

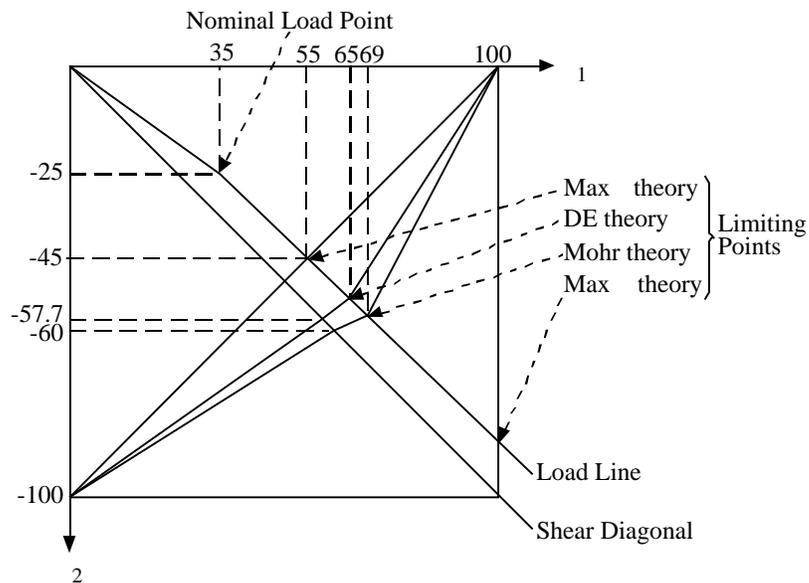
DEPARTMENT OF NUCLEAR ENGINEERING

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

NOTE L.4

“INTRODUCTION TO STRUCTURAL MECHANICS”

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L. Wolf

Revised by

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“INTRODUCTION TO STRUCTURAL MECHANICS”

M. S. Kazimi, N.E. Todreas and L. Wolf

1. DEFINITION OF CONCEPTS

Structural mechanics is the body of knowledge describing the relations between external forces, internal forces and deformation of structural materials. It is therefore necessary to clarify the various terms that are commonly used to describe these quantities. In large part, structural mechanics refers to solid mechanics because a solid is the only form of matter that can sustain loads parallel to the surface. However, some considerations of fluid-like behavior (creep) are also part of structural mechanics.

Forces are vector quantities, thus having direction and magnitude. They have special names (Fig. 1) depending upon their relationship to a reference plane:

- a) Compressive forces act normal and into the plane;
- b) Tensile forces act normal and out of the plane; and
- c) Shear forces act parallel to the plane.

Pairs of oppositely directed forces produce twisting effects called moments.

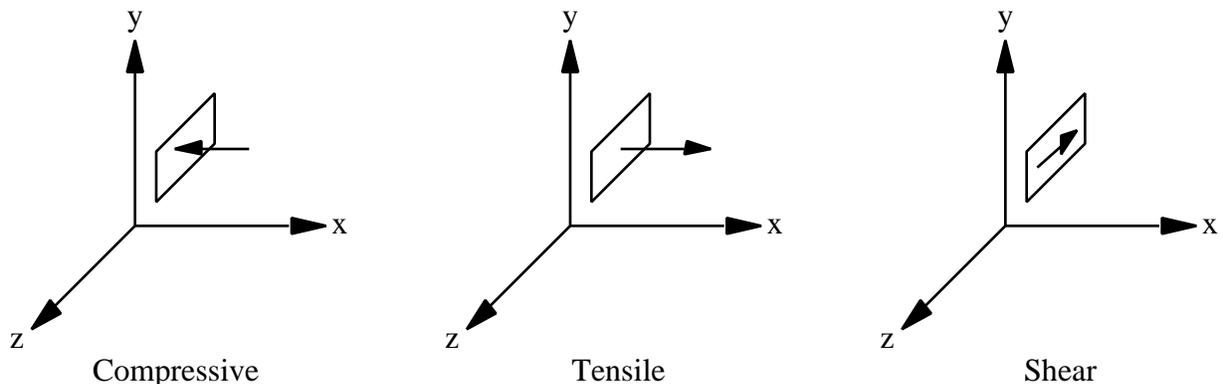


Figure 1. Definition of Forces.

The mathematics of stress analysis requires the definition of coordinate systems. Fig. 2 illustrates a right-handed system of rectangular coordinates.

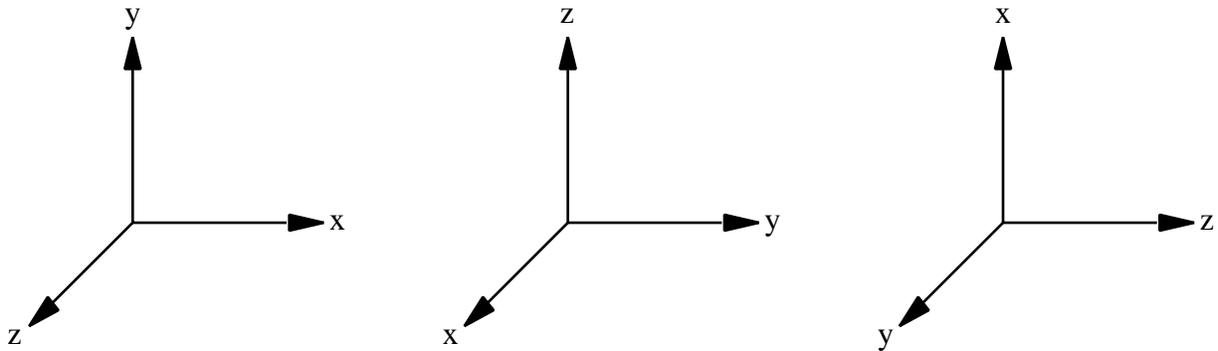


Figure 2. Right-handed System of Rectangular Coordinates.

In the general case, a body as shown in Fig. 3 consisting of an isolated group of particles will be acted upon by both external or surface forces, and internal or body forces (gravity, centrifugal, magnetic attractions, etc.)

If the surface and body forces are in balance, the body is in static equilibrium. If not, accelerations will be present, giving rise to inertia forces. By D'Alembert's principle, the resultant of these inertial forces is such that when added to the original system, the equation of equilibrium is satisfied.

The system of particles of Fig. 3 is said to be in equilibrium if every one of its constitutive particles is in equilibrium. Consequently, the resulting force on each particle is zero, and hence the vector sum of all the forces shown in Fig. 3 is zero. Finally, since we observe that the internal forces occur in self-canceling pairs, the first necessary condition for equilibrium becomes that the vector sum of the external forces must be zero.

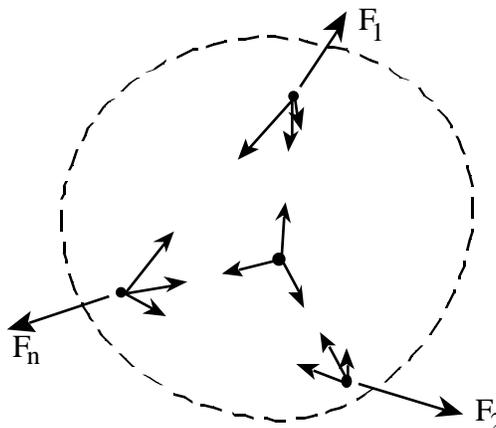


Figure 3. An Isolated System of Particles Showing External and Internal Forces (Ref. 1, Fig 1.12).

$$\vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = \sum_n \vec{F}_n = 0 \quad (1.1a)$$

The total moment of all the forces in Fig. 3 about an arbitrary point 0 must be zero since the vector sum of forces acting on each particle is zero. Again, observe that internal forces occur in self-canceling pairs along the same line of action. This leads to the second condition for equilibrium: the total moment of all the external forces about an arbitrary point 0 must be zero.

$$\vec{r}_1 \vec{F}_1 + \vec{r}_2 \vec{F}_2 + \dots + \vec{r}_n \vec{F}_n = \sum_n \vec{r}_n \vec{F}_n = 0 \quad (1.1b)$$

where \vec{r}_n extends from point 0 to an arbitrary point on the line of action of force \vec{F}_n .

In Fig. 4A, an arbitrary plane, aa, divides a body in equilibrium into regions I and II. Since the force acting upon the entire body is in equilibrium, the forces acting on part I alone must be in equilibrium.

In general, the equilibrium of part I will require the presence of forces acting on plane aa. These internal forces applied *to* part I *by* part II are distributed continuously over the cut surface, but, in general, will vary over the surface in both direction and intensity.

