} = 40MW$. Is this enough to drive 25% of the total current? If not what bootstrap fraction can be driven with $40MW$.

E. The last part of the problem examines whether or not the bootstrap current profile closely overlaps the full current profile. On the same graph plot two curves as a function of ρ using the results obtained for the case $f_B = 0.75$. The two curves correspond to (1) $\langle J_B \rangle$ vs. ρ and (2) $\langle J_\phi \rangle$ vs. ρ. Is there good overlap?