22.615, MDH Theory of Fusion Systems
Prof.
Final Exam Solution

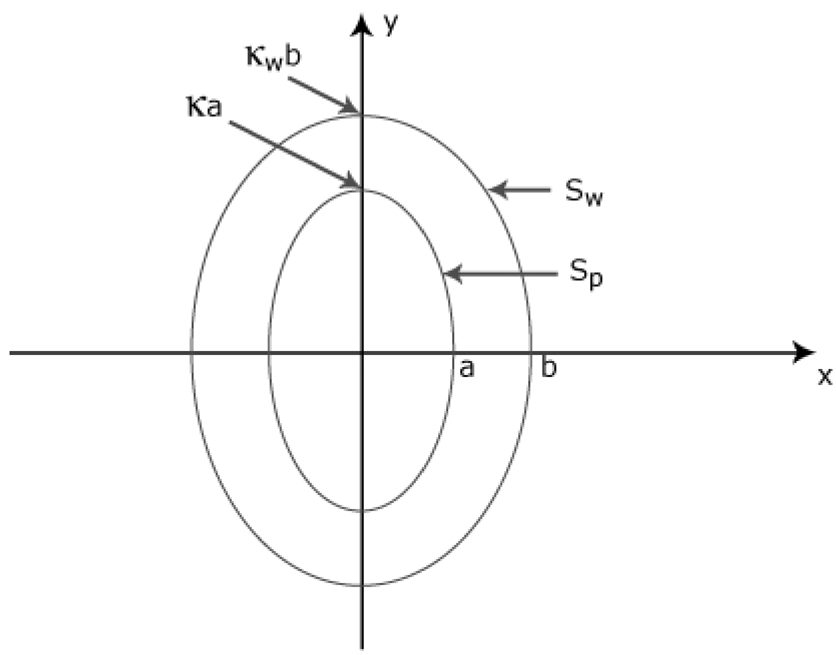


Figure: 1

1a. $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=+\mu_0 R_0 J_0$$

$$\psi(S)=0 \qquad S: \frac{x^2}{a^2}+\frac{y^2}{\kappa^2 a^2}=1$$

Solution: $$\psi=A\left(\frac{x^2}{a^2}+\frac{y^2}{\kappa^2 a^2}-1\right)$$

$$\nabla^2\psi=2A\left(\frac{1}{a^2}+\frac{1}{\kappa^2 a^2}\right)=\frac{2A}{a^2}\left(\frac{1+\kappa^2}{\kappa^2}\right)=\mu_0 R_0 J_0$$

$$A=\frac{\mu_0 R_0 J_0 a^2}{2}\frac{\kappa^2}{1+\kappa^2}$$

$$\psi=\frac{\mu_0 R_0 a^2 J_0}{2}\frac{\kappa^2}{1+\kappa^2}\left[\frac{x^2}{a^2}+\frac{y^2}{\kappa^2 a^2}-1\right]$$

$$\psi = \psi_a\left(\frac{x^2}{a^2} + \frac{y^2}{\kappa^2 a^2} - 1\right)$$

$$\psi_a = \frac{\mu_0\ R_0\ a^2\ J_0}{2}\frac{\kappa^2}{1+\kappa^2}\ ,\ \mu_0\ J_0 = \frac{2\left(1+\kappa^2\right)}{\kappa^2}\frac{\psi_a}{R_0 a^2}$$

1b. $$\underline{B}_p = \frac{1}{R_0}\nabla\psi\times\underline{e}_z = \frac{1}{R_0}\frac{\partial\psi}{\partial y}\underline{e}_x - \frac{1}{R_0}\frac{\partial\psi}{\partial x}e_y$$

$$B_x = \frac{1}{R_0}\frac{\partial\psi}{\partial y} = \frac{2\psi_a}{R_0\kappa^2 a^2}y$$

$$B_y = -\frac{1}{R_0}\frac{\partial\psi}{\partial x} = -\frac{2\psi_a}{R_0 a^2}x$$