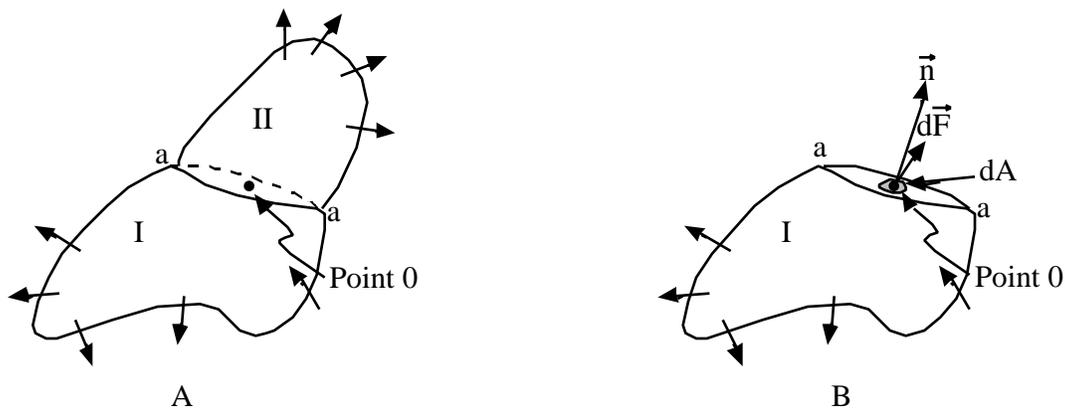


Figure 4. Examination of Internal Forces of a Body in Equilibrium.

Stress is the term used to define the intensity and direction of the internal forces acting at a particular point on a given plane.

Metals are composed of grains of material having directional and boundary characteristics. However, these grains are usually microscopic and when a larger portion of the material is considered, these random variations average out to produce a macroscopically uniform material.

Macroscopic uniformity = homogenous.

If there are no macroscopic direction properties the material is isotropic.

Definition of Stress (mathematically), (Fig. 4B) [see Ref. 1, p. 203]

$$\begin{aligned} \vec{T} &= \text{stress at point } 0 \text{ on plane } aa \text{ whose normal is } \vec{n} \text{ passing through point } 0 \\ &= \lim_{dA \rightarrow 0} \frac{d\vec{F}}{dA} \text{ where } d\vec{F} \text{ is a force acting on area } dA. \end{aligned}$$

[Reference 1 uses the notation \vec{T} to introduce the concept that \vec{T} is a stress vector]

NOTE: Stress is a point value.

The stress acting at a point on a specific plane is a vector. Its direction is the limiting direction of force $d\vec{F}$ as area dA approaches zero. It is customary to resolve the stress vector into two components whose scalar magnitudes are:

normal stress component : acting perpendicular to the plane
shear stress component : acting in the plane.

1.1 Concept of State of Stress

The selection of different cutting planes through point 0 would, in general, result in stresses differing in both direction and magnitude. Stress is thus a second-order tensor quantity, because not only are magnitude and direction involved but also the orientation of the plane on which the stress acts is involved.

NOTE: A complete description of the magnitudes and directions of stresses on all possible planes through point 0 constitutes the state of stress at point 0.

NOTE: A knowledge of maximum stresses alone is not always sufficient to provide the best evaluation of the strength of a member. The orientation of these stresses is also important.

In general, the overall stress state must be determined first and the maximum stress values derived from this information. The state of stress at a point can normally be determined by computing the stresses acting on certain conveniently oriented planes passing through the point of interest. Stresses acting on any other planes can then be determined by means of simple, standardized analytical or graphical methods. Therefore, it is convenient to consider the three mutually perpendicular planes as faces of a cube of infinitesimal size which surround the point at which the stress state is to be determined.

Figure 5 illustrates the general state of 3D stress at an arbitrary point by illustrating the stress components on the faces of an infinitesimal cubic element around the point.

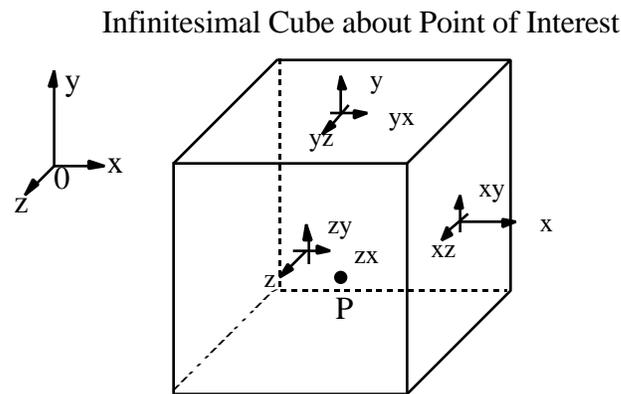


Figure 5. Stress Element Showing General State of 3D Stress at a Point Located Away from the Origin.

Notation Convention for Fig. 5:

Normal stresses are designated by a single subscript corresponding to the outward drawn normal to the plane that it acts upon.

The rationale behind the double-subscript notation for shear stresses is that the first designates the plane of the face and the second the direction of the stress. The plane of the face is represented by the axis which is normal to it, instead of the two perpendicular axes lying in the plane.

Stress components are positive when a positively-directed force component acts on a positive face or a negatively-directed force component acts on a negative face. When a positively-directed force component acts on a negative face or a negatively-directed force component acts on a positive face, the resulting stress component will be negative. A face is positive when its outwardly-directed normal vector points in the direction of the positive coordinate axis (Ref. 1, pp. 206-207). All stresses shown in Fig. 5 are positive. Normal stresses are positive for tensile stress and negative for compressive stress. Figure 6 illustrates positive and negative shear stresses.

NOTE: y_x equals x_y .

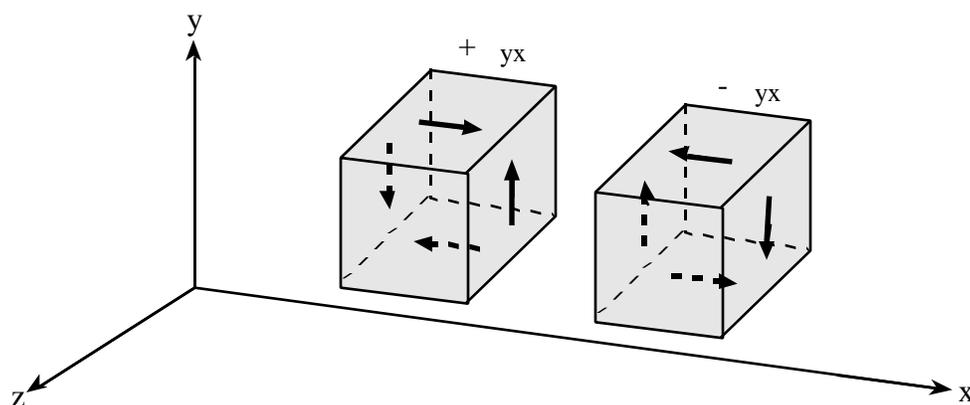


Figure 6. Definition of Positive and Negative y_x .

Writing the state of stress as tensor S:

$$S = \begin{pmatrix} x & xy & xz \\ yx & y & yz \\ zx & zy & z \end{pmatrix} \quad \text{9-components} \quad (1.2)$$

However, we have three equal pairs of shear stress:

$$xy = yx, \quad xz = zx, \quad yz = zy \quad (1.3)$$

Therefore, six quantities are sufficient to describe the stresses acting on the coordinate planes through a point, i.e., the triaxial state of stress at a point. If these six stresses are known at a point, it is possible to compute from simple equilibrium concepts the stresses on any plane passing through the point [Ref. 2, p. 79].

1.2 Principal Stresses, Planes and Directions

The tensor S becomes a symmetric tensor if Eq. 1.3 is introduced into Eq. 1.2. A fundamental property of a symmetrical tensor (symmetrical about its principal diagonal) is that there exists an orthogonal set of axes 1, 2, 3 (called principal axes) with respect to which the tensor elements are all zero except for those on the principal diagonal:

$$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (1.4)$$

Hence, when the tensor represents the state of stress at a point, there always exists a set of mutually perpendicular planes on which only normal stress acts. These planes of zero shear stress are called principal planes, the directions of their outer normals are called principal directions, and the stresses acting on these planes are called principal stresses. An element whose faces are principal planes is called a principal element.

For the general case, the principal axes are usually numbered so that:

$$1 \quad 2 \quad 3$$

1.3 Basic Considerations of Strain

The concept of strain is of fundamental importance to the engineer with respect to the consideration of deflections and deformation.

A component may prove unsatisfactory in service as a result of excessive deformations, although the associated stresses are well within the allowable limits from the standpoint of fracture or yielding.

Strain is a directly measurable quantity, stress is not.

Concept of Strain and State of Strain

Any physical body subjected to forces, i.e., stresses, deforms under the action of these forces.

Strain is the direction and intensity of the deformation at any given point with respect to a specific plane passing through that point. Strain is therefore a quantity analogous to stress.

State of strain is a complete definition of the magnitude and direction of the deformation at a given point with respect to all planes passing through the point. Thus, state of strain is a tensor and is analogous to state of stress.

For convenience, strains are always resolved into normal components, ϵ_x and ϵ_y , and shear components, γ_{xy} (Figs. 7 & 8). In these figures the original shape of the body is denoted by solid lines and the deformed shape by the dashed lines. The change in length in the x-direction is dx , while the change in the y-direction is dy . Hence, ϵ_x , ϵ_y and γ_{xy} are written as indicated in these figures.

