Quantization of the electromagnetic field

The classical electromagnetic field

Maxwell Equations

Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$
Maxwell-Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$
Ampere's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

Maxwell Equations

- In empty space $\quad (c = 1/\sqrt{\mu_0 \varepsilon_0})$

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Maxwell-Faraday equation	$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$
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Wave Equations

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

$$\nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0$$

Derivation of wave equations

- Curl of Maxwell Faraday equation

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c}\frac{\partial \nabla \times \mathbf{B}}{\partial t}$$

- Use Ampere's Law
  and vector identity $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right)$$

Derivation of wave equations

- Use Gauss Law

$$\nabla(\cancel{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right)$$

- Obtain wave equation

$$-\nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t}$$

Wave equation

$$\nabla^2 E \quad \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

- Eigenvalue equation from separation of variables: $\mathbf{E}(\vec{x},t) = \sum_m f_m(t)\vec{u}_m(\vec{x})$

$$\nabla^2 u_m = -k_m^2 u_m \qquad \frac{d^2 f_m}{dt^2} + c^2 k_m^2 f_m(t) = 0$$

Normal modes

- $\{u_m\}$ are eigenfunctions of the wave equation
- Boundary conditions (from Maxwell eqs.)

$$\nabla \cdot u_m = 0, \quad \vec{n} \times u_m = 0$$

- Orthonormality condition

$$\int \vec{u}_m(x)\vec{u}_n(x)d^3x = \delta_{n,m}$$

- They form a basis.

B-field

- Electric field in $\{u_m\}$ basis:

$$\mathbf{E}(\vec{x},t) = \sum_m f_m(t)\vec{u}_m(\vec{x})$$

- Magnetic field in $\{u_m\}$ basis

$$\mathbf{B}(x,t) = \sum_m h_m(t)\left(\nabla \times u_m(x)\right)$$

B-field solution

- What are the coefficients h_n?
  - We still need to satisfy Maxwell equations:

$$\nabla \times E = -\frac{1}{c}\partial_t B \quad \rightarrow$$

$$\sum_n f_n(t)\nabla \times u_n = -\frac{1}{c}\sum_n \partial_t h_n(t)\nabla \times u_n$$

  - Solution: $\frac{d\,h_n}{d\,t} = -c f_n$

Eigenvalues of h_n

- Find equation for h_n only: Ampere's law

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\sum_n h_n(t)\nabla \times (\nabla \times u_n) = \frac{1}{c}\sum_n \frac{d\,f_n}{d\,t} u_n$$

$$\rightarrow \quad -\sum_n h_n \nabla^2 u_n = \frac{1}{c}\sum_n \frac{d\,f_n}{d\,t} u_n$$

Eigen-equations

- Eigenvalue equation for h_n

$$\frac{d^2}{dt^2}h_n(t) + c^2k_n^2h_n(t) = 0$$

- Eigenvalue equation for f_n

$$\frac{d^2f_n}{dt^2} + c^2k_n^2f_n(t) = 0$$

E.M. field Hamiltonian

- Total energy:

$$\mathcal{H} \propto \frac{1}{2}\int (E^2 + B^2) d^3x$$

- Substituting, integrating and using orthonormality conditions:

$$\mathcal{H} = \frac{1}{8\pi}\sum_{n,m}\left(f_n f_m \int u_n(x) u_m(x) d^3x + h_n h_m \int (\nabla \times u_n)\cdot(\nabla \times u_m) d^3x \right)$$

$$\mathcal{H} = \sum_n \frac{1}{8\pi}(f_n^2 + k_n^2 h_n^2)$$

E.M. field as H.O.

- Hamiltonian looks very similar to a sum of harmonic oscillators:

$$\mathcal{H}_{h.o.} = \frac{1}{2}\sum_{\mathrm{n}}(\mathrm{p}_{\mathrm{n}}^2 + \omega_{\mathrm{n}}^2 \mathrm{q}_{\mathrm{n}}^2) \;\Leftrightarrow\; \mathcal{H}_{\mathrm{e.m.}} = \frac{1}{2}\sum_{\mathrm{n}}\frac{1}{4\pi}(\mathrm{f}_{\mathrm{n}}^2 + \mathrm{k}_{\mathrm{n}}^2 \mathrm{h}_{\mathrm{n}}^2)$$

- h_n is derivative of f_n

  $\Longrightarrow$ identify with momentum

Quantized electromagnetic field

Operators

- We associate quantum operators to the coefficients $f_n, \quad f_n \rightarrow \hat{f}_n$
- We write this operator in terms of annihilation and creation operators

$$\hat{f}_n = \sqrt{2\pi\omega_n\hbar}(a_n^\dagger + a_n)$$

that create or destroy one mode of the e.m. field

Operator fields

- Electric field

$$\mathbf{E}(x,t) = \sum_n \sqrt{2\hbar\pi\omega_n}[a_n^\dagger(t) + a_n(t)]\mathbf{u}_n(x)$$

