22.615, MDH Theory of Fusion Systems
Prof.
MHD Take Home Final
Distributed: Tuesday May 6, 2003
Due: Thursday May 15, 2003

Introduction

In class it was pointed out that elongated tokamaks have improved stability with respect to $n \geq 1$ modes. However, if the elongation is too large, the tokamak will be unstable to an n=0 vertical instability. This mode can be stabilized by the presence of a perfectly conducting wall sufficiently close to the plasma. However, for a realistic wall with finite resistivity the mode again becomes unstable, although with a much slower growth time, comparable to the wall diffusion time. Even so, the slower growing mode can then, in both principle and practice, be feedback stabilized.

The key issue, therefore, in designing an experiment is to make sure that the resistive wall surrounding the plasma is sufficiently close that it would provide n=0 stability if it were perfectly conducting. This is necessary so that the feedback system only has to stabilize the slow growing, n=0, resistive wall mode. If the wall is too far away the mode grows on the ideal MHD time scale which is far too fast for feedback to be practical.

The purpose of the final exam problem is to examine the stability of an elongated tokamak surrounded by a perfectly conducting wall against the n=0 vertical instability. Specifically we want to determine a relationship between the critical elongation for stability vs wall position; that is, κ vs b/a.

1. Calculate the equilibrium

Experience has shown that the essence of the n=0 vertical instability can be ascertained by ignoring toroidal effects $(a/R_0 \rightarrow 0)$ and assuming the plasma pressure is zero $(p \rightarrow 0)$. These effects do not play a dominant role. Under the above assumptions the Grad-Shafranov equation reduces to

$$\nabla^2 \psi = -\mu_0 R_0 J_z \left(\psi \right)$$

$$B_p = \frac{1}{R_0} \nabla \psi \times e_z$$

For simplicity the plasma current is assumed to be a constant.

$$J_z \left(\psi \right) = -J_0 = \text{ constant}$$

The geometry is shown in Fig. 1. The plasma is assumed to be elliptical in shape with a horizontal radius 'a' and vertical radius κa. Surrounding the plasma is a perfectly conducting wall, elliptical in shape, with a horizontal radius 'b' and vertical radius $\kappa_w b$. Note that to make the problem tractable analytically, value of κ_w will be chosen so that the wall and plasma surfaces correspond to confocal ellipses. The

relation between κ_w and κ will be determined in part (3). For now, focus on the plasma equilibrium

a) Solve for the equilibrium ψ assuming the arbitrary constant in the flux is chosen such that $\psi = 0$ on the plasma surface. Hint: use rectangular x, y coordinates.

b) Solve for B_x and B_y.

It is also possible to calculate J_x, J_y, and B_{z2} (where $B_z = B_0 + B_{z2}$). However, trust me that these quantities make contributions to the stability that are smaller by $\varepsilon^2 = (a/R_0)^2$ than the dominant contributions. Hence, they can be neglected everywhere and there is no need to calculate them.

2. Calculate the fluid Energy

The first step in the stability analysis of the n=0 mode is to evaluate the fluid energy. This task is made easy by using a simple trial function for the plasma displacement rather than considering the full 2-D eigenfunction equation. For the trial function assume the plasma undergoes a pure, uniform vertical displacement:

$$\xi = \xi_0 e_y$$

where $\xi_0 = $ constant.

a) Evaluate δW_F using this trial function. Use any form of δW_F that you like although keep in mind that some forms will be much simpler to use than others. In other words, think a little before diving into algebra. Your answer for δW_F should be negative or else you have made a mistake.

3. Calculate the plasma-vacuum boundary condition

The next step in the problem is the calculation of the vacuum energy. Part (3) of the final exam sets up the calculation of the vacuum fields in terms of a single scalar $\widetilde{A}$ including appropriate boundary conditions. Part (4) involves the actual evaluation of δW_V .

In class we usually analyzed the vacuum energy by introducing a scalar potential $\tilde{\phi}$. For the present 2-D problem, however, it is more convenient to introduce a scalar vector potential as follows

$$\widetilde{B} = \nabla \widetilde{A}(x, \, y) \times e_z$$

Note that $\nabla \cdot \widetilde{B} = 0$ automatically.

a) What partial differential equation in $(x, \, y)$ coordinates does $\widetilde{A}$ satisfy?

b) Write an integral expression for δW_V in terms of $\widetilde{A}$.

c) For this and all following parts of the calculation of δW_v it will be convenient to switch from rectangular coordinates to a set of elliptic coordinates u, v defined by

$$x = c \sinh u \cos v$$

$$y = c \cosh u \sin v$$

where c= constant. The basic physical parameters describing the geometry are a, κ and a wall spacing parameter 'w' defined in terms of the ratio of the vertical wall height to vertical plasma height

$$w = \frac{\kappa_w b}{\kappa a}$$

This is the critical parameter for a vertical displacement. The plasma surface S_P is defined as $u = u_1$. If the wall is confocal ellipse then the wall surface S_W is defined as $u = u_2$. Find expressions for c, u_1, κ_w, b, and u_2 in terms of a, κ, and w.

d) The boundary condition on the conducting wall surface S_W is given by $n \cdot \widetilde{B}\big|_{S_W} = 0$. Express this condition in terms of $\widetilde{A}$.

e) The boundary condition on the plasma-vacuum interfaces S_P is given by $n \cdot \widetilde{B}\big|_{S_P} = n \cdot \nabla \times (\xi \times B)\big|_{S_P}$. After some algebra, this condition turns out to be relatively simple and can be expressed in terms of $\widetilde{A}$ as follows.

$$\widetilde{A}\big|_{S_P} = K \sin v$$

Find the constant K.

4. Calculate the vacuum energy

You are now ready to calculate the vacuum energy.

a) Convert the differential equation for $\widetilde{A}$ from rectangular (x, y) coordinates to elliptic (u, v) coordinates

b) Solve for $\widetilde{A}$ (u, v) using the boundary conditions obtained in part (3)

c) Substitute back and derive an expression for δW_V.

5. The final result

Combine terms to obtain the total δW. Set expression to zero to determine the condition for marginal stability. With a little algebra you should be able to obtain a simple marginal stability condition of the form.

$$w=w(\kappa)$$

Calculate and plot this function.