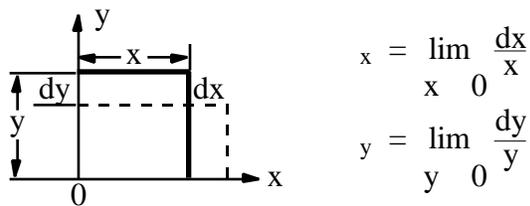


Figure 7. Deformation of a Body where the x-Dimension is Extended and the y-Dimension is Contracted.

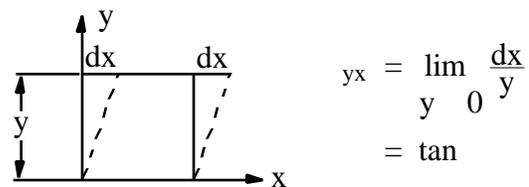


Figure 8. Plane Shear Strain.

Subscript notation for strains corresponds to that used with stresses. Specifically,

- γ_{yx} : shear strain resulting from taking adjacent planes perpendicular to the y-axis and displacing them relative to each other in the x-direction (Fig. 9).
- ϵ_x, ϵ_y : normal strains in x- and y-directions, respectively.

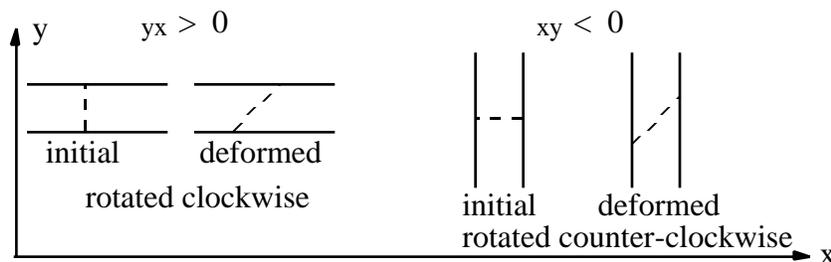


Figure 9. Strains resulting from Shear Stresses $\pm \gamma_{xy}$.

Sign conventions for strain also follow directly from those for stress: positive normal stress produces positive normal strain and vice versa. In the above example (Fig. 7), $\epsilon_x > 0$, whereas $\epsilon_y < 0$. Adopting the positive clockwise convention for shear components, $\gamma_{xy} < 0$, $\gamma_{yx} > 0$. In Fig. 8, the shear is γ_{yx} and the rotation is clockwise.

NOTE: Half of γ_{xy} , γ_{xz} , γ_{yz} is analogous to ϵ_x , ϵ_z and ϵ_y , whereas ϵ_x is analogous to ϵ_x .

$$T = \begin{pmatrix} \epsilon_x & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\ \frac{1}{2} \gamma_{yx} & \epsilon_y & \frac{1}{2} \gamma_{yz} \\ \frac{1}{2} \gamma_{zx} & \frac{1}{2} \gamma_{zy} & \epsilon_z \end{pmatrix}$$

It may be helpful in appreciating the physical significance of the fact that ϵ_x is analogous to $\epsilon_x/2$ rather than with ϵ_x itself to consider Fig. 10. Here it is seen that each side of an element changes in slope by an angle $\epsilon_x/2$.

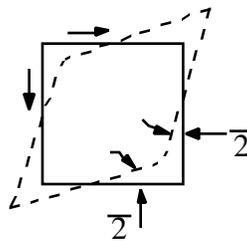


Figure 10. State of Pure Shear Strain

1.4 Plane Stress

In Fig. 11, all stresses on a stress element act on only two pairs of faces. Further, these stresses are practically constant along the z-axis. This two-dimensional case is called biaxial stress or plane stress.

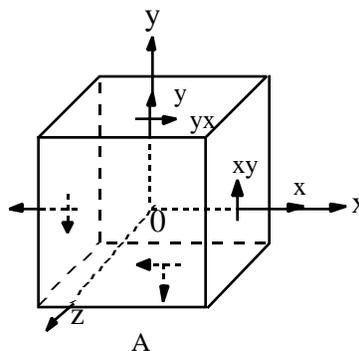


Figure 11. State of Plane Stress

The stress tensor for stresses in horizontal (x) and vertical (y) directions in the plane xy is:

$$S = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix}$$

When the angle of the cutting plane through point 0 is varied from horizontal to vertical, the shear stresses acting on the cutting plane vary from positive to negative. Since shear stress is a continuous function of the cutting plane angle, there must be some intermediate plane in which the shear stress is zero. The normal stresses in the intermediate plane are the principle stresses. The stress tensor is:

$$S' = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

where the zero principal stress is always σ_3 . These principal stresses occur along particular axes in a particular plane. By convention, σ_1 is taken as the larger of the two principal stresses. Hence

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 = 0$$

In general we wish to know the stress components in a set of orthogonal axes in the xy plane, but rotated at an arbitrary angle, θ , with respect to the x, y axes. We wish to express these stress components in terms of the stress terms referred to the x, y axes, i.e., in terms of σ_x , σ_y , τ_{xy} and the angle of rotation θ . Fig. 12 indicates this new set of axes, x_1 and y_1 , rotated by θ from the original set of axes x and y. The determination of the new stress components σ_{x_1} , σ_{y_1} , $\tau_{x_1y_1}$ is accomplished by considering the equilibrium of the small wedge centered on the point of interest whose faces are along the x-axis, y-axis and perpendicular to the x_1 axis. For the general case of expressing stresses σ_{x_1} and $\tau_{x_1y_1}$ at point 0 in plane x_1, y_1 in terms of stresses σ_x , σ_y , τ_{xy} and the angle θ in plane x, y by force balances in the x_1 and y_1 directions, we obtain (see Ref. 1, p. 217):

$$\sigma_{x_1} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta \quad (1.5a)$$

$$\tau_{x_1y_1} = (\sigma_y - \sigma_x) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \quad (1.5b)$$

$$\sigma_{y_1} = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2 \tau_{xy} \sin \theta \cos \theta \quad (1.5c)$$

where the angle θ is defined in Fig. 12.

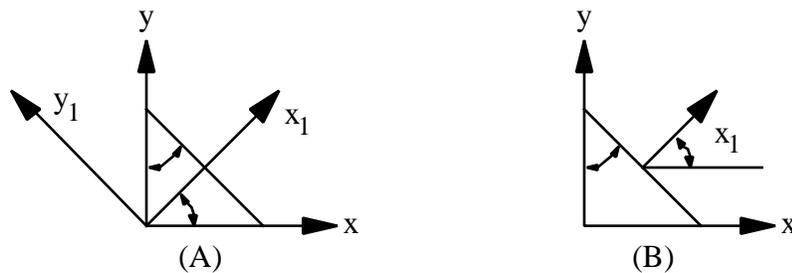


Figure 12. (A) Orientation of Plane x_1y_1 at Angle θ to Plane xy;
(B) Wedge for Equilibrium Analysis.

These are the transformation equations of stress for plane stress. Their significance is that stress components σ_{x_1} , σ_{y_1} and $\tau_{x_1y_1}$ at point 0 in a plane at an arbitrary angle θ to the plane xy are uniquely determined by the stress components σ_x , σ_y and τ_{xy} at point 0.

Eq. 1.5 is commonly written in terms of 2θ . Using the identities:

$$\sin^2 \theta = \frac{(1 - \cos 2\theta)}{2}; \quad \cos^2 \theta = \frac{(1 + \cos 2\theta)}{2}; \quad 2 \sin \theta \cos \theta = \sin 2\theta$$

we get

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad (1.6a)$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad (1.6b)$$

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad (1.6c)^*$$

The orientation of the principal planes, in this two dimensional system is found by equating $\tau_{x_1y_1}$ to zero and solving for the angle θ .

1.5 Mohr's Circle

Equations 1.6a,b,c, taken together, can be represented by a circle, called Mohr's circle of stress. To illustrate the representation of these relations, eliminate the function of the angle 2θ from Eq. 1.6a and 1.6b by squaring both sides of both equations and adding them. Before Eq. 1.6a is squared, the term $(\sigma_x + \sigma_y)/2$ is transposed to the left side. The overall result is (where plane y_1x_1 is an arbitrarily located plane, so that σ_{x_1} is now written as σ_1 and $\tau_{x_1y_1}$ as τ):

$$[\sigma_1 - 1/2(\sigma_x + \sigma_y)]^2 + \tau^2 = 1/4(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 \quad (1.7)$$

Now construct Mohr's circle on the σ and τ plane. The principal stresses are those on the σ -axis when $\tau = 0$. Hence, Eq. 1.7 yields the principal normal stresses σ_1 and σ_2 as:

$$\sigma_{1,2} = \underbrace{\frac{\sigma_x + \sigma_y}{2}}_{\text{center of circle}} \pm \underbrace{\sqrt{\frac{\tau_{xy}^2}{4} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}_{\text{radius}} \quad (1.8)$$

The maximum shear stress is the radius of the circle and occurs in the plane represented by a vertical orientation of the radius.

