Chapter 2

Motion of Charged Particles in Fields

Plasmas are complicated because motions of electrons and ions are determined by the electric and magnetic fields but *also change* the fields by the currents they carry.

For now we shall ignore the second part of the problem and assume that *Fields are Prescribed.*

Even so, calculating the motion of a charged particle can be quite hard.

Equation of motion:

$$\underbrace{m\frac{d\mathbf{v}}{dt}}_{\text{Rate of change of momentum}} = \underbrace{\underset{\text{charge}}{q}\;(\underset{\text{E-field}}{\mathbf{E}} + \underset{\text{velocity}}{\mathbf{v}} \wedge \underset{\text{B-field}}{\mathbf{B}})}_{\text{Lorentz Force}} \tag{2.1}$$

Have to solve this differential equation, to get position $\mathbf{r}$ and velocity ($\mathbf{v}=\dot{\mathbf{r}}$) given $\mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t)$.

Approach: Start simple, gradually generalize.

2.1 Uniform B field, $\mathbf{E} = 0$.

$$m\dot{\mathbf{v}} = q\mathbf{v} \wedge \mathbf{B} \tag{2.2}$$

2.1.1 Qualitatively

in the plane perpendicular to B: Accel. is perp to $\mathbf{v}$ so particle moves in a circle whose radius r_L is such as to satisfy

$$m r_L \Omega^2 = m\frac{v_\perp^2}{r_L} = |q| v_\perp B \tag{2.3}$$

Ω is the angular (velocity) frequency

1st equality shows $\Omega^2 = v_\perp^2/r_L^2$ $\;(r_L = v_\perp/\Omega)$

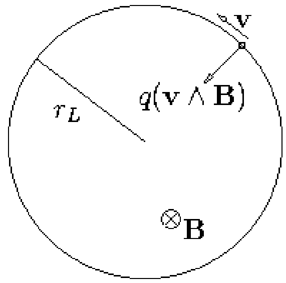

Figure 2.1: Circular orbit in uniform magnetic field.

Hence second gives $m\frac{v_\perp}{\Omega}\Omega^2 = |q|v_\perp B$

$$\text{i.e.} \quad \Omega = \frac{|\text{q}|\text{B}}{\text{m}} \quad . \tag{2.4}$$

Particle moves in a circular orbit with

$$\text{angular velocity} \quad \Omega = \frac{|q|B}{m} \quad \text{the "Cyclotron Frequency"} \tag{2.5}$$

$$\text{and radius} \quad r_l = \frac{v_\perp}{\Omega} \quad \text{the "Larmor Radius.} \tag{2.6}$$

2.1.2 By Vector Algebra

- Particle Energy is constant. *proof*: take $\mathbf{v}$. Eq. of motion then

$$m\mathbf{v}.\dot{\mathbf{v}} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v}.(\mathbf{v} \wedge \mathbf{B}) = 0. \tag{2.7}$$

- Parallel and Perpendicular motions separate. $v_\parallel$ = constant because accel ($\propto \mathbf{v} \wedge \mathbf{B}$) is perpendicular to $\mathbf{B}$.

Perpendicular Dynamics:

Take B in $\hat{z}$ direction and write components

$$m\dot{v}_x = qv_yB \quad , \quad m\dot{v}_y = -qv_xB \tag{2.8}$$

Hence

$$\ddot{v}_x = \frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x = -\Omega^2 v_x \tag{2.9}$$

Solution: $v_x = v_\perp \cos \Omega t$ (choose zero of time)

Substitute back:

$$v_y = \frac{m}{qB}\dot{v}_x = -\frac{|q|}{q}v_\perp \sin \Omega t \tag{2.10}$$

Integrate:

$$x = x_0 + \frac{v_\perp}{\Omega} \sin \Omega t \quad , \quad y = y_0 + \frac{q}{|q|} \frac{v_\perp}{\Omega} \cos \Omega t \tag{2.11}$$

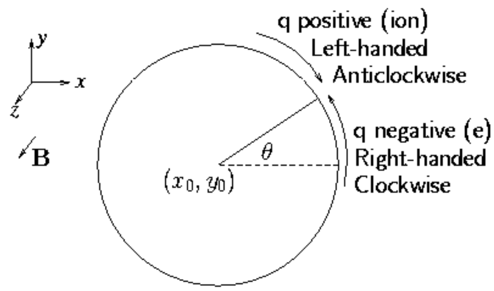

Figure 2.2: Gyro center (x_0, y_0) and orbit

This is the equation of a circle with center $\mathbf{r}_0 = (x_0, y_0)$ and radius $r_L = v_\perp/\Omega$: Gyro Radius. [Angle is $\theta = \Omega t$]

Direction of rotation is as indicated opposite for opposite sign of charge:

Ions rotate anticlockwise. Electrons clockwise about the magnetic field.

The current carried by the plasma always is in such a direction as to *reduce* the magnetic field.

This is the property of a magnetic material which is *"Diagmagnetic"*.

When $v_\parallel$ is non-zero the total motion is along a helix.

2.2 Uniform B and non-zero E

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{2.12}$$

Parallel motion: Before, when $\mathbf{E} = 0$ this was $v_\parallel$ = const. Now it is clearly

$$\dot{v}_\parallel = \frac{qE_\parallel}{m} \tag{2.13}$$

Constant acceleration along the field.

Perpendicular Motion

Qualitatively:

Speed of positive particle is greater at top than bottom so radius of curvature is greater. Result is that guiding center moves perpendicular to both **E** and **B**. It 'drifts' across the field.