- Magnetic field

$$\mathbf{B}(x,t) = \sum_n ic_n\sqrt{\frac{2\pi\hbar}{\omega_n}}[a_n^\dagger - a_n]\nabla \times \mathbf{u}_n(x)$$

Hamiltonian

- The Hamiltonian is then simply expressed in terms of the a_n operators

$$\mathcal{H} = \sum_n \omega_n \left(a_n^\dagger a_n + \frac{1}{2} \right)$$

- The frequencies are

$$\omega_n(k) = c|\vec{k}_n|$$

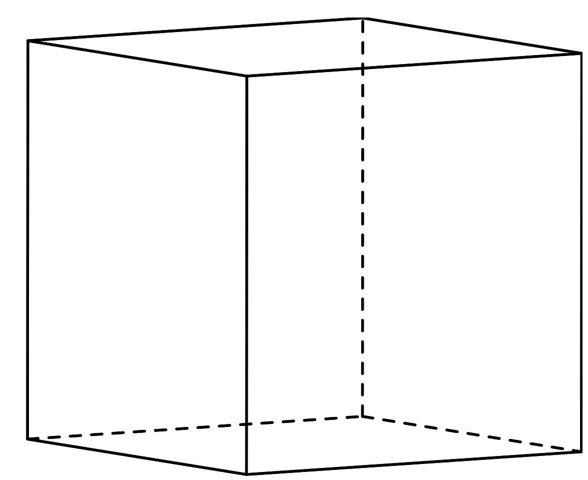
Gauges

Lorentz (scalar potential $\varphi = 0$)
Coulomb (vector potential $\nabla \cdot \vec{A} = 0$)

Zero-Point Energy

Field in cavity

- Field in a cavity of volume $V = L_x L_y L_z$



- Given the boundary conditions,
  the normal modes are:

$$u_{n,\alpha} = A_\alpha \cos(k_{n,x} r_x) \sin(k_{n,y} r_y) \sin(k_{n,z} r_z)$$

- with $k_{n,\alpha} = \frac{n_\alpha \pi}{L_\alpha}, \quad n_\alpha \in \mathcal{N}$

Polarization

- Because of the boundary condition,

$$\nabla \cdot \vec{u}_n = 0$$

- the coefficients A must satisfy:

$$A_x k_{n,x} + A_y k_{n,y} + A_z k_{n,z} = 0$$

- For each set $\{n_x, n_y, n_z\}$ there are 2 solutions

Two polarizations per each mode

Electric field in cavity

- The electric field has a simple form

$$E(x,t) = \sum_{\alpha=1,2} (\mathcal{E}_\alpha + \mathcal{E}_\alpha^\dagger)$$

- with $\mathcal{E}_\alpha^\dagger = \hat{e}_\alpha \sum_n \mathcal{E}_n a_n^\dagger e^{i(\vec{k}_n \cdot \vec{r} - \omega t)}$

- and $\mathcal{E}_n = \sqrt{\frac{\hbar\omega_n}{2\epsilon_0 V}}$ the field of one photon of frequency ω_n

Energy density

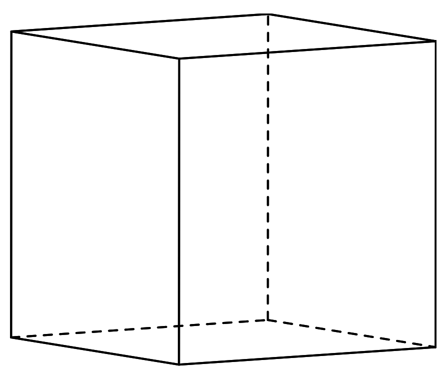
$$E = \langle \mathcal{H} \rangle = 2 \sum_{k=1}^{k_c} \hbar \omega_k \left\langle a_k^\dagger a_k + \frac{1}{2} \right\rangle$$

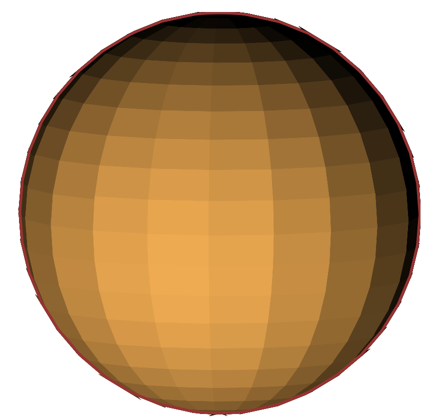
- The Zero-point energy density is then

$$E_0 = \frac{2}{V} \sum_{k=1}^{k_c} \frac{1}{2} \hbar \omega_k$$

Energy density

- If cavity is large, wavevector is almost continuous





$$\sum_{k>0} \longrightarrow \frac{1}{8}\int d^3k\,\rho(k)$$

Zero-point energy

- Integrating over the positive octant

$$E_0 = \frac{2}{V}\frac{2V}{\pi^3}\frac{4\pi}{8}\int_{k=0}^{k_c} dk\frac{1}{2}\hbar k^3 c$$