* σ_{y_1} can also be obtained from σ_{x_1} by substituting $\theta + 90^\circ$ for θ .

$$\sigma_{\max} = \pm \sqrt{\frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad (1.9)$$

The angle between the principal axes x_1y_1 and the arbitrary axis, xy can be determined from Eq. 1.6b by taking x_1y_1 as principal axes. Hence, from Eq. 1.6b with $\tau_{x_1y_1} = 0$ we obtain:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (1.10)$$

From our wedge we know an arbitrary plane is located an angle θ from the general x -, y -axis set and θ ranges from zero to 180° . Hence, for the circle with its 360° range of rotation, planes separated by θ in the wedge are separated by 2θ on Mohr's circle. Taking the separation between the principal axis and the x -axis where the shear stress is τ_{xy} , we see that Fig. 13A also illustrates the relation of Eq. 1.10. Hence, Mohr's circle is constructed as illustrated in Fig. 13A.

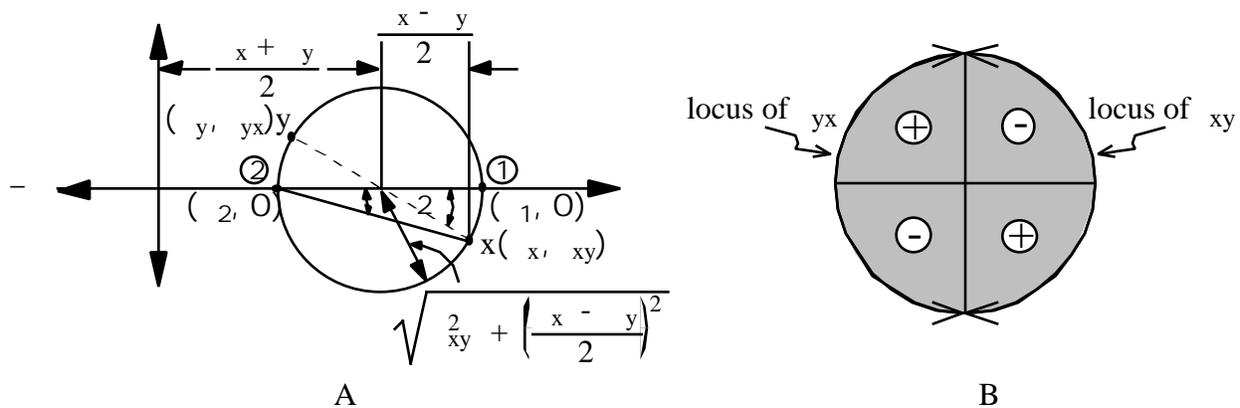


Figure 13. (A) Mohr's Circle of Stress; (B) Shear Stress Sign Convection for Mohr's Circle.*

When the principal stresses are known and it is desired to find the stresses acting on a plane oriented at angle θ from the principal plane numbered 1 in Fig. 14, Eq. 1.6 becomes:

$$\sigma = \frac{\sigma_1 + \sigma_2}{2} \pm \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta \quad (1.11)$$

and

$$\tau = -\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta \quad (1.12)$$

* For shear stresses, the sign convection is complicated since τ_{yx} and τ_{xy} are equal and of the same sign (see Figs 5 and 6, and Eq. 1.3), yet one must be plotted in the convention positive y -direction and the other in the negative y -direction (note the $-y$ -axis replaces the y -axis in Fig. 13A). The convention adopted is to use the four quadrants of the circle as follows: positive τ_{xy} is plotted in the fourth quadrant and positive τ_{yx} in the second quadrant. Analogously, negative τ_{xy} lies in the first quadrant. Further, the shear stress at 90° on the circle is positive while that at 270° is negative. [Ref. 1, p. 220]

Note that the negative shear stress of Eq. 1.12 is plotted in the first and third quadrants of Fig. 14, consistent with this sign convention.

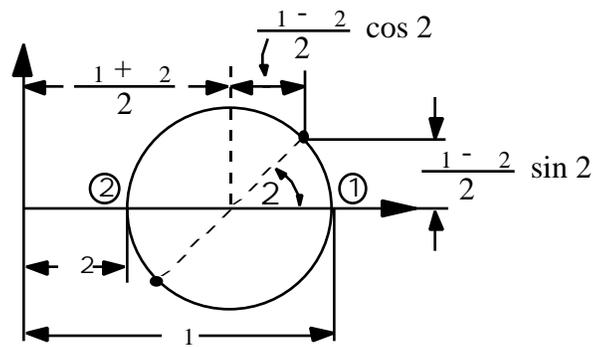


Figure 14. Mohr's Circle Representation of Eqs. 1.11 & 1.12.

Example 1:

Consider a cylindrical element of large radius with axial (x direction) and torque loadings which produce shear and normal stress components as illustrated in Fig. 15. Construct Mohr's circle for this element and determine the magnitude and orientation of the principal stresses.

From this problem statement, we know the coordinates σ_x, τ_{xy} are 60, 40 ksi and the coordinates σ_y, τ_{yx} are 0, 40 ksi. Further, from Eq. 1.10, $2\theta = \tan^{-1} 2(40)/(60) = 53^\circ$. From these values and the layout of Fig. 13, we can construct Mohr's circle for this example as shown in Fig. 16. Points 1 and 2 are stresses on principal planes, i.e., principal stresses are 80 and -20 ksi and points S and S' are maximum and minimum shear stress, i.e., 50 and -50 ksi. These results follow from Eqs. 1.8 and 1.9 as:

From Eq. 1.8
$$\sigma_{1, 2} = \frac{60 + 0}{2} \pm \sqrt{40^2 + \left(\frac{60 - 0}{2}\right)^2} = 80, -20 \text{ ksi}$$

From Eq. 1.9
$$\tau_{\max} = \pm \sqrt{40^2 + \left(\frac{60 - 0}{2}\right)^2} = \pm 50 \text{ ksi}$$

From Fig. 13A at plane of $\tau_{\max} = (60 + 0)/2 = 30 \text{ ksi}$, i.e., at S and S'

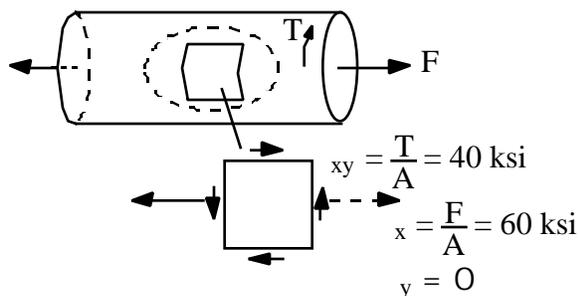


Figure 15. Example 1 Loadings.

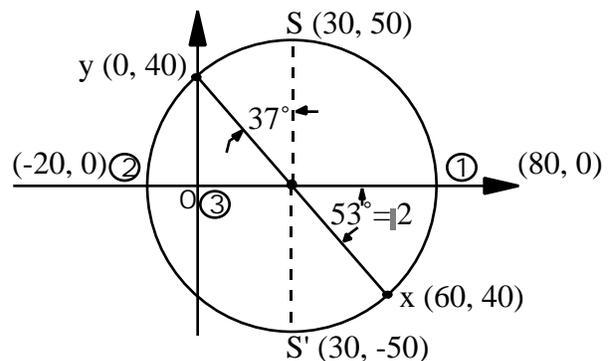


Figure 16. Mohr's Circle for Example 1.

The principal element, i.e., its faces are principal planes, and the maximum shear element are shown in Fig. 17.

NOTE: The angle between points y and S is the complement of 53° or 37° , and plane y is rotated $37^\circ/2$ counter-clockwise from plane S.

NOTE: The procedure for determining the orientation of these elements: point 1 is 53° counter-clockwise from point x on the circle; hence the plane represented by 1 is half of 53° counter-clockwise from plane x. Similarly, the S plane is $143^\circ/2$ counter-clockwise of plane x, or $37^\circ/2$ clockwise of plane y.