Algebraically: It is clear that if we can find a constant velocity $\mathbf{v}_d$ that satisfies

$$\mathbf{E} + \mathbf{v}_d \wedge \mathbf{B} = 0 \tag{2.14}$$

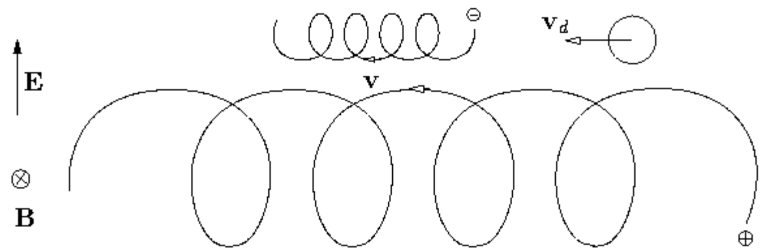

Figure 2.3: $\mathbf{E} \wedge \mathbf{B}$ drift orbit

then the sum of this drift velocity plus the velocity

$$\mathbf{v}_L = \frac{d}{dt}[\mathbf{r}_L e^{i\Omega(t-t_0)}] \tag{2.15}$$

which we calculated for the $\mathbf{E} = 0$ gyration will satisfy the equation of motion.
Take $\wedge\mathbf{B}$ the above equation:

$$0 = \mathbf{E} \wedge \mathbf{B} + (\mathbf{v}_d \wedge \mathbf{B}) \wedge \mathbf{B} = \mathbf{E} \wedge \mathbf{B} + (v_d.\mathbf{B})\mathbf{B} - B^2\mathbf{v}_d \tag{2.16}$$

so that

$$\mathbf{v}_d = \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.17}$$

does satisfy it.
Hence the full solution is

$$\mathbf{v} = \underset{\text{parallel}}{\mathbf{v}_\parallel} + \underset{\text{cross-field drift}}{\mathbf{v}_d} + \underset{\text{Gyration}}{\mathbf{v}_L} \tag{2.18}$$

where

$$\dot{v}_\parallel = \frac{qE_\parallel}{m} \tag{2.19}$$

and
$\mathbf{v}_d$ (eq 2.17) is the "E × B drift" of the gyrocenter.

Comments on E × B drift:

1. It is *independent* of the properties of the drifting particle (q, m, v, whatever).
2. Hence it is in the *same* direction for electrons and ions.
3. Underlying physics for this is that in the frame moving at the E × B drift E = 0. We have 'transformed away' the electric field.
4. Formula given above is exact except for the fact that relativistic effects have been ignored. They would be important if $v_d \sim c$.

2.2.1 Drift due to Gravity or other Forces

Suppose particle is subject to some other force, such as gravity. Write it **F** so that

$$m\dot{\mathbf{v}} = \mathbf{F} + q\ \mathbf{v} \wedge \mathbf{B} = q(\frac{1}{q}\mathbf{F} + \mathbf{v} \wedge \mathbf{B}) \tag{2.20}$$

This is just like the Electric field case except with $\mathbf{F}/q$ replacing $\mathbf{E}$.

The drift is therefore

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F} \wedge \mathbf{B}}{B^2} \tag{2.21}$$

In this case, if force on electrons and ions is same, they drift in *opposite* directions.

This general formula can be used to get the drift velocity in some other cases of interest (see later).

2.3 Non-Uniform B Field

If B-lines are straight but the magnitude of B varies in space we get orbits that look qualitatively similar to the E ⊥ B case:

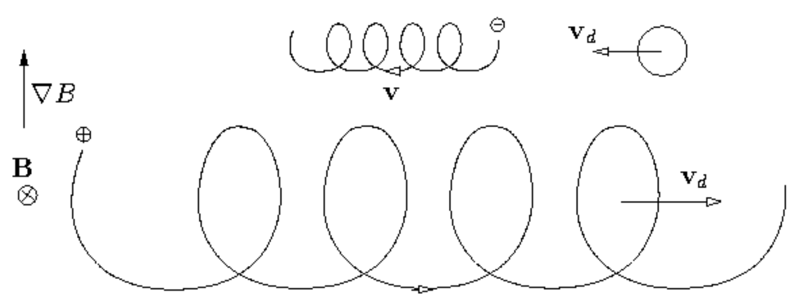

Figure 2.4: ∇B drift orbit

Curvature of orbit is greater where B is greater causing loop to be small on that side. Result is a drift perpendicular to both **B** and ∇B. Notice, though, that electrons and ions go in *opposite* directions (unlike $\mathbf{E} \wedge \mathbf{B}$).

Algebra

We try to find a decomposition of the velocity as before into $\mathbf{v} = \mathbf{v}_d + \mathbf{v}_L$ where $\mathbf{v}_d$ is constant.

We shall find that this can be done only approximately. Also we must have a simple expression for B. This we get by assuming that the Larmor radius is much smaller than the scale length of B variation i.e.,

$$r_L << B/|\nabla B| \tag{2.22}$$

in which case we can express the field approximately as the first two terms in a Taylor expression:

$$\mathbf{B} \simeq \mathbf{B}_0 + (\mathbf{r}.\nabla)\mathbf{B} \tag{2.23}$$

Then substituting the decomposed velocity we get:

$$m\frac{d\mathbf{v}}{dt} = m\dot{\mathbf{v}}_L = q(\mathbf{v} \wedge \mathbf{B}) = q[\mathbf{v}_L \wedge \mathbf{B}_0 + \mathbf{v}_d \wedge \mathbf{B}_0 + (\mathbf{v}_L + \mathbf{v}_d) \wedge (\mathbf{r}.\nabla)\mathbf{B}] \tag{2.24}$$

$$\text{or} \quad 0 = \mathbf{v}_d \wedge \mathbf{B}_0 + \mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B} + \mathbf{v}_d \wedge (\mathbf{r}.\nabla)\mathbf{B} \tag{2.25}$$

Now we shall find that v_d/v_L is also small, like $r|\nabla B|/B$. Therefore the last term here is second order but the first two are first order. So we drop the last term.

Now the awkward part is that $\mathbf{v}_L$ and $\mathbf{r}_L$ are periodic. Substitute for $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L$ so

$$0 = \mathbf{v}_d \wedge \mathbf{B}_0 + \mathbf{v}_L \wedge (\mathbf{r}_L.\nabla)\mathbf{B} + \mathbf{v}_L \wedge (\mathbf{r}_0.\nabla)\mathbf{B} \tag{2.26}$$

We now average over a cyclotron period. The last term is $\propto e^{-i\Omega t}$ so it averages to zero:

$$0 = \mathbf{v}_d \wedge \mathbf{B} + \langle \mathbf{v}_L \wedge (\mathbf{r}_L.\nabla)\mathbf{B} \rangle \ . \tag{2.27}$$

To perform the average use

$$\mathbf{r}_L = (x_L, y_L) = \frac{v_\perp}{\Omega}\left(\sin\Omega t, \frac{q}{|q|}\cos\Omega t\right) \tag{2.28}$$

$$\mathbf{v}_L = (\dot{x}_L, \dot{y}_L) = v_\perp\left(\cos\Omega t, \frac{-q}{|q|}\sin\Omega t\right) \tag{2.29}$$

$$\text{So} \quad [v_L \wedge (\mathbf{r}.\nabla)\mathbf{B}]_x = v_y y \frac{d}{dy} B \tag{2.30}$$

$$[\mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B}]_y = -v_x y \frac{d}{dy} B \tag{2.31}$$

(Taking ∇B to be in the y-direction).