- setting a cutoff k_c,we have

$$E_0 = \frac{c\hbar}{2\pi^2}\int_{k=0}^{k_c} dk k^3 = \frac{\hbar c k_c^4}{8\pi^2}$$

Zero-point energy

- It's huge!

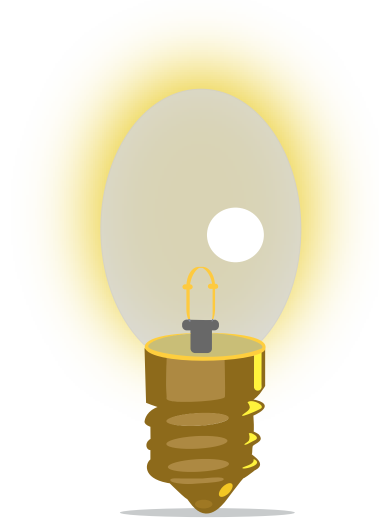


Image by MIT OpenCourseWare.

Cutoff at visible frequency

$$\lambda_c = 2\pi/k = 0.4 \times 10^{-6} m$$

$$2.7 \times 10^{-8} \mathrm{J/m^3} \ @ \ 1\mathrm{m}$$

$$23 \ \mathrm{J/m^3}$$

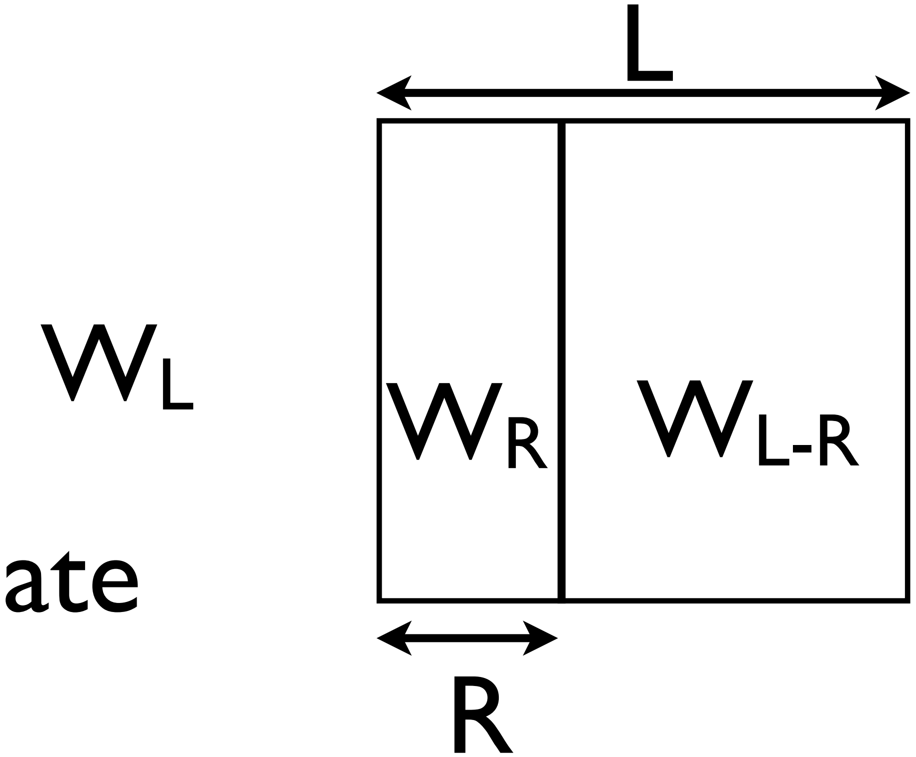
- But is it ever seen?