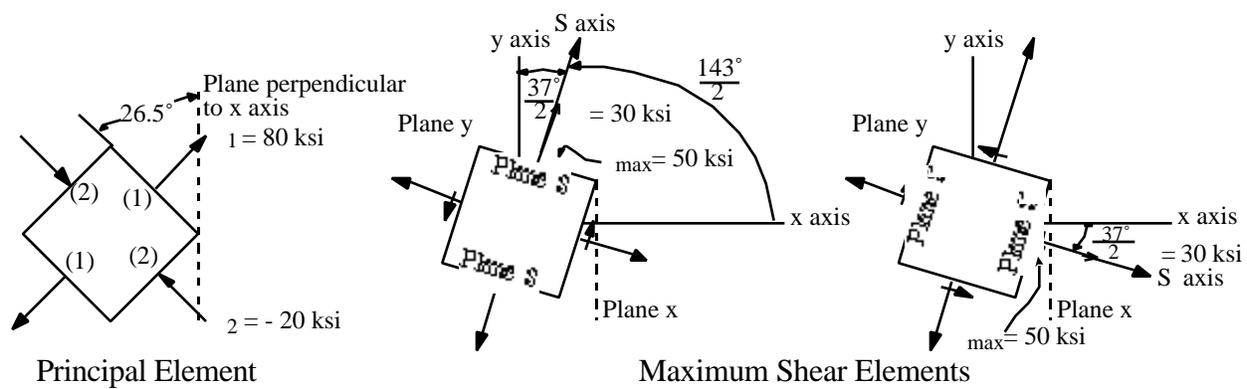


Figure 17. Key Elements for Example 1.

Mohr's circle clearly illustrates these additional points:

1. the principal stresses represent the extreme values of normal stress at the point in question;
2. $\sigma_{\max} = \frac{1}{2} (\sigma_1 - \sigma_2)$;
3. planes of maximum shear are always 45° removed from the principal planes; and
4. normal stresses acting on maximum shear planes S and S' are equal to the algebraic average of principal stresses σ_1 and σ_2 , i.e., $1/2 (\sigma_1 + \sigma_2)$.

1.6 Octahedral Planes and Stresses

Figure 18 illustrates the orientation of one of the eight octahedral planes which are associated with a given stress state. Each of the octahedral planes cut across one of the corners of a principal element, so that the eight planes together form an octahedron.

The stresses acting on these planes have interesting and significant characteristics. First of all, identical normal stresses act on all eight planes. By themselves the normal stresses are therefore said to be hydrostatic and tend to compress or enlarge the octahedron but not distort it.

$$\tau_{oct} = \frac{\tau_1 + \tau_2 + \tau_3}{3} \quad (1.13a)$$

Shear stresses are also identical. These distort the octahedron without changing its volume. Although the octahedral shear stress is smaller than the highest principal shear stress, it constitutes a single value that is influenced by all three principal shear stresses. Thus, it is important as a criterion for predicting yielding of a stressed material.

$$\tau_{oct} = \frac{1}{3} [(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2]^{1/2} \quad (1.13b)$$

In cases in which τ_x , τ_y , τ_z , τ_{xy} , τ_{xz} and τ_{yz} are known:

$$\tau_{oct} = \frac{\tau_x + \tau_y + \tau_z}{3} \quad (1.14a)$$

$$\tau_{oct} = \frac{1}{3} [(\tau_x - \tau_y)^2 + (\tau_y - \tau_z)^2 + (\tau_z - \tau_x)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)]^{1/2} \quad (1.14b)$$

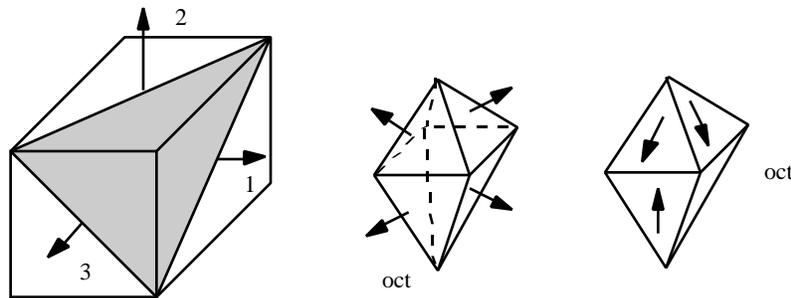


Figure 18. Octahedral Planes Associated with a Given Stress State.

1.7 Principal Strains and Planes

Having observed the correspondence between strain and stress, it is evident that with suitable axis transformation one obtains an expression for the strain tensor T' which is identical to that of stress tensor S' except that the terms in the principal diagonal are ϵ_1 , ϵ_2 , ϵ_3 . Hence, recalling from Section 1.3 that $\tau/2$ corresponds to ϵ , strain relations analogous to Eqs. 1.8, 1.9 and 1.10 can be written as follows:

$$\epsilon_1, \epsilon_2 = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \tau_{xy}^2} \quad (1.15)$$

$$\tau_{max} = \pm 2 \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \tau_{xy}^2} \quad (1.16)$$

$$2\theta = \tan^{-1} \frac{\tau_{xy}}{\epsilon_x - \epsilon_y} \quad (1.17)$$

Conclusion

The preceding sections *dealt separately* with the concepts of stress and strain at a point. These considerations involving stresses and strains separately are general in character and applicable to bodies composed of any continuous distribution of matter.

NOTE: No material properties were involved in the relationships, hence they are applicable to water, oil, as well as materials like steel and aluminum.

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2. ELASTIC STRESS-STRAIN RELATIONS

The relationships between these quantities are of direct importance to the engineer concerned with design and stress analysis. Generally two principal types of problems exist:

1. Determination of the stress state at a point from a known strain state—the problem encountered when stresses are to be computed from experimentally determined strains.
2. Determination of the state of strain at a point from a known stress state—the problem commonly encountered in design, where a part is assured to carry certain loads, and strains must be computed with regard to critical clearances and stiffnesses.

We limit ourselves to solids loaded in the elastic range. Furthermore, we shall consider only materials which are isotropic, i.e., materials having the same elastic properties in all directions. Most engineering materials can be considered as isotropic. Notable exceptions are wood and reinforced concrete.

2.1 Generalized Hooke's Law

Let us consider the various components of stress one at a time and add all their strain effects. For a uni-axial normal stress in the x direction, σ_x , the resulting normal strain is

$$\epsilon_x = \frac{\sigma_x}{E} \quad (2.1)$$

where E is Young's modulus or the modulus of elasticity.

Additionally this stress produces lateral contraction, i.e., ϵ_y and ϵ_z , which is a fixed fraction of the longitudinal strain, i.e.,

$$\epsilon_y = \epsilon_z = -\nu \epsilon_x = -\nu \frac{\sigma_x}{E} \quad (2.2)$$

This fixed fraction is called Poisson's ratio, ν . Analogous results are obtained from strains due to σ_y and σ_z .

The shear-stress components produce only their corresponding shear-strain components that are expressed as:

$$\epsilon_{zx} = \frac{\tau_{zx}}{G}, \quad \epsilon_{xy} = \frac{\tau_{xy}}{G}, \quad \epsilon_{yz} = \frac{\tau_{yz}}{G} \quad (2.3a,b,c)$$

where the constant of proportionality, G, is called the shear modulus.

For a linear-elastic isotropic material with all components of stress present:

$$\epsilon_x = \frac{1}{E} [\sigma_x - \nu (\sigma_y + \sigma_z)] \quad (2.4a)$$

$$\epsilon_y = \frac{1}{E} [\sigma_y - \nu (\sigma_z + \sigma_x)] \quad (2.4b)$$

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_x + \sigma_y)] \quad (2.4c)$$

$$\tau_{xy} = \frac{\tau_{xy}}{G} \quad (2.5a) \text{ same as (2.3)}$$

$$\tau_{yz} = \frac{\tau_{yz}}{G} \quad (2.5b)$$

$$\tau_{zx} = \frac{\tau_{zx}}{G} \quad (2.5c)$$

These equations are the generalized Hooke's law.

It can also be shown (Ref 1, p. 285) that for an isotropic materials, the properties G , E and ν are related as:

$$G = \frac{E}{2(1 + \nu)} \quad (2.6)$$

Hence,

$$\tau_{xy} = \frac{2(1 + \nu)}{E} \tau_{xy} \quad (2.7a)$$

$$\tau_{yz} = \frac{2(1 + \nu)}{E} \tau_{yz} \quad (2.7b)$$

$$\tau_{zx} = \frac{2(1 + \nu)}{E} \tau_{zx} \quad (2.7c)$$

Equations 2.4 and 2.5 may be solved to obtain stress components as a function of strains:

$$\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)] \quad (2.8a)$$

$$\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\epsilon_y + \nu(\epsilon_z + \epsilon_x)] \quad (2.8b)$$

$$\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y)] \quad (2.8c)$$

$$\tau_{xy} = \frac{E}{2(1 + \nu)} \tau_{xy} = G \tau_{xy} \quad (2.9a)$$

$$\tau_{yz} = \frac{E}{2(1 + \nu)} \tau_{yz} = G \tau_{yz} \quad (2.9b)$$

$$\tau_{zx} = \frac{E}{2(1 + \nu)} \tau_{zx} = G \tau_{zx} \quad (2.9c)$$

For the first three relationships one may find:

$$\sigma_x = \frac{E}{(1 + \nu)} \left[\epsilon_x + \frac{\nu}{(1 - 2\nu)} (\epsilon_x + \epsilon_y + \epsilon_z) \right] \quad (2.10a)$$

$$\sigma_y = \frac{E}{(1 + \nu)} \left[\epsilon_y + \frac{\nu}{(1 - 2\nu)} (\epsilon_x + \epsilon_y + \epsilon_z) \right] \quad (2.10b)$$

$$\sigma_z = \frac{E}{(1 + \nu)} \left[\epsilon_z + \frac{\nu}{(1 - 2\nu)} (\epsilon_x + \epsilon_y + \epsilon_z) \right]. \quad (2.10c)$$