Then

$$\langle v_y y \rangle = -\langle \cos\Omega t \sin\Omega t \rangle \frac{v_\perp^2}{\Omega} = 0 \tag{2.32}$$

$$\langle v_x y \rangle = \frac{q}{|q|}\langle \cos\Omega t \cos\Omega t \rangle \frac{v_\perp^2}{\Omega} = \frac{1}{2}\frac{v_\perp^2}{\Omega}\frac{q}{|q|} \tag{2.33}$$

So

$$\langle \mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B} \rangle = -\frac{q}{|q|}\frac{1}{2}\frac{v_\perp^2}{\Omega}\nabla B \tag{2.34}$$

Substitute in:

$$0 = \mathbf{v}_d \wedge \mathbf{B} - \frac{q}{|q|}\frac{v_\perp^2}{2\Omega}\nabla B \tag{2.35}$$

and solve as before to get

$$\mathbf{v}_d = \frac{\left(\frac{-1}{|q|}\frac{v_\perp^2}{2\Omega}\ \nabla B\right)\wedge\mathbf{B}}{B^2} = \frac{q}{|q|}\frac{v_\perp^2}{2\Omega}\frac{\mathbf{B}\wedge\nabla B}{B^2} \tag{2.36}$$

or equivalently

$$\mathbf{v}_d = \frac{1}{q}\frac{mv_\perp^2}{2B}\frac{\mathbf{B}\wedge\nabla B}{B^2} \tag{2.37}$$

This is called the 'Grad B drift'.

2.4 Curvature Drift

When the B-field lines are curved and the particle has a velocity $v_\parallel$ along the field, another drift occurs.

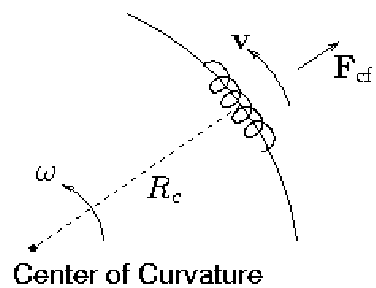

Figure 2.5: Curvature and Centrifugal Force

Take $|B|$ constant; radius of curvature R_e.

To 1st order the particle just spirals along the field.

In the frame of the guiding center a force appears because the plasma is rotating about the center of curvature.

This centrifugal force is F_{cf}

$$F_{cf} = m\frac{v_\parallel^2}{R_c} \quad \text{pointing outward} \tag{2.38}$$

as a vector

$$\mathbf{F}_{cf} = mv_\parallel^2\frac{\mathbf{R}_c}{R_c^2} \tag{2.39}$$

[There is also a coriolis force $2m(\omega\wedge\mathbf{v})$ but this averages to zero over a gyroperiod.]

Use the previous formula for a force

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F}_{cf}\wedge\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2} \tag{2.40}$$

This is the "Curvature Drift".

It is often convenient to have this expressed in terms of the field gradients. So we relate $\mathbf{R}_c$ to ∇B etc. as follows:

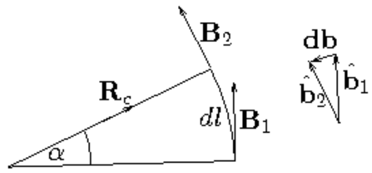

Figure 2.6: Differential expression of curvature

(Carets denote unit vectors)

From the diagram

$$\mathbf{db} = \hat{\mathbf{b}}_2 - \hat{\mathbf{b}}_1 = -\hat{\mathbf{R}}_c \alpha \tag{2.41}$$

and

$$d\ell = \propto R_c \tag{2.42}$$

So

$$\frac{d\mathbf{b}}{dl} = -\frac{\hat{\mathbf{R}}_c}{R_c} = -\frac{\mathbf{R}_c}{R_c^2} \tag{2.43}$$

But (by definition)

$$\frac{d\mathbf{b}}{dl} = (\hat{\mathbf{B}}.\nabla)\hat{\mathbf{b}} \tag{2.44}$$

So the curvature drift can be written

$$\mathbf{v}_d = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c}{R_c^2} \wedge \frac{\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{q}\frac{\mathbf{B} \wedge (\hat{\mathbf{b}}.\nabla)\hat{\mathbf{b}}}{B^2} \tag{2.45}$$

2.4.1 Vacuum Fields

Relation between ∇B & $\mathbf{R}_c$ drifts

The curvature and ∇B are related because of Maxwell's equations, their relation depends on the current density $\mathbf{j}$. A particular case of interest is $\mathbf{j} = 0$: vacuum fields.

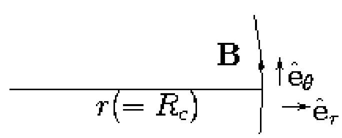

Figure 2.7: Local polar coordinates in a vacuum field

$$\nabla \wedge \mathbf{B} = 0 \quad \text{(static case)} \tag{2.46}$$

Consider the z-component

$$0 = (\nabla \wedge \mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) \quad (B_r = 0 \text{ by choice}). \tag{2.47}$$

$$= \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \tag{2.48}$$

or, in other words,

$$(\nabla B)_r = -\frac{B}{R_c} \tag{2.49}$$

[Note also $0 = (\nabla \wedge \mathbf{B})_\theta = \partial B_\theta/\partial z \;: (\nabla B)_z = 0$]
and hence $(\nabla B)_{\text{perp}} = -B\ \mathbf{R}_c/R_c^2$.
Thus the grad B drift can be written:

$$\mathbf{v}_{\nabla B} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B}\wedge\nabla B}{B^3} = \frac{mV_\perp^2}{2q}\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2B^2} \tag{2.50}$$

and the total drift across a vacuum field becomes

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2B^2}\ . \tag{2.51}$$

Notice the following:

1. R_c & ∇B drifts are in the *same* direction.
2. They are in *opposite* directions for opposite charges.
3. They are proportional to particle *energies*
4. Curvature $\leftrightarrow$ Parallel Energy ($\times$ 2)
   $\nabla B \leftrightarrow$ Perpendicular Energy
5. As a result one can very quickly calculate the average drift over a thermal distribution of particles because

$$\langle\frac{1}{2}mv_\parallel^2\rangle = \frac{T}{2} \tag{2.52}$$

$$\langle\frac{1}{2}mv_\perp^2\rangle = T \qquad \text{2 degrees of freedom} \tag{2.53}$$

Therefore

$$\langle\mathbf{v}_R + \mathbf{v}_{\nabla B}\rangle = \frac{2T}{q}\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2B^2}\left(= \frac{2T}{q}\frac{\mathbf{B}\wedge\left(\hat{\mathbf{b}}.\nabla\right)\hat{\mathbf{b}}}{B^2}\right) \tag{2.54}$$

2.5 Interlude: Toroidal Confinement of Single Particles

Since particles can move freely along a magnetic field even if not across it, we cannot obviously confine the particles in a straight magnetic field. Obvious idea: bend the field lines into circles so that they have no ends.