2. $$\underline{B}_1 = \nabla\times\left(\underline{\xi}\times B\right) = \underline{B}\cdot\nabla\underline{\xi} - \underline{\xi}\cdot\nabla\underline{B} - \underline{B}\cdot\underline{\xi} = -\xi_0\,\underline{e}_y\cdot\nabla\underline{B}_p = -\xi_0\frac{\partial B_p}{\partial y}$$

$$\underline{B}_1 = -\xi_0\left(\frac{\partial B_x}{\partial y}\underline{e}_x + \frac{\partial B_y}{\partial y}\underline{e}_y\right) \qquad \mu_0 J_z = -\mu_0 J_0$$

$$\delta W_F = \frac{1}{2}\int d\underline{r}\left[\frac{\underline{B}_1^2}{\mu_0} - \underline{\xi}\cdot\underline{J}\times\underline{B}_1 + \underbrace{\gamma p\left(\nabla\cdot\underline{\xi}\right)^2}_{=0} + \underbrace{\left(\underline{\xi}\cdot\nabla p\right)\nabla\cdot\underline{\xi}}_{=0}\right]$$

$$\delta W_F = \frac{1}{2}\int d\,r\left[\frac{\underline{B}_1^2}{\mu_0} - \underline{\xi}\cdot\underline{J}\times\underline{B}_1\right]$$

$$\frac{\underline{B}_1^2}{\mu_0} = \frac{\xi_0^2}{\mu_0}\left[\left(\frac{\partial B_x}{\partial y}\right)^2 + \left(\frac{\partial B_y}{\partial y}\right)^2\right] = \frac{4\psi_a^2\xi_0^2}{\mu_0 R_0^2 a^4\kappa^4}$$

$$-\underline{\xi}\cdot\underline{J}\times\underline{B}_1 = -\xi_0\,\underline{e}_y\cdot\left(J_z\underline{e}_z\right)\times\left(-\xi_0\frac{\partial B}{\partial y}\times\underline{e}_x - \xi_0\frac{\partial B}{\partial y}\times\underline{e}_y\right)$$

$$= \xi_0^2 J_z\frac{\partial B_x}{\partial y} = -\xi_0^2 J_0\frac{\partial B_x}{\partial y}$$

$$= -\frac{\xi_0^2}{\mu_0}\frac{2\left(1+\kappa^2\right)}{\kappa^2}\frac{\psi_a}{R_0 a^2}\cdot\frac{2\psi_a}{R_0\kappa^2 a^2} = -\frac{4\psi_a^2\xi_0^2}{\mu_0 R_0^2 a^4\kappa^4}\left(1+\kappa^2\right)$$

$$\delta W_F = \frac{4\psi_a^2\xi_0^2}{2\mu_0 R_0^2 a^4 \kappa^4}\int d\underline{r}\left[\cancel{1} - \cancel{1} - \kappa^2\right] = \frac{2\psi_a^2\xi_0^2}{\mu_0 R_0^2 a^4 \kappa^4}\left(-\kappa^2\right) 2\pi R_0 \pi a^2 \kappa$$

$$\boxed{\delta W_F = -\frac{4\pi^2\psi_a^2\xi_0^2}{\mu_0 R_0 a^2}\frac{1}{\kappa} = -W_0} \qquad W_0 = \frac{4\pi^2\psi_a^2\xi_0^2}{\mu_0 R_0 a^2 \kappa}$$

3a. $\underline{\tilde{B}} = \nabla\tilde{A} \times \underline{e}_z \qquad \nabla \cdot \underline{\tilde{B}} = 0$

$$\nabla \times \underline{\tilde{B}} = 0 = \nabla \times \left(\nabla\tilde{A} \times \underline{e}_z\right) = \underbrace{\underline{e}_z \cdot \nabla\tilde{A}}_{=0} - \underbrace{\nabla\tilde{A} \cdot \nabla\underline{e}_z}_{=0} - \underline{e}_z \nabla \cdot \nabla\tilde{A} + \nabla\tilde{A}\underbrace{\nabla \cdot \underline{e}_z}_{=0}$$