For special case in which the x, y, z axes coincide with principal axes 1, 2, 3, we can simplify the strain set, Eqs. 2.4 and 2.5, and the stress set Eqs. 2.8 and 2.9, by virtue of all shear strains and shear stresses being equal to zero.

$$\sigma_1 = \frac{1}{E} [\sigma_1 - \nu (\sigma_2 + \sigma_3)] \quad (2.11a)$$

$$\sigma_2 = \frac{1}{E} [\sigma_2 - \nu (\sigma_3 + \sigma_1)] \quad (2.11b)$$

$$\sigma_3 = \frac{1}{E} [\sigma_3 - \nu (\sigma_1 + \sigma_2)] \quad (2.11c)$$

$$\epsilon_1 = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu) \epsilon_1 + \nu (\epsilon_2 + \epsilon_3)] \quad (2.12a)$$

$$\epsilon_2 = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu) \epsilon_2 + \nu (\epsilon_3 + \epsilon_1)] \quad (2.12b)$$

$$\epsilon_3 = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu) \epsilon_3 + \nu (\epsilon_1 + \epsilon_2)]. \quad (2.12c)$$

For biaxial-stress state, one of the principal stresses (say $\sigma_3 = 0$), Eqs. 2.11a,b,c become:

$$\sigma_1 = \frac{1}{E} (\sigma_1 - \nu \sigma_2) \quad (2.13a)$$

$$\sigma_2 = \frac{1}{E} (\sigma_2 - \nu \sigma_1) \quad (2.13b)$$

$$\sigma_3 = -\frac{\nu}{E} (\sigma_1 + \sigma_2). \quad (2.13c)$$

In simplifying Eqs. 2.12a,b,c for the case of $\sigma_3 = 0$, we note from Eq. 2.12c that for ϵ_3 to be zero,

$$\epsilon_3 = -\frac{\nu}{1 - \nu} (\epsilon_1 + \epsilon_2). \quad (2.14)$$

Substituting this expression into the first two of Eqs. 2.12a,b,c gives:

$$\sigma_1 = \frac{E}{1 - \nu} (\epsilon_1 + \nu \epsilon_2) \quad (2.15a)$$

$$\sigma_2 = \frac{E}{1 - \nu} (\epsilon_2 + \nu \epsilon_1) \quad (2.15b)$$

$$\sigma_3 = 0. \quad (2.15c)$$

In case of uniaxial stress Eqs. 2.13 and 2.15 must, of course reduce to:

$$\epsilon_1 = \frac{1}{E} \sigma_1 \quad (2.16a)$$

$$\epsilon_2 = \epsilon_3 = -\frac{\nu}{E} \sigma_1 \quad (2.16b)$$

$$\sigma_1 = E \epsilon_1 \quad (2.17a)$$

$$\sigma_2 = \sigma_3 = 0. \quad (2.17b)$$

2.2 Modulus of Volume Expansion (Bulk Modulus)

k may be defined as the ratio between hydrostatic stress (in which $\sigma_1 = \sigma_2 = \sigma_3$) and volumetric strain (change in volume divided by initial volume), i.e.,

$$k = \frac{\sigma}{(\Delta V/V)}. \quad (2.18)$$

NOTE: Hydrostatic compressive stress exists within the fluid of a pressurized hydraulic cylinder, in a rock at the bottom of the ocean or far under the earth's surface, etc.

Hydrostatic tension can be created at the interior of a solid sphere by the sudden application of uniform heat to the surface, the expansion of the surface layer subjecting the interior material to triaxial tension. For $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, Eqs. 2.11a,b,c show that:

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon = \frac{\sigma}{E}(1 - 2\nu).$$

This state of uniform triaxial strain is characterized by the absence of shearing deformation; an elemental cube, for example, would change in size but remain a cube. The size of an elemental cube initially of unit dimension would change from 1³ to $(1 + \epsilon)^3$ or to $1 + 3\epsilon + 3\epsilon^2 + \epsilon^3$. If we consider normal structural materials, ϵ is a quantity sufficiently small, so that ϵ^2 and ϵ^3 are completely negligible, and the volumetric change is from 1 to $1 + 3\epsilon$. The volumetric strain, $\Delta V/V$, is thus equal to 3ϵ or to:

$$\frac{\Delta V}{V} = 3\epsilon = \frac{3(1 - 2\nu)\sigma}{E}. \quad (2.19)$$

Hence,

$$k = \frac{\sigma}{\frac{\Delta V}{V}} = \frac{E}{3(1 - 2\nu)}. \quad (2.20)$$

Now $\nu = 0.5$, so that k cannot become negative. A simple physical model of a representative atomic crystalline structure gives:

$$\nu = 1/3, \quad (2.21)$$

so that

$$\boxed{k = E}. \quad (2.22)$$

3. THIN-WALLED CYLINDERS AND SPHERE

3.1 Stresses

- Stresses in a Thin-Walled Cylinder:

Consider a portion of the cylinder sufficiently remote from the ends to avoid end effects. The equation of equilibrium of radial forces (Fig. 19) acting on the element is:

$$p_i (r_i d \, dL) = 2 \, t_{,av} t dL \left(\sin \frac{d}{2} \right) \tag{3.1}$$

yielding
$$t_{,av} = \frac{p_i r_i}{t} . \tag{3.2}$$

when $\sin (d / 2)$ is approximated by $d / 2$, which is valid for small d . This is a correct expression for the average tangential stress (hoop stress) regardless of cylinder thickness.

NOTE: It is only when this equation is used as an approximation for the maximum tangential stress, $t_{,max}$, that the thickness t must be very small in comparison with the cylinder diameter.

Equation 3.1 can be modified to give a good quick estimate of the maximum tangential stress due to internal pressure for cylinders of moderate thickness by replacing r_i with r_{av} , which we label R :

$$R = r_i + \frac{t}{2} . \tag{3.3}$$

Thus,

$$t_{,max} = \frac{p_i R}{t} \tag{3.4}$$

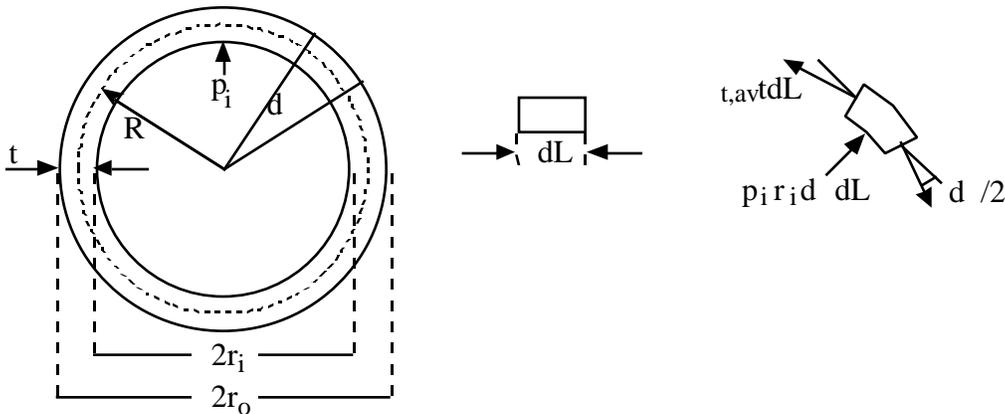


Figure 19. Radial Forces in an Internally Pressurized Thin-Walled Cylinder.

The following are errors in estimating t_{\max} from Eqs. 3.2 or 3.4 compared to the thick-walled cylinder results:

| t/r_i | % Error Using Eq. 3.2 | % Error Using Eq. 3.4 |
|---------|--------------------------|--------------------------|
| 0.1 | 5 (low) | 0.5 (low) |
| 0.2 | 10 (low) | 1.0 (low) |
| 0.5 | 23 (low) | 3.8 (low) |
| 1.0 | 40 (low) | 10.2 (low) |

If the ends are closed, the cylinder is also subjected to an axial force of magnitude $p_i r_i^2$. This is distributed over an area of cross section that can be expressed as:

$$A = (r_o^2 - r_i^2) = 2 R t . \quad (3.5)$$

Thus the average axial tensile stress can be expressed by:

$$\sigma_{a,av} = \frac{p_i r_i^2}{r_o^2 - r_i^2} = \frac{p_i r_i^2}{2 r_{av} t} . \quad (3.6)$$

For the thin-walled case, $r_i \approx r_{av} \approx R$ since $r_i = R(1 - 0.5t/R)$. Hence, Eq. 3.6 reduces to:

$$\sigma_{a,av} = \frac{p_i R}{2t} . \quad (3.7)$$

Thus the axial stress is approximately half the tangential stress.