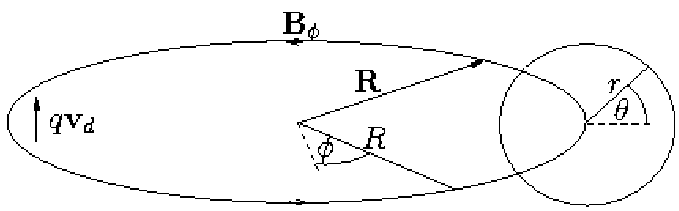

Figure 2.8: Toroidal field geometry

<u>Problem</u>

Curvature & ∇B drifts

$$\mathbf{v}_d = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}\wedge\mathbf{B}}{R^2B^2} \tag{2.55}$$

$$|\mathbf{v}_d| = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{1}{BR} \tag{2.56}$$

Ions drift *up*. Electrons down. There is no confinement. When there is finite density things

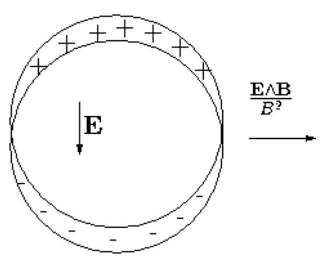

Figure 2.9: Charge separation due to vertical drift

are even worse because charge separation occurs $\rightarrow$ E $\rightarrow$ $E \wedge B$ $\rightarrow$ Outward Motion.

2.5.1 How to solve this problem?

Consider a beam of electrons $v_\parallel \neq 0$ $v_\perp = 0$. Drift is

$$v_d = \frac{mv_\parallel^2}{q}\frac{1}{B_T R} \tag{2.57}$$

What B_z is required to cancel this?

Adding B_z gives a compensating vertical velocity

$$v = v_\parallel \frac{B_z}{B_T} \qquad \text{for } \mathrm{B_z} \ << \ \mathrm{B_T} \tag{2.58}$$

We want total

$$v_z = 0 = v_\parallel \frac{B_z}{B_T} + \frac{m v_\parallel^2}{q} \frac{q}{B_T R} \tag{2.59}$$

So $B_z = -m v_\parallel / R q$ is the right amount of field.

Note that this is such as to make

$$r_L(B_z) = \frac{|m v_\parallel|}{|q B_z|} \ = R \ \ . \tag{2.60}$$

But B_z required depends on $v_\parallel$ and q so we can't compensate for all particles simultaneously. Vertical field along cannot do it.

2.5.2 The Solution: Rotational Transform

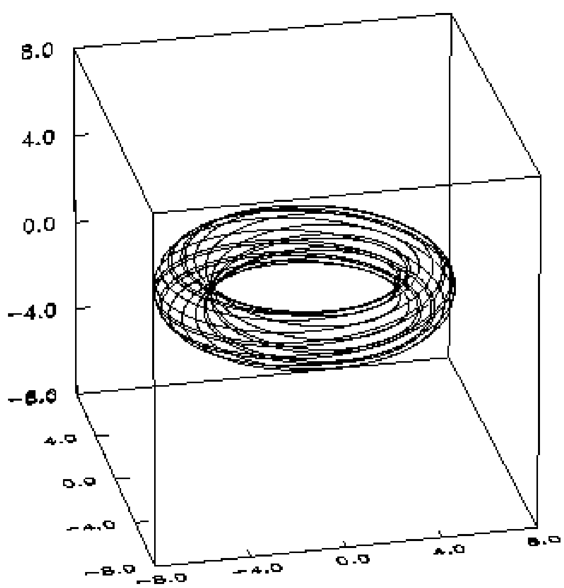

Figure 2.10: Tokamak field lines with rotational transform

Toroidal Coordinate system (r, θ, ϕ) (minor radius, poloidal angle, toroidal angle), see figure 2.8.

Suppose we have a *poloidal field* B_θ

Field Lines become helical and wind around the torus: figure 2.10.

In the poloidal cross-section the field describes a circle as it goes round in ϕ.
Equation of motion of a particle *exactly* following the field is:

$$r\frac{d\theta}{dt} = \frac{B_\theta}{B_\phi}v_\phi = \frac{B_\theta}{B_\phi}\frac{B_\phi}{B}v_\parallel = \frac{B_\theta}{B}v_\parallel \tag{2.61}$$

and

$$r = \text{constant.} \tag{2.62}$$

Now add on to this motion the cross field drift in the $\hat{\mathbf{z}}$ direction.

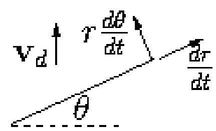

Figure 2.11: Components of velocity

$$r\frac{d\theta}{dt} = \frac{B_\theta}{B}v_\parallel + v_d\cos\theta \tag{2.63}$$

$$\frac{dr}{dt} = v_d\sin\theta \tag{2.64}$$

Take ratio, to eliminate time:

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u_d\sin\theta}{\frac{B_\theta}{B}v_\parallel + v_d\cos\theta} \tag{2.65}$$

Take $B_\theta, B, v_\parallel, v_d$ to be constants, then we can integrate this orbit equation:

$$[\ln r] = [-\ln|\frac{B_\theta v_\parallel}{B} + v_d\cos\theta|]\ . \tag{2.66}$$

Take $r = r_0$ when $\cos\theta = 0$ ($\theta = \frac{\pi}{2}$) then

$$r = r_0/\left[1 + \frac{Bv_d}{b_\theta v_\parallel}\cos\theta\right] \tag{2.67}$$

If $\frac{Bv_d}{B_\theta v_\parallel} << 1$ this is approximately

$$r = r_0 - \Delta\cos\theta \tag{2.68}$$

where $\Delta = \frac{Bv_d}{B_\theta v_\parallel}r_0$
This is approximately a circular orbit shifted by a distance Δ:

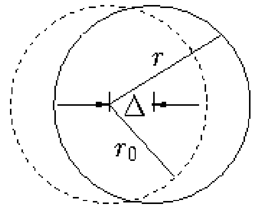

Figure 2.12: Shifted, approximately circular orbit

Substitute for v_d

$$\Delta \simeq r_0 \frac{B}{B_\theta}\frac{1}{q}\frac{(mv_\parallel^2 + \frac{1}{2}mv_\perp^2)}{v_\parallel}\frac{1}{B_\phi R} \tag{2.69}$$

$$\simeq \frac{1}{qB_\theta}\frac{mv_\parallel^2 + \frac{1}{2}mv_\perp^2}{v_\parallel}\frac{r_p}{R} \tag{2.70}$$

$$\text{If } v_\perp = 0 \qquad \Delta = \frac{mv_\parallel}{qB_\theta}\frac{r_0}{R} = r_{L\theta}\frac{r_0}{R}, \tag{2.71}$$

where $r_{L\theta}$ is the Larmor Radius in a field $B_\theta \times r/R$.