Casimir Effect

- Dutch theoretical physicist Hendrik Casimir (1909–2000) first predicted in 1948 that when two mirrors face each other in vacuum, fluctuations in the vacuum exert "radiation pressure" on them

Casimir Effect

- Cavity bounded by conductive walls



- Add a conductive plate @ distance R
- Change in energy is:

$$\Delta W = (W_R + W_{L-R}) - W_L$$

Casimir effect

- Each term is calculated from zero-point energy
- Continuous approximation is not valid if R is small
- Thus the difference ΔW is not zero

$$\Delta W = -\hbar c \frac{\pi^2}{720} \frac{L^2}{R^3}$$

Casimir Force

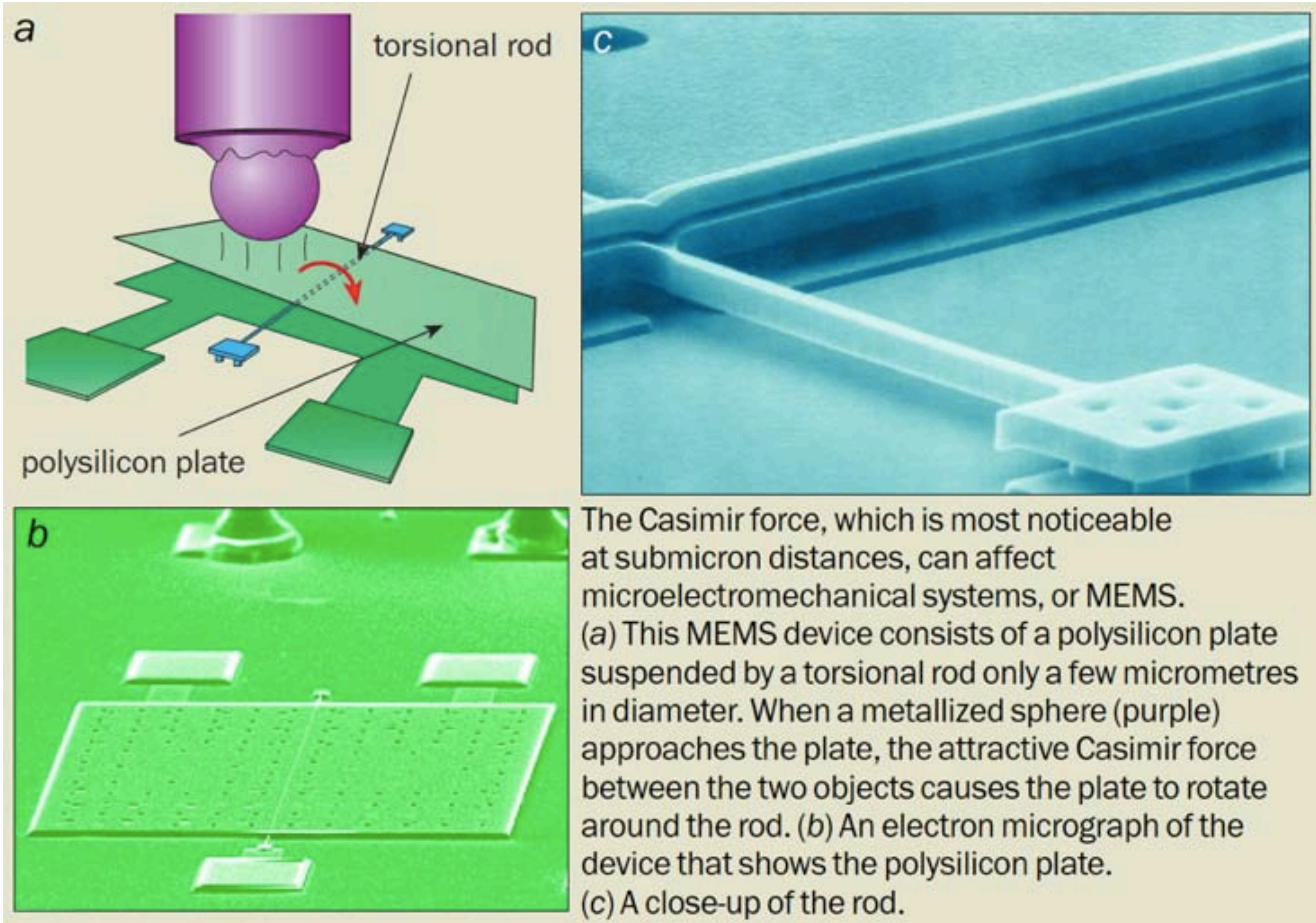
- The difference in energy corresponds to an attractive force

$$F = -\frac{\partial \Delta W}{\partial R} = -\hbar c \frac{\pi^2}{240} \frac{L^2}{R^4}$$

- or a pressure

$$P = -\frac{\pi^2}{240} \frac{\hbar c}{R^4}$$

Casimir in MEMS



The Casimir force, which is most noticeable at submicron distances, can affect microelectromechanical systems, or MEMS. (*a*) This MEMS device consists of a polysilicon plate suspended by a torsional rod only a few micrometres in diameter. When a metallized sphere (purple) approaches the plate, the attractive Casimir force between the two objects causes the plate to rotate around the rod. (*b*) An electron micrograph of the device that shows the polysilicon plate. (*c*) A close-up of the rod.

Quantum Mechanical Actuation of Microelectromechanical Systems by the Casimir Force

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