- Stresses in a Thin-Walled Sphere:

From a force balance these stresses can be determined to be

$$\sigma_{a,av} = \sigma_{t,av} = \frac{pR}{2t} \quad (3.8)$$

3.2 Deformation and Strains

The deformation of the diameter of a thin-walled cylinder or sphere caused by the internal pressure is to be determined. Note that the state of stress at the inner surface is triaxial and is not plane stress. The principal stresses of the inner surface are $\sigma_1 = \sigma_r$, $\sigma_2 = \sigma_z$ and $\sigma_3 = -p$. However, $\sigma_3 = -p$ is so small compared to σ_1 and σ_2 that it can be neglected when considering strains. *Thus, the state of stress in thin-walled pressure vessels is usually considered to be plane stress.*

The deformation of such pressure vessels is influenced by the tangential and axial (transverse for the sphere) stresses, and hence use must be made of the following relations obtained from Eqs. 2.4, 2.5 and 2.8, respectively, with $\sigma_y = 0$.

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} \sigma_z \quad (3.9a)$$

$$\epsilon_y = -\frac{\nu}{E} (\sigma_x + \sigma_z) \quad (3.9b)$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E} \sigma_x \quad (3.9c)$$

$$= \frac{\nu \sigma_y}{G} \quad (3.9d)$$

$$\epsilon_x = \frac{E}{(1 - \nu^2)} (\epsilon_x + \nu \epsilon_z) \quad (3.10a)$$

$$\epsilon_z = \frac{E}{(1 - \nu^2)} (\epsilon_z + \nu \epsilon_x) \quad (3.10b)$$

to express Hooke's law for plane stress. Equations 3.10a,b are obtained from Eqs. 2.8a and 2.8c upon inspection of ϵ_y evaluated from Eq. 2.8b with σ_y taken as zero.

Let σ_x , σ_z , and ϵ_x and ϵ_z represent the tangential and axial stress and strain, respectively, in the wall. The substitution of these symbols in Eqs. 3.9a,b,c, i.e., $\sigma_x = \sigma_x$ and $\sigma_z = \sigma_z$, gives:

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} \sigma_z \quad (3.11)$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E} \sigma_x \quad (3.12)$$

- Closed-End Cylinder:

For the strains in the closed-end cylinder, the values of ϵ_x and ϵ_z as derived in Eqs. 3.4 and 3.8, respectively, are substituted into Eqs. 3.11 and 3.12 to give:

$$\epsilon_x = \frac{1}{E} \left(\frac{pR}{t} - \frac{\nu pR}{2t} \right) = \frac{pR}{2Et} (2 - \nu) \quad (3.13)$$

$$\epsilon_z = \frac{1}{E} \left(\frac{\nu pR}{2t} - \frac{pR}{t} \right) = \frac{\nu pR}{2Et} (1 - 2\nu). \quad (3.14)$$

Let the change in length of radius R be Δr when the internal pressure is applied. Then the change in length of the circumference is $2\Delta r$. But the circumferential strain, ϵ_x , is, by definition, given by the following equation:

$$\epsilon_x = \frac{2\Delta r}{2R} = \frac{\Delta r}{R}. \quad (3.15)$$

By combining Eqs. 3.15 and 3.13 we get:

$$\Delta r = \frac{pR^2}{2Et} (2 - \nu). \quad (3.16)$$

The change in length, ℓ , for a closed-end cylinder is equal to:

$$\ell = z\ell \quad (3.17)$$

or

$$\ell = \frac{pR\ell}{2Et} (1 - 2) \quad (3.18)$$

• Sphere:

$$z = \frac{pR}{2t} \quad (3.19)$$

Thus, from Eq. 3.11:

$$= \frac{1}{E} \left(\frac{pR}{2t} - \frac{pR}{2t} \right) = \frac{pR}{2Et} (1 -) \quad (3.20)$$

By combining Eq. 3.20 with Eq. 3.15, we obtain:

$$r = \frac{pR^2}{2Et} (1 -) \quad (3.21)$$

3.3 End Effects for the Closed-End Cylinder

Figure 20 illustrates a cylinder closed by thin-walled hemispherical shells. They are joined together at AA and BB by rivets or welds. The dashed lines show the displacements due to internal pressure, p . These displacements are given by Eqs. 3.16 and 3.21 as:

$$R_c = \frac{pR^2}{2Et} (2 -) \quad (3.22a)$$

$$R_s = \frac{pR^2}{2Et} (1 -) \quad (3.22b)$$

The value of R_c is more than twice that of R_s for the same thickness, t , and as a result the deformed cylinder and hemisphere do not match boundaries. To match boundaries, rather large shearing forces, V , and moments, M , must develop at the joints as Fig. 20 depicts (only those on the cylinder are shown; equal but opposite shears and moments exist on the ends of the hemispheres). This shear force is considerably minimized in most reactor pressure vessels by sizing the hemisphere thickness much smaller than the cylinder thickness. For example, if the hemisphere thickness is 120 mm and the cylinder thickness is 220 mm, for $\nu = 0.3$, then the ratio R_c to R_s is

$$\frac{R_c}{R_s} = \frac{t_s}{t_c} \left(\frac{2 - \nu}{1 - \nu} \right) = 1.32$$

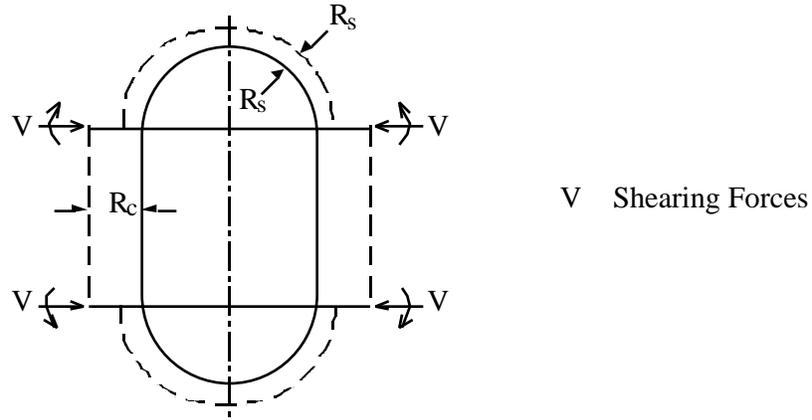


Figure 20. Discontinuities in Strains for the Cylinder (R_c) and the Sphere (R_s).

The general solution of the discontinuity stresses induced by matching cylindrical and hemisphere shapes is not covered in these notes. However, a fundamental input to this solution is the matching of both displacements and deflections of the two cylinders of different wall thicknesses at the plane they join. Thus, the relationship between load, stress, displacement and deflection for cylinders and hemispheres are derived. These relations are presented in Tables 1 and 2, respectively.

Table 1. Relationship Between Load, Stresses, Displacement and Deflection for the Cylinder (equations cited are from Ref. 2)

| Load | Stresses t, θ, r $, z, r$ | Displacement u_o | Deflection δ |
|--|---|---|---|
| p | $\frac{pR}{t}, \frac{pR}{2t}, -\frac{p}{2}$ | $\frac{pR^2}{2tE} \left(2 - \frac{t}{R} \right)$ | 0 |
| M_o Edge Moment per Unit Perimeter Length | $\frac{EM_o}{2^2DR} \pm \frac{6M_o}{t^2}, \pm \frac{6M_o}{t^2}, 0$ (Eq. 7.3.22, p. 7.3-12) Note: $u_o = \frac{M_o}{2^2D}$; so first term $= \frac{Eu_o}{R}$. | $\frac{M_o}{2^2D}$ (Eq. 7.3.18, p. 7.3.7) units $\frac{F}{L^2} \frac{L/L}{L^2} = L$ | $-\frac{M_o}{D}$ (Eq. 7.3.19, p. 7.3.8) dimensionless |

Table 1 continued on next page

Table 1. Relationship Between Load, Stresses, Displaced and Deflection for the Cylinder (cont'd)

| | | | |
|--|--|--|--|
| Q_o Edge Shear Force per Unit Perimeter Length | $\frac{EQ_o}{2^3 D}, 0, 0$ (Eq. 7.3.22, p. 7.3-12) Note: $u_o = \frac{Q_o}{2^3 D}$; so first term $= \frac{u_o E}{R}$ | $\frac{Q_o}{2^3 D}$ (Eq. 7.3.18, p. 7.3.7) units $\frac{F/L}{L^3} FL = L$ | $-\frac{Q_o}{2^2 D}$ (Eq. 7.3.19, p. 7.3.8) dimensionless |
| where Plate Flexural Rigidity: $D = \frac{Et^3}{12(1 - \nu^2)}, \quad \text{units} \left(\frac{F}{L^2} L^3 \right)$ $\kappa = \frac{Et}{4R^2 D} = \frac{3(1 - \nu^2)}{R^2 t^2} \left(\frac{1}{L^2} \frac{1}{L^2} \right)$ Equations and pages cited are from Ref. 2. | | | |