Provided Δ is small, particles *will* be confined. Obviously the important thing is the poloidal rotation of the field lines: Rotational Transform.

Rotational Transform

$$\text{rotational transform} \equiv \frac{\text{poloidal angle}}{\text{1 toroidal rotation}} \tag{2.72}$$

$$(\text{transform}/2\pi =) \quad \iota \equiv \frac{\text{poloidal angle}}{\text{toroidal angle}} \quad . \tag{2.73}$$

(Originally, ι was used to denote the transform. Since about 1990 it has been used to denote the transform divided by 2π which is the inverse of the safety factor.)

'Safety Factor'

$$`q_s' = \frac{1}{\iota} = \frac{\text{toroidal angle}}{\text{poloidal angle}} \quad . \tag{2.74}$$

Actually the value of these ratios may vary as one moves around the magnetic field. Definition strictly requires one should take the limit of a large no. of rotations.

q_s is a topological number: number of rotations the long way per rotation the short way.

Cylindrical approx.:

$$q_s = \frac{rB_\phi}{RB_\theta} \tag{2.75}$$

In terms of safety factor the orbit shift can be written

$$|\Delta| = r_{L\theta}\ \frac{r}{R} = r_{L\phi}\frac{B_\phi r}{B_\theta R} = r_L q_s \tag{2.76}$$

(assuming $B_\phi >> B_\theta$).

2.6 The Mirror Effect of Parallel Field Gradients: $\mathbf{E} = 0$, $\nabla B \parallel \mathbf{B}$

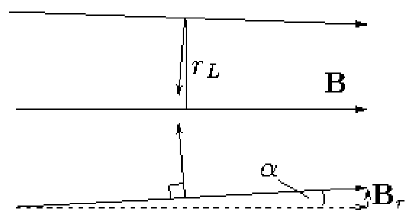

Figure 2.13: Basis of parallel mirror force

In the above situation there is a net force along **B**.
Force is

$$< F_\parallel > \; = \; -|q\mathbf{v} \wedge \mathbf{B}|\ \sin\alpha = \; -|q| v_\perp B \sin\alpha \tag{2.77}$$

$$\sin\alpha \; = \; \frac{-B_r}{B} \tag{2.78}$$

Calculate B_r as function of B_z from $\nabla.\mathbf{B} = 0$.

$$\nabla.\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}B_z = 0\ . \tag{2.79}$$

Hence

$$rB_r = -\int r\frac{\partial B_z}{\partial z}dr \tag{2.80}$$

Suppose r_L is small enough that $\frac{\partial B_z}{\partial z} \simeq$ const.

$$[rB_r]_0^{r_L} \simeq \int_0^{r_L} rdr\ \frac{\partial B_z}{\partial z} = -\frac{1}{2}\ r_L^2\frac{\partial B_z}{\partial z} \tag{2.81}$$

So

$$B_r(r_L) = -\frac{1}{2}r_L\frac{\partial B_z}{\partial z} \tag{2.82}$$

$$\sin\alpha = -\frac{B_r}{B} = +\frac{r_L}{2}\frac{1}{2}\frac{\partial B_z}{\partial z} \tag{2.83}$$

Hence

$$< F_\parallel >= -|q| \; \frac{v_\perp r_L}{2}\frac{\partial B_z}{\partial z} = -\frac{\frac{1}{2}mv_\perp^2}{B}\frac{\partial B_z}{\partial z}. \tag{2.84}$$

As particle enters increasing field region it experiences a net parallel *retarding* force.

Define *Magnetic Moment*

$$\mu \equiv \frac{1}{2}mv_\perp^2/B \; . \tag{2.85}$$

Note this is consistent with loop current definition

$$\mu = AI = \pi r_L^2 . \frac{|q|v_\perp}{2\pi r_L} = \frac{|q|r_L v_\perp}{2} \tag{2.86}$$

Force is $F_\parallel = \mu.\nabla_\parallel \mathbf{B}$

This is force on a 'magnetic dipole' of moment μ.

$$F_\parallel = \mu.\nabla_\parallel \mathbf{B} \tag{2.87}$$

Our μ always points along $\mathbf{B}$ but in opposite direction.

2.6.1 Force on an Elementary Magnetic Moment Circuit

Consider a plane rectangular circuit carrying current I having elementary area $dxdy = dA$. Regard this as a vector pointing in the $\mathbf{z}$ direction $\mathbf{dA}$. The force on this circuit in a field $\mathbf{B}(\mathbf{r})$ is $\mathbf{F}$ such that

$$\begin{aligned}
F_x &= Idy[B_z(x+dx) - B_z(x)] = Idydx\frac{\partial B_z}{\partial x} && (2.88)\\
F_y &= -Idx[B_z(y+dy) - B_z(y)] = Idydx\frac{\partial B_z}{\partial y} && (2.89)\\
F_z &= -Idx[B_y(y+dy) - B_y(y)] - Idy[B_x(x+dx) - B_x(x)] && (2.90)\\
&= -Idxdy\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right] = Idydx\frac{\partial B_z}{\partial z} && (2.91)
\end{aligned}$$

(Using $\nabla.\mathbf{B} = 0$).

Hence, summarizing: $\mathbf{F} = Idydx\nabla B_z$. Now define $\mu = I\mathbf{dA} = Idydx\hat{\mathbf{z}}$ and take it constant. Then clearly the force can be written

$$\mathbf{F} = \nabla(\mathbf{B}.\mu) \quad [\text{Strictly} = (\nabla\mathbf{B}).\mu] \tag{2.92}$$

μ is the (vector) magnetic moment of the circuit.