$$\therefore \boxed{\nabla^2\tilde{A} = 0}$$

3b. $$\delta W_v = \frac{1}{2\mu_0}\int \underline{\tilde{B}}^2 d\underline{r}$$

$$\underline{\tilde{B}}^2 = \left(\nabla\tilde{A} \times \underline{e}_z\right) \cdot \left(\nabla\tilde{A} \times \underline{e}_z\right) = \left(\nabla\tilde{A}\right)^2$$

$$\boxed{\delta W_v = \frac{1}{2\mu_0}\int \left(\nabla\tilde{A}\right)^2 d\underline{r} = \frac{1}{2\mu_0}\int\left[\left(\frac{\partial\tilde{A}}{\partial x}\right)^2 + \left(\frac{\partial\tilde{A}}{\partial y}\right)^2\right] d\underline{r}}$$

3c. (See attached sheet for summary of coordinate transformation)

$$\underline{n} \cdot \underline{\tilde{B}} = \underline{n} \cdot \nabla\tilde{A} \times \underline{e}_z = -\underline{n} \cdot \underline{e}_z \times \nabla\tilde{A} = -\underline{n} \times \underline{e}_z \cdot \nabla\tilde{A} = \underline{t} \cdot \nabla\tilde{A} = \frac{1}{a\Delta\rho}\frac{\partial\tilde{A}}{\partial\theta}$$

$$\underline{n} \cdot \underline{\tilde{B}}\Big|_{S_w} = \frac{1}{\Delta b}\frac{\partial\tilde{A}}{\partial\theta}\bigg|_{\rho = b/a} = 0$$

$$\boxed{\therefore \tilde{A}\left(\frac{b}{a}, \theta\right) = 0}$$

Boundary Conditions

$$\underline{n} \cdot \underline{\tilde{B}}\Big|_{S_p} = \underline{n} \cdot \nabla\tilde{A} \times \underline{e}_z$$

$$\underline{n} \cdot \underline{\tilde{B}} = \underline{n} \cdot \left(-\xi_0 \frac{\partial \underline{B}}{\partial y}\right) = -\xi_0 \frac{C^{1/2}}{\Delta^{1/2}}\left(\cosh u \cos v \frac{\partial B_x}{\partial y} + \sinh u \sin v \frac{\partial B_y}{\partial y}\right)$$

$$= -\xi_0 \frac{C^{1/2}}{\Delta^{1/2}}\left[\cosh u \cos v \frac{\partial}{\partial y}\frac{2\psi_a}{R_0\kappa^2 a^2} y + \sinh u \sin v \frac{\partial}{\partial y}\left(-\frac{2\psi_a}{R_0 a^2} x\right)\right]$$

$$= -\xi_0 \frac{C^{1/2}}{\Delta^{1/2}} \frac{2\psi_a}{R_0 \kappa^2 a^2} \cosh u_1 \cos v$$

$$= -\frac{2\psi_a \xi_0}{R_0 \kappa^2 a^2} \frac{1}{(C\Delta)^{1/2}} \kappa a \cos v$$

$$\boxed{\underline{n} \cdot \underline{\widetilde{B}} = -\frac{2\psi_a \xi_0}{R_0 \kappa a} \frac{1}{(C\Delta)^{1/2}} \cos v}$$

$$\underline{n} \cdot \nabla \widetilde{A} \times \underline{e}_z = -\underline{n} \cdot \underline{e}_z \times \nabla \widetilde{A} = -\underline{n} \times \underline{e}_z \cdot \nabla \widetilde{A} = \underline{t} \cdot \nabla \widetilde{A} = \frac{1}{(C\Delta)^{1/2}} \frac{\partial \widetilde{A}}{\partial v}$$

$$\left.\frac{\partial \widetilde{A}}{\partial v}\right|_{u_1} = -\frac{2\psi_0 \xi_0}{R_0 \kappa a} \cos v$$

$$\boxed{\widetilde{A}(u_1, v) = -\frac{2\psi_0 \xi_0}{R_0 a \kappa} \sin v}$$

Vacuum Vector Potential

$$\frac{\partial^2 \widetilde{A}}{\partial u^2} + \frac{\partial^2 \widetilde{A}}{\partial v^2} = 0$$