The shape of the circuit does not matter since any circuit can be considered to be composed of the sum of many rectangular circuits. So in general

$$\mu = I\mathbf{dA} \tag{2.93}$$

and force is

$$\mathbf{F} = \nabla(\mathbf{B}.\mu) \qquad (\mu \text{ constant}), \tag{2.94}$$

We shall show in a moment that $|\mu|$ is constant for a circulating particle, regard as an elementary circuit. Also, μ for a particle always points in the -**B** direction. [Note that this means that the effect of particles on the field is to *decrease* it.] Hence the force may be written

$$\mathbf{F} = -\mu \nabla B \tag{2.95}$$

This gives us both:

- *Magnetic Mirror Force:*

$$F_{\parallel} = -\mu \nabla_{\parallel} B \tag{2.96}$$

  and

- *Grad B Drift:*

$$\mathbf{v}_{\nabla B} = \frac{1}{q} \frac{\mathbf{F} \wedge \mathbf{B}}{B^2} = \frac{\mu}{q} \frac{\mathbf{B} \wedge \nabla B}{B^2}. \tag{2.97}$$

2.6.2 μ is a constant of the motion

'Adiabatic Invariant'

Proof from $F_{\parallel}$

Parallel equation of motion

$$m \frac{dv_{\parallel}}{dt} = F_{\parallel} = -\mu \frac{dB}{dz} \tag{2.98}$$

So

$$m v_{\parallel} \frac{dv_{\parallel}}{dt} = \frac{d}{dt}(\frac{1}{2} m v_{\parallel}^2) = -\mu v_z \frac{dB}{dz} = -\mu \frac{dB}{dt} \tag{2.99}$$

or

$$\frac{d}{dt}(\frac{1}{2} m v_{\parallel}^2) + \mu \frac{dB}{dt} = 0 \quad . \tag{2.100}$$

Conservation of Total KE

$$\frac{d}{dt}(\frac{1}{2} m v_{\parallel}^2 + \frac{1}{2} m v_{\perp}^2) = 0 \tag{2.101}$$

$$= \quad \frac{d}{dt}(\frac{1}{2} m v_{\parallel}^2 + \mu B) = 0 \tag{2.102}$$

Combine

$$\frac{d}{dt}(\mu B) - \mu \frac{dB}{dt} = 0 \tag{2.103}$$

$$= \frac{d\mu}{dt} = 0 \qquad \text{As required} \tag{2.104}$$

Angular Momentum

of particle about the guiding center is

$$r_L m v_\perp = \frac{m v_\perp}{|q| B} m v_\perp = \frac{2m}{|q|} \frac{\frac{1}{2} m v_\perp^2}{B} \quad (2.105)$$

$$= \frac{2m}{|q|} \mu \quad . \quad (2.106)$$

Conservation of magnetic moment is basically conservation of angular momentum about the guiding center.

Proof direct from Angular Momentum

Consider angular momentum about G.C. Because θ is ignorable (locally) Canonical angular momentum is conserved.

$$p = [\mathbf{r} \wedge (m\mathbf{v} + q\mathbf{A})]_z \qquad \text{conserved.} \quad (2.107)$$

Here $\mathbf{A}$ is the vector potential such that $\mathbf{B} = \nabla \wedge \mathbf{A}$

the definition of the vector potential means that

$$B_z = \frac{1}{r} \frac{\partial (r A_\theta))}{\partial r} \quad (2.108)$$

$$\Rightarrow r_L A_\theta(r_L) = \int_0^{r_L} r.B_z dr = \frac{r_L^2}{2} B_z = \frac{\mu m}{|q|} \quad (2.109)$$

Hence

$$p = \frac{-q}{|q|} r_L v_\perp m + q \frac{m\mu}{|q|} \quad (2.110)$$

$$= -\frac{q}{|q|} m\mu. \quad (2.111)$$

So p = const $\leftrightarrow$ μ = constant.

Conservation of μ is basically conservation of angular momentum of particle about G.C.

2.6.3 Mirror Trapping

$F_\parallel$ may be enough to reflect particles back. But may not!

Let's calculate whether it will:

Suppose reflection occurs.

At reflection point $v_{\parallel r} = 0$.

Energy conservation

$$\frac{1}{2} m (v_{\perp 0}^2 + v_{\parallel 0}^2) = \frac{1}{2} m v_{\perp r}^2 \quad (2.112)$$

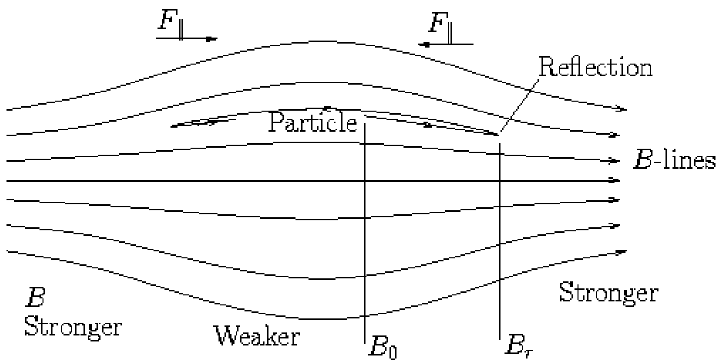

Figure 2.14: Magnetic Mirror

μ conservation

$$\frac{\frac{1}{2}mv_{\perp 0}^2}{B_0} = \frac{\frac{1}{2}mv_{\perp r}^2}{B_r} \tag{2.113}$$

Hence

$$v_{\perp 0}^2 + v_{\parallel 0}^2 = \frac{B_r}{B_0} v_{\perp 0}^2 \tag{2.114}$$

$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{\parallel o}^2} \tag{2.115}$$

2.6.4 Pitch Angle θ

$$\tan\theta = \frac{v_\perp}{v_\parallel} \tag{2.116}$$

$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{\parallel 0}^2} = \sin^2\theta_0 \tag{2.117}$$

So, given a pitch angle θ_0, reflection takes place where $B_0/B_r = \sin^2\theta_0$.
If θ_0 is too small no reflection can occur.
Critical angle θ_c is obviously

$$\theta_c = \sin^{-1}(B_0/B_1)^{\frac{1}{2}} \tag{2.118}$$

Loss Cone is all $\theta < \theta_c$.
Importance of Mirror Ratio: $R_m = B_1/B_0$.