$$\widetilde{A}(u_2, v) = 0 \qquad \widetilde{A}(u_1, v) = -\frac{2\psi_0 \xi_0}{R_0 a \kappa} \sin v$$

$$\widetilde{A} = K \sinh(u - u_2) \sin v = -\frac{2\psi_0 \xi_0}{R_0 a \kappa} \frac{\sinh(u - u_2)}{\sinh(u - u_2)} \sin v$$

$$\frac{\partial \widetilde{A}}{\partial u} = -\frac{2\psi_0 \xi}{R_0 a \kappa} \frac{\cosh(u - u_2)}{\sinh(u_1 - u_2)} \sin v$$

Vacuum Energy

$$\delta W_v = \frac{1}{2\mu_0} \int \widetilde{B}_1^2 \, d\underline{r} = \frac{1}{2\mu_0} \int \left(\nabla \widetilde{A}\right)^2 d\underline{r} = \frac{1}{2\mu_0} \int \Big[\underbrace{\nabla \cdot \left(\widetilde{A} \nabla \widetilde{A}\right)}_{} - \underbrace{\widetilde{A} \nabla^2 \widetilde{A}}_{=0}\Big] d\underline{r}$$

$$= -\frac{1}{2\mu_0} \int \underline{n} \cdot \nabla \widetilde{A} \, \widetilde{A} \, dS_p$$

$$\boxed{\delta W_v = -\frac{1}{2\mu_0} \int \widetilde{A} \, \underline{n} \cdot \nabla \widetilde{A} \, dS_p}$$

$$\tilde{A}\,\underline{n}\cdot\nabla\tilde{A}\,dS_p = \frac{1}{(C\Delta)^{1/2}}\frac{\partial\tilde{A}}{\partial u}\tilde{A}\;\;2\pi R_0\,(C\Delta)^{1/2}\,dv$$

$$= 2\pi R_0\left[\tilde{A}\frac{\partial\tilde{A}}{\partial u}\,dv\right]$$

Then

$$\delta W_v = -\frac{1}{2\mu_0}2\pi R_0\int_0^{2\pi}\left(\tilde{A}\frac{\partial\tilde{A}}{\partial u}\right)_{u_1}dv$$

$$= -\frac{\pi R_0}{\mu_0}\frac{4\psi_a^2\xi_0^2}{R_0^2a^2\kappa^2}\int_0^{2\pi}\sin v\,\frac{\cosh(u_1-u_2)}{\sinh(u_1-u_2)}\sin v\,dv$$

$$\boxed{\delta W_v = +\frac{4\pi^2\psi_a^2\xi_0^2}{\mu_0R_0a^2\kappa^2}\;\cos h(u_2-u_1)}$$

Complete δW

$$\delta W = \frac{4\pi^2\psi_a^2\xi_0^2}{\mu_0R_0a^2\kappa^2}\left[-\kappa+\coth(u_2-u_1)\right]$$

Marginal Stability

$$\kappa = \coth(u_2-u_1)$$

$$\cosh u_2 - u_1 = \cosh u_2\cosh u_1 - \sinh u_2\sinh u_1$$
$$\sinh u_2 - u_1 = \sinh u_2\cosh u_1 - \cosh u_2\sinh u_1$$

$$\coth(u_2-u_1) = \frac{1-\tanh u_2\tanh u_1}{\tanh u_2-\tanh u_1} = \frac{1-\dfrac{1}{\kappa_w}\dfrac{1}{\kappa}}{\dfrac{1}{\kappa_w}-\dfrac{1}{\kappa}}$$

$$= \frac{\kappa_w\kappa-1}{\kappa-\kappa_w}$$

Therefore $\quad \kappa = \dfrac{\kappa_w\kappa-1}{\kappa-\kappa_w}$

$$\kappa^2-\kappa\kappa_w = \kappa\kappa_w-1$$

$$\kappa^2-2\kappa\kappa_w+1=0$$

$$\boxed{\kappa_w = \frac{\kappa^2 + 1}{2\kappa}}$$

Equate Expression for κ_w

$$\left(\frac{\kappa^2 + 1}{\partial\kappa}\right)^2 = \frac{W^2\kappa^2}{\left(W^2 - 1\right)\kappa^2 + 1}$$