2.6.5 Other Features of Mirror Motions

Flux enclosed by gyro orbit is constant.

$$\Phi = \pi r_L^2 B = \frac{\pi m^2 v_\perp^2}{q^2 B^2} B \tag{2.119}$$

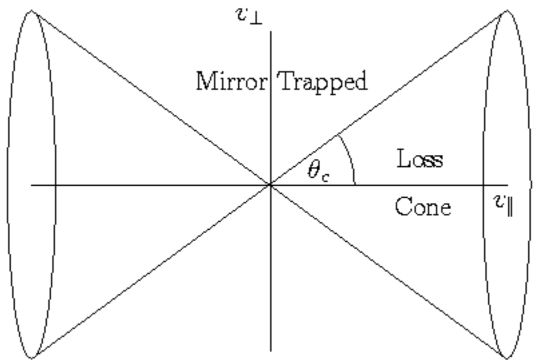

Figure 2.15: Critical angle θ_c divides velocity space into a loss-cone and a region of mirror-trapping

$$= \frac{2\pi m}{q^2}\frac{\frac{1}{2}mv_\perp^2}{B} \tag{2.120}$$

$$= \frac{2\pi m}{q^2}\mu = \text{constant.} \tag{2.121}$$

Note that if B changes 'suddenly' μ might not be conserved.

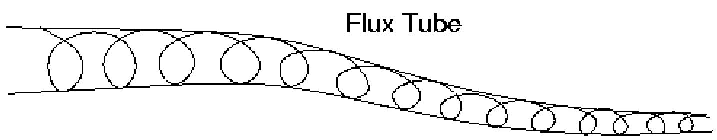

Figure 2.16: Flux tube described by orbit

Basic requirement

$$r_L << B/|\nabla B| \tag{2.122}$$

Slow variation of B (relative to r_L).

2.7 Time Varying B Field (E inductive)

Particle can gain energy from the inductive $\mathbf{E}$ field

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.123}$$

$$\text{or } \oint \mathbf{E}.\mathbf{dl} = -\int_s \dot{\mathbf{B}}.\mathbf{ds} = -\frac{d\Phi}{dt} \tag{2.124}$$

Hence work done on particle in 1 revolution is

$$\delta w = -\oint |q|\mathbf{E}.\mathbf{d}\ell = +|q|\int_s \dot{\mathbf{B}}.\mathbf{ds} = +|q|\frac{d\Phi}{dt} = |q|\dot{B}\pi r_L^2 \tag{2.125}$$

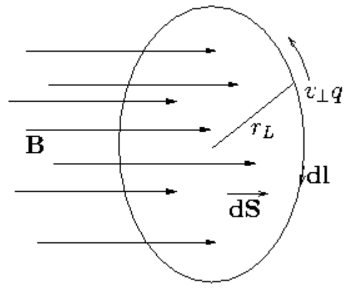

Figure 2.17: Particle orbits round **B** so as to perform a line integral of the Electric field

($\mathbf{d}\ell$ and $\mathbf{v}_\perp q$ are in opposition directions).

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = |q|\dot{B}\pi r_L^2 = \frac{2\pi\dot{B}m}{|q|B}\frac{\frac{1}{2}mv_\perp^2}{B} \tag{2.126}$$

$$= \frac{2\pi\dot{B}}{|\Omega|}\mu. \tag{2.127}$$

Hence

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{|\Omega|}{2\pi}\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu\frac{db}{dt} \tag{2.128}$$

but also

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}\left(\mu B\right)\ . \tag{2.129}$$

Hence

$$\frac{d\mu}{dt} = 0. \tag{2.130}$$

Notice that since $\Phi = \frac{2\pi m}{q^2}\mu$, this is just another way of saying that the flux through the gyro orbit is conserved.

Notice also *energy increase*. Method of 'heating'. Adiabatic Compression.

2.8 Time Varying E-field (E, B uniform)

Recall the $\mathbf{E}\wedge\mathbf{B}$ drift:

$$\mathbf{v}_{E\wedge B} = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \tag{2.131}$$

when E varies so does $\mathbf{v}_{E\wedge B}$. Thus the guiding centre experiences an acceleration

$$\dot{\mathbf{v}}_{E\wedge B} = \frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right) \tag{2.132}$$

In the frame of the guiding centre which is accelerating, a force is felt.

$$\mathbf{F}_a = -m\frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right) \qquad \text{(Pushed back into seat! } - \text{ve.)} \tag{2.133}$$

This force produces another drift

$$\mathbf{v}_D = \frac{1}{q}\frac{\mathbf{F}_a \wedge \mathbf{B}}{B^2} = \frac{m}{qB^2}\frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right)\wedge\mathbf{B} \tag{2.134}$$

$$= -\frac{m}{qB}\frac{d}{dt}\Big((\mathbf{E}.\mathbf{B})\,\mathbf{B} - B^2\mathbf{E}\Big) \tag{2.135}$$

$$= \frac{m}{qB^2}\dot{\mathbf{E}}_\perp \tag{2.136}$$

This is called the *'polarization drift'*.

$$\mathbf{v}_D = \mathbf{v}_{E\wedge B} \quad + \quad \mathbf{v}_p = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \quad + \frac{m}{qB^2}\;\dot{\mathbf{E}}_\perp \tag{2.137}$$

$$= \frac{E\wedge B}{B^2} \quad + \quad \frac{1}{\Omega B}\dot{\mathbf{E}}_\perp \tag{2.138}$$

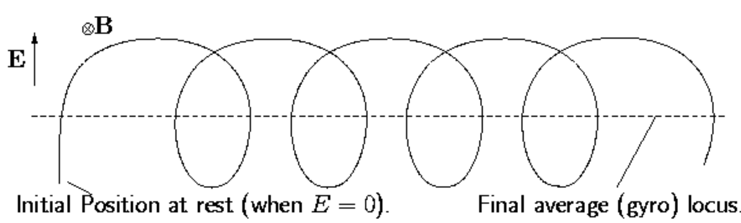

Figure 2.18: Suddenly turning on an electric field causes a shift of the gyrocenter in the direction of force. This is the polarization drift.