$$\left[W^2\kappa^2 - \kappa^2 + 1\right]\left(\frac{\kappa^2 + 1}{\partial\kappa}\right) = W^2\kappa^2$$

$$W^2\left[\kappa^2\left(\frac{\kappa^2 + 1}{\partial\kappa}\right)^2 - \kappa^2\right] = \left(\kappa^2 - 1\right)\left(\frac{\kappa^2 + 1}{\partial\kappa}\right)^2$$

$$\frac{W^2}{\not{4}}\left[\kappa^4 + 2\kappa^2 + 1 - 4\kappa^2\right] = \frac{\left(\kappa^2 - 1\right)\left(\kappa^2 + 1\right)}{\not{4}\kappa^2}$$

$$W^2\left[\kappa^2 - 1\right]^2 = \frac{\left(\kappa^2 - 1\right)\left(\kappa^2 + 1\right)^2}{\kappa^2}$$

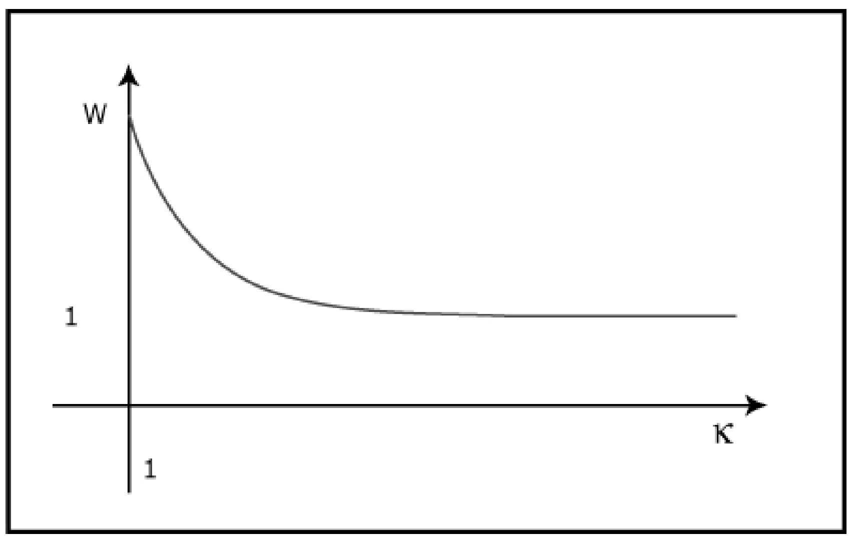
$$W^2 = \frac{\left(\kappa^2 + 1\right)^2}{\kappa^2\left(\kappa^2 - 1\right)}$$

$$\boxed{W = \frac{\kappa^2 + 1}{\kappa^2\left(\kappa^2 - 1\right)^{1/2}}}$$

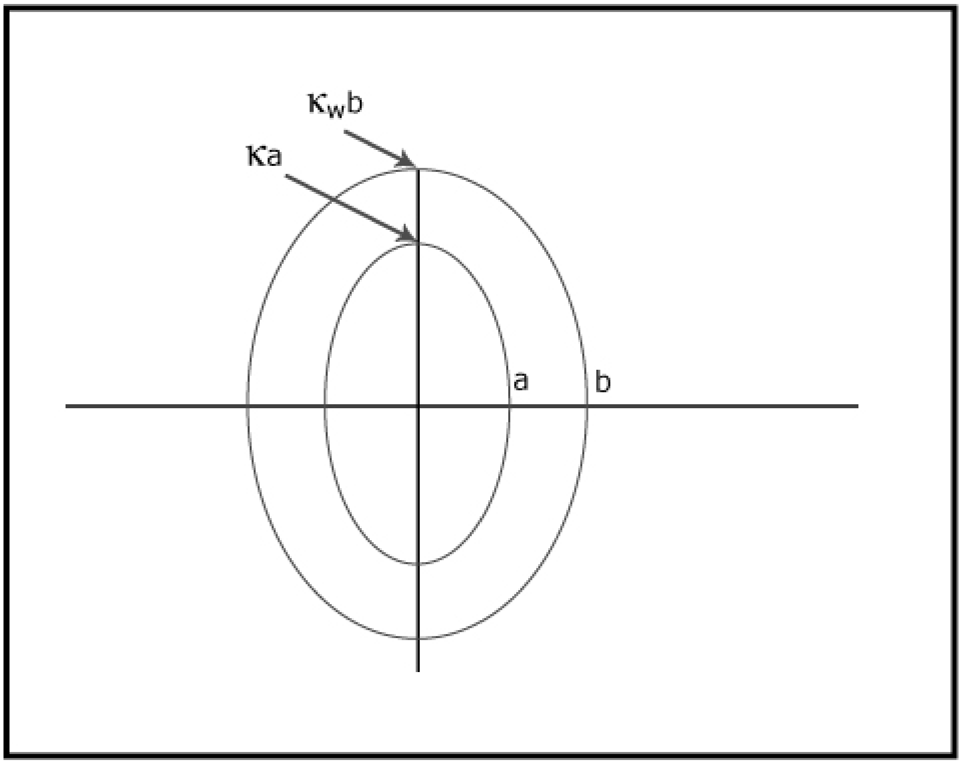
$\kappa \to 1 \quad W \to \infty$

$\kappa \to \infty \quad W \to 1$

$\kappa = 2 \qquad W = \dfrac{5}{2\sqrt{3}} = 1.44$



Elliptic Coordinator



Basic Parameters

$$a, \kappa, W = \kappa_w b / \kappa a$$

Express C, u_1, u_2, b, κ_w in terms of these parameters

$$x = C \sinh u \cos v$$

$$y = C \cosh u \sin v$$

C and u_1

$$C \sinh u_1 = a$$

$$C \cosh u_1 = \kappa a$$

$$\boxed{\tanh u_1 = \frac{1}{\kappa}}$$

$$\boxed{C^2 = \left(\kappa^2 - 1\right) a^2}$$

u_2, b, κ_w

$$C\sinh u_2 = b$$

$$C\cosh u_2 = \kappa_w b$$

$$\frac{b\kappa_w}{a\kappa} = W$$

$$C^2 = \left(\kappa_w^2 - 1\right) b^2 = \frac{\left(\kappa_w^2 - 1\right) a^2 \kappa^2 W^2}{\kappa_w^2}$$

$$\left(\kappa^2 - 1\right) \cancel{a^2} = \left(\kappa_w^2 - 1\right) \frac{\cancel{a^2} \kappa^2 W^2}{\kappa_w^2}$$

$$\kappa^2 \kappa_w^2 - \kappa_w^2 = W^2 \kappa^2 \kappa_w^2 - \kappa^2 W^2$$

$$\kappa_w^2 \left[\kappa^2 - 1 - W^2 \kappa^2\right] = -\kappa^2 W^2$$

$$-\kappa_w^2 \left[\left(W^2 - 1\right) \kappa^2 + 1\right] = -\kappa^2 W^2$$

$$\boxed{\kappa_w^2 = \frac{W^2 \kappa^2}{\left(W^2 - 1\right) \kappa^2 + 1}}$$

$$b^2 = \frac{a^2 W^2 \kappa^2}{\kappa_w^2} = \frac{a^2 W^2 \kappa^2}{W^2 \kappa^2} \left[\left(W^2 - 1\right) \kappa^2 + 1\right]$$

$$\boxed{b^2 = a^2 \left[\left(W^2 - 1\right) \kappa^2 + 1\right]}$$

$$\boxed{\tanh u_2 = \frac{1}{\kappa_w} = \frac{\left[\left(W^2 - 1\right) \kappa^2 + 1\right]^{1/2}}{W\kappa}}$$