Start-up effect: When we 'switch on' an electric field the average position (gyro center) of an initially stationary particle shifts over by $\sim \frac{1}{2}$ the orbit size. The polarization drift is this polarization effect on the medium.
Total *shift* due to $\mathbf{v}_p$ is

$$\Delta\mathbf{r}\int\mathbf{v}_p dt = \frac{m}{qB^2}\int\hat{\mathbf{E}}_\perp dt = \frac{m}{qB^2}\left[\Delta\mathbf{E}_\perp\right] \quad . \tag{2.139}$$

2.8.1 Direct Derivation of $\frac{d\mathbf{E}}{dt}$ effect: 'Polarization Drift'

Consider an oscillatory field $\mathbf{E} = \mathbf{E}e^{-i\omega t}$ $(\perp r_0\mathbf{B})$

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right) \tag{2.140}$$

$$= q\left(\mathbf{E}e^{-i\omega t} + \mathbf{v}\wedge\mathbf{B}\right) \tag{2.141}$$

Try for a solution in the form

$$\mathbf{v} = \mathbf{v}_D e^{-i\omega t} + \mathbf{v}_L \tag{2.142}$$

where, as usual, $\mathbf{v}_L$ satisfies $m\dot{\mathbf{v}}_L = q\mathbf{v}_L \wedge \mathbf{B}$

Then

$$(1) \qquad m(-i\omega\mathbf{v}_D = q\left(\mathbf{E} + \mathbf{v}_D \wedge \mathbf{B}\right) \qquad x\ell^{-i\omega t} \tag{2.143}$$

Solve for $\mathbf{v}_D$: Take $\wedge\mathbf{B}$ this equation:

$$(2) \qquad -mi\omega\left(\mathbf{v}_D \wedge \mathbf{B}\right) = q\Big(\mathbf{E} \wedge \mathbf{B} + \Big(\mathbf{B}^2.\mathbf{v}|D\Big)\,\mathbf{B} - B^2\mathbf{v}_D\Big) \tag{2.144}$$

add $mi\omega \times (1)$ to $q \times (2)$ to eliminate $\mathbf{v}_D \wedge \mathbf{B}$.

$$m^2\omega^2\mathbf{v}_D + q^2(\mathbf{E} \wedge \mathbf{B} - B^2\mathbf{v}_D) = mi\omega q\mathbf{E} \tag{2.145}$$

$$\text{or :} \qquad \mathbf{v}_D\left[1 - \frac{m^2\omega^2}{q^2B^2}\right] = -\frac{mi\omega}{qB^2}\mathbf{E} + \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.146}$$

$$\text{i.e.} \qquad \mathbf{v}_D\left[1 - \frac{\omega^2}{\Omega^2}\right] = -\frac{i\omega q}{\Omega B|q|}\mathbf{E} + \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.147}$$

Since $-i\omega \leftrightarrow \frac{\partial}{\partial t}$ this is the same formula as we had before: the sum of polarization and $\mathbf{E} \wedge \mathbf{B}$ drifts *except* for the $[1 - \omega^2\Omega^2]$ term.

This term comes from the change in $\mathbf{v}_D$ with time (accel).

Thus our earlier expression was only approximate. A good approx if $\omega << \Omega$.

2.9 Non Uniform E (Finite Larmor Radius)

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E}(\mathbf{r}) + \mathbf{v} \wedge \mathbf{B}\right) \tag{2.148}$$

Seek the usual soltuion $\mathbf{v} = \mathbf{v}_D + \mathbf{v}_g$.

Then average out over a gyro orbit

$$\langle m\frac{dv_D}{dt}\rangle = 0 = \langle q\left(\mathbf{E}(\mathbf{r}) + \mathbf{v} \wedge \mathbf{B}\right)\rangle \tag{2.149}$$

$$= q\left[\langle\mathbf{E}(\mathbf{r})\rangle + \mathbf{v}_D \wedge \mathbf{B}\right] \tag{2.150}$$

Hence drift is obviously

$$\mathbf{v}_D = \frac{\langle\mathbf{E}(\mathbf{r})\rangle \wedge \mathbf{B}}{B^2} \tag{2.151}$$

So we just need to find the *average* E field experienced.

Expand $\mathbf{E}$ as a Taylor series about the G.C.

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 + (\mathbf{r}.\nabla)\,\mathbf{E} + \left(\frac{x^2\partial^2}{2!\partial x^2} + \frac{y^2}{2!}\frac{\partial^2}{\partial y^2}\right)\mathbf{E} + \text{cross terms } + . \tag{2.152}$$

(E.g. cross terms are $xy\frac{\partial^2}{\partial x \partial y}\mathbf{E}$).

Average over a gyro orbit: $\mathbf{r} = r_L(\cos\theta, \sin\theta, 0)$.

Average of cross terms = 0.

Then

$$\langle \mathbf{E}(\mathbf{r}) \rangle = \mathbf{E} + (\langle \mathbf{r}_L \rangle . \nabla)\mathbf{E} + \frac{\langle r_L^2 \rangle}{2!} \nabla^2 \mathbf{E}. \tag{2.153}$$

linear term $\langle r_L \rangle = 0$. So

$$\langle \mathbf{E}(\mathbf{r}) \rangle \simeq \mathbf{E} + \frac{r_L^2}{4} \nabla^2 E \tag{2.154}$$

Hence $\mathbf{E} \wedge \mathbf{B}$ with 1st finite-Larmor-radius correction is

$$\mathbf{v}_{E\wedge B} = \left(1 + \frac{r_L^2}{r}\nabla^2\right)\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}. \tag{2.155}$$

[Note: Grad B drift is a finite Larmor effect already.]

Second and Third Adiabatic Invariants

There are additional approximately conserved quantities like μ in some geometries.

2.10 Summary of Drifts

$$\mathbf{v}_E = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \qquad \text{Electric Field} \tag{2.156}$$

$$\mathbf{v}_F = \frac{1}{q}\frac{\mathbf{F}\wedge\mathbf{B}}{B^2} \qquad \text{General Force} \tag{2.157}$$

$$\mathbf{v}_E = \left(1 + \frac{r_L^2}{4}\nabla^2\right)\frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \qquad \text{Nonuniform E} \tag{2.158}$$

$$\mathbf{v}_{\nabla B} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B}\wedge\nabla B}{B^3} \qquad \text{GradB} \tag{2.159}$$

$$\mathbf{v}_R = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2 B^2} \qquad \text{Curvature} \tag{2.160}$$

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2 B^2} \qquad \text{Vacuum Fields.} \tag{2.161}$$

$$\mathbf{v}_p = \frac{q}{|q|}\frac{\dot{E}_\perp}{|\Omega| B} \qquad \text{Polarization} \tag{2.162}$$

Mirror Motion

$$\mu \equiv \frac{mv_\perp^2}{2B} \qquad \text{is constant} \tag{2.163}$$

Force is $\mathbf{F} = -\mu\nabla B$.