Derivation

$$x = C\sinh u \cos v$$

$$y = C\cosh u \sin v$$

$$dx = C\cosh u \cos v\, du - c\sinh u \sin v\, dv \qquad \cosh u \cos v - \sinh u \sin v$$

$$dy = C\sinh u \sin v\, du + c\cosh \cos v\, dv \qquad \sinh u \sin v + \cosh u \cos v$$

$$C\left(\cosh^2 u \cos^2 v + \sinh^2 u \sin^2 v\right) du = \cosh u \cos v\, dx + \sinh u \sin v\, dy$$

$$C\left(\cosh^2 u\cos^2 v + \sinh^2 u\sin^2 v\right)dv = -\sinh u\sin v\,dx + \cosh u\cos v\,dy$$

$$\cosh^2 u\cos^2 v + \sinh^2 u\sin^2 v = \cosh^2 u\cos^2 v + \cosh^2 u\sin^2 v - \sin^2 v$$

$$= \cosh^2 u - \sin^2 v$$

$$du = \frac{\cosh u\cos v\,dx + \sinh u\sin v\,dy}{C\left(\cosh^2 u - \sin^2 u\right)}$$

$$dv = \frac{-\sinh u\sin v\,dx + \cosh u\cos v\,dy}{C\left(\cosh^2 u - \sin^2 v\right)}$$

$$u_x = \frac{\cosh u\cos v}{C\left(\cosh^2 u - \sin^2 v\right)} \qquad u_y = \frac{\sinh u\sin v}{C\left(\cosh^2 u - \sin^2 v\right)}$$

$$v_x = \frac{-\sinh u\sin v}{C\left(\cosh^2 u - \sin^2 v\right)} \qquad v_y = \frac{\cosh u\cos v}{C\left(\cosh^2 u - \sin^2 v\right)}$$

$$\frac{\partial}{\partial x} = u \times \frac{\partial}{\partial u} + v \times \frac{\partial}{\partial v} \qquad \frac{\partial}{\partial y} = u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial x} = \frac{1}{C\left(\cosh^2 u - \sin^2 v\right)}\left[\cosh u\cos v\frac{\partial}{\partial u} - \sinh u\sin v\frac{\partial}{\partial v}\right]$$

$$\frac{\partial}{\partial y} = \frac{1}{C\left(\cosh^2 u - \sin^2 v\right)}\left[\sinh u\sin v\frac{\partial}{\partial u} + \cosh u\cos v\frac{\partial}{\partial v}\right]$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \Delta = C\left(\cosh^2 u - \sin^2 v\right)$$

$$= \frac{\partial}{\partial x}\left(u \times \frac{\partial}{\partial u} + v \times \frac{\partial}{\partial v}\right) + \frac{\partial}{\partial y}\left(u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}\right)$$

$$= u_{xx}\frac{\partial}{\partial u} + v_{xx}\frac{\partial}{\partial v} + \frac{1}{\Delta}\left[u_x \cosh u\cos v\frac{\partial^2}{\partial u^2} - u_x \sinh u\sin v\frac{\partial^2}{\partial u\partial v}\right]$$

$$+\frac{1}{\Delta}\left[v_x \cosh u\cos v\frac{\partial^2}{\partial u\partial v} - v_x \sinh u\sin v\frac{\partial^2}{\partial v^2}\right]$$

$$+u_{yy}\frac{\partial}{\partial u} + v_{yy}\frac{\partial}{\partial v} + \frac{1}{\Delta}\left[u_y \sinh u\sin v\frac{\partial^2}{\partial u^2} + u_y \cosh u\cos v\frac{\partial^2}{\partial u\partial v}\right]$$

$$+\frac{1}{\Delta}\left[v_y \sinh u \sin v \frac{\partial^2}{\partial u \partial v} + v_y \cosh u \cos v \frac{\partial^2}{\partial v^2}\right]$$

Use $v_x = -u_y \qquad v_y = u_x$

$$\nabla^2_{(1)} = u_{xx}\frac{\partial}{\partial u} - u_{xy}\frac{\partial}{\partial v} + u_{yy}\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial v}$$

$$= v_{xy}\frac{\partial}{\partial u} - v_{xy}\frac{\partial}{\partial u} = 0$$

$$\nabla^2 = \frac{1}{\Delta}\left\{\left(u_x \cosh u \cos v + u_y \sinh u \sin v\right)\frac{\partial^2}{\partial u^2}\right.$$

$$\left(u_x \cosh u \cos v + u_y \sinh u \sin v\right)\frac{\partial^2}{\partial v^2}$$

$$+\left(-u_x \sinh u \sin v - u_y \cosh u \cos v + u_y \cosh u \cos v + u_x \sinh u \sin v\right)\frac{\partial^2}{\partial u \partial v}$$

$$= \frac{1}{\Delta^2}\left(\cosh^2 u \cos^2 v + \sinh^2 u \sin^2 v\right)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$$

$$\boxed{\nabla^2 = \frac{1}{C^2}\frac{1}{\cosh^2 u - \sin^2 v}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)}$$

n, t

$$\underline{n}^1 = \nabla u = u_x \underline{e}_x + u_y \underline{e}_y$$

$$= \frac{\cosh u \cos v}{\Delta}\underline{e}_x + \frac{\sinh u \sin v}{\Delta}\underline{e}_y$$

$$\boxed{\underline{n}\cdot\nabla = \frac{C^{1/2}}{\Delta^{1/2}}\left[\cosh u \cos v\, \underline{e}_x - \sinh u \sin v\, \underline{e}_y\right]}$$

$$\underline{n}\cdot\nabla = \frac{C^{1/2}}{\Delta^{1/2}}\left[\cosh u \cos v \frac{\partial}{\partial x} + \sinh u \sin v \frac{\partial}{\partial y}\right]$$

$$= \frac{C^{1/2}}{\Delta\Delta^{1/2}}\left[\cosh u \cos v\left(\cosh u \cos v \frac{\partial}{\partial u} - \sinh u \sin v \frac{\partial}{\partial v}\right)\right]$$

$$+\sinh u \sin v\left(\sinh u \sin v \frac{\partial}{\partial u} + \cosh u \cos v \frac{\partial}{\partial v}\right)$$

$$\boxed{\underline{n}\cdot\nabla = \frac{1}{(C\Delta)^{1/2}}\frac{\partial}{\partial u}}$$

$$\boxed{\underline{t} = \underline{e}_z \times \underline{n} = \frac{C^{1/2}}{\Delta^{1/2}}\left[\cosh u \cos v\, \underline{e}_y - \sinh u \sin v\, \underline{e}_x\right]}$$

$$\underline{t}\cdot\nabla = \frac{C^{1/2}}{\Delta^{1/2}}\left[\cosh u \cos v \frac{\partial}{\partial y} - \sinh u \sin v \frac{\partial}{\partial x}\right]$$

$$= \frac{C^{1/2}}{\Delta\Delta^{1/2}}\left[\cosh u \cos v\left(\sinh u \sin v \frac{\partial}{\partial u} + \cosh u \cos v \frac{\partial}{\partial v}\right)\right]$$

$$-\sinh u \sin v\left(\cosh u \cos v \frac{\partial}{\partial u} - \sinh u \sin v \frac{\partial}{\partial v}\right)$$

$$\boxed{\underline{t}\cdot\nabla = \frac{1}{(C\Delta)^{1/2}}\frac{\partial}{\partial v}}$$

Plasma Area

$$du\,dv = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} dx\,dy = \left(u_x v_y - u_y v_x\right) dx\,dy$$

$$= \frac{1}{\Delta}\left[\cosh^2 u \cos^2 v + \sinh^2 u \sin^2 v\right] dx\,dy$$

$$= \frac{1}{C}\, dx\,dy$$

$$\boxed{dx\,dy = C\,du\,dv}$$

Surface Area

$$dS = 2\pi R_0\, dl = 2\pi R_0 \left(x_v^2 + y_v^2\right)^{1/2} dv$$

$$= 2\pi R_0\left[C^2 \sinh^2 u \sin^2 v + C^2 \cosh^2 u \cos^2 v\right]^{1/2} dv$$

$$dS_p = 2\pi R_0 \left(C\Delta\right)^{1/2} dv$$