Table 2. Relationship Between Load, Stresses, Displaced and Deflection for the Hemisphere

| Load | Stresses t, ℓ, r | Displacement o | Deflection o |
|--|--|--|-----------------------|
| p | $\frac{pR}{2t}, \frac{pR}{2t}, -\frac{p}{2}$ | $\frac{pR^2}{2tE} (1 - \nu) + \frac{t}{R}$ | 0 |
| M_o Edge Moment per Unit Perimeter Length | $\frac{2^2 M_o}{tR} \pm \frac{6M_o}{t^2}, \pm \frac{6M_o}{t^2}, 0$ Note: $\frac{2^2 M_o}{t} = \frac{o}{E}$; so first term $= \frac{E o}{R}$ | $\frac{2^2 M_o}{Et}$ | $\frac{4^3 M_o}{REt}$ |
| Q_o Edge Shear Force per Unit Perimeter Length | $\frac{2 Q_o}{t}, 0, 0$ Note: $\frac{2 Q_o}{t} = \frac{E o}{R}$ | $\frac{2R Q_o}{Et}$ | $\frac{2^2 Q_o}{Et}$ |
| where $\kappa = R; \quad \kappa = \frac{3(1 - \nu^2)}{R^2 t^2}$ | | | |

4. THICK-WALLED CYLINDER UNDER RADIAL PRESSURE [Ref. 1, pp. 293-300]

- Long Cylinder Plane Strain

The cylinder is axially restrained ($\epsilon_z = 0$) at either end and subjected to uniform radial pressure, not body forces. The cylinder cross section and stresses on an element are illustrated in Fig. 21.

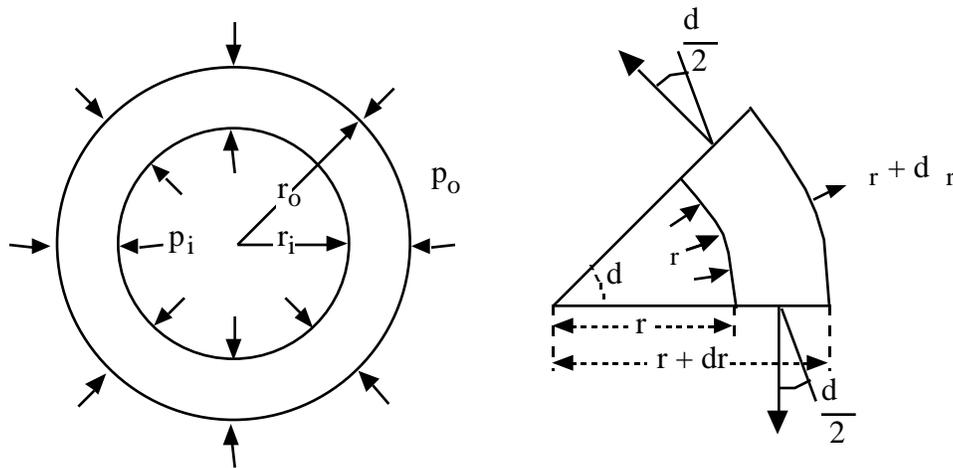


Figure 21. Force Balance and Displacements in a Pressurized Cylinder.

The relevant elastic equation set when azimuthal symmetry prevails is

- Strain–Displacement Relations

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \frac{dv}{dr} - \frac{v}{r} \quad (4.1a,b,c)$$

where u is the outward radial displacement and v is the axial direction displacement. Since the effect of internal pressure is to move the material in the radial direction without any rotation, both v and ϵ_r should be zero. This will be demonstrated formally from the boundary conditions.

- Stress–Equilibrium Relations (for forces in the r and θ directions)

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad \frac{d\sigma_\theta}{dr} + \frac{2\sigma_r}{r} = 0. \quad (4.2a,b)$$

- Stress–Strain Relations

$$\epsilon_r = \frac{1}{E} \left[\sigma_r - \nu (\sigma_\theta + \sigma_z) \right] \quad (4.3a)$$

$$\epsilon_\theta = \frac{1}{E} \left[\nu \sigma_r - (\sigma_z + \sigma_\theta) \right] \quad (4.3b)$$

$$\epsilon_z = 0 = \frac{1}{E} \left[\nu (\sigma_r + \sigma_\theta) - \sigma_z \right] \quad (4.3c)$$

The strain-displacement relations can, by elimination of the displacement variables, be combined as a compatibility (of strain) equation:

$$\frac{d}{dr} + \frac{-}{r} = 0 \quad (4.4)$$

Finally the boundary conditions are:

$$r(r_i) = -p_i ; \quad r(r_o) = -p_o \quad (4.5a)$$

$$r(r_i) = r(r_o) = 0 \quad (4.5b)$$

Note that the radial dimension is assumed to be free of displacement constraints.

4.1 Displacement Approach

When displacements are known, it is preferable to use this solution approach. In this case we solve for the displacements. Proceeding, solve Eq. 4.3 for stresses and substitute them into the stress–equilibrium relations of Eq. 4.2 to obtain:

$$(1 -) \left(\frac{d}{dr} + \frac{r -}{r} \right) + \left(\frac{d}{dr} + \frac{-}{r} \right) = 0 \quad (4.6a)$$

$$\frac{d}{dr} + \frac{2}{r} = 0 \quad (4.6b)$$

Substituting Eq. 4.1 in Eq. 4.6 and simplification show that the displacements u and v are governed by the equations:

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru) \right] = 0 \quad (4.7a)$$

$$\frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} = \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (rv) \right] = 0 . \quad (4.7b)$$

Successive integration of these equations yields the displacement solutions

$$u = \frac{C_1 r}{2} + \frac{C_2}{r} ; \quad v = \frac{C_3 r}{2} + \frac{C_4}{r} \quad (4.8a,b)$$

where the constants of integration are to be determined by the stress boundary conditions (Eqs. 4.5a,b). With the use of Eqs. 4.8a,b, the strain and stress solutions are obtained from Eq. 4.1 and

$$r = \frac{E}{(1 +)(1 - 2)} [(1 -) r +] \quad (4.9a)$$

$$= \frac{E}{(1 +)(1 - 2)} [(1 -) + r] \quad (4.9b)$$

$$r = G r \quad (4.9c)$$

as

$$r = \frac{C_1}{2} - \frac{C_2}{r^2}, \quad = \frac{C_1}{2} + \frac{C_2}{r^2}, \quad r = -\frac{2C_4}{r^2} \quad (4.10a,b,c)$$

$$r = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[\frac{C_1}{2} - (1 - 2\nu) \frac{C_2}{r^2} \right] \quad (4.11a)$$

$$= \frac{E}{(1 + \nu)(1 - 2\nu)} \left[\frac{C_1}{2} + (1 - 2\nu) \frac{C_2}{r^2} \right] \quad (4.11b)$$

$$z = \left(\frac{r}{r_0} + \nu \right) \frac{EC_1}{(1 + \nu)(1 - 2\nu)} \quad (4.11c)$$

$$r = -\frac{2GC_4}{r^2}. \quad (4.11d)$$

By virtue of the boundary conditions, Eqs. 4.5a,b, the first and last of Eq. 4.11 yield the following equations for the determination of the constants C_1 and C_2 .

$$\frac{C_1}{2} - (1 - 2\nu) \frac{C_2}{r_i^2} = -\frac{(1 + \nu)(1 - 2\nu)}{E} p_i \quad (4.12a)$$

$$\frac{C_1}{2} - (1 - 2\nu) \frac{C_2}{r_0^2} = -\frac{(1 + \nu)(1 - 2\nu)}{E} p_o \quad (4.12b)$$

$$C_4 = 0 \quad (4.12c)$$

The solutions for C_1 and C_2 are

$$\frac{C_1}{2} = \frac{(1 + \nu)(1 - 2\nu)(p_i r_i^2 - p_o r_0^2)}{E(r_0^2 - r_i^2)} \quad (4.13a)$$

$$C_2 = \frac{(1 + \nu)(p_i - p_o) r_i^2 r_0^2}{E(r_0^2 - r_i^2)}. \quad (4.13b)$$

As to the constant C_3 , it remains undetermined. However, with $C_4 = 0$, Eq. 4.8 shows that $v = C_3 r/2$ which corresponds to a rigid-body rotation about the axis of the cylinder. Since this rotation does not contribute to the strains, C_3 is taken as zero. Hence Eqs. 4.11a,b,c and Eq. 4.8a become:

$$r = \frac{1}{(r_o/r_i)^2 - 1} \left\{ \left[1 - \frac{(r_o/r_i)^2}{(r/r_i)^2} \right] p_i - \left[1 - \frac{1}{(r/r_i)^2} \right] \left(\frac{r_o}{r_i} \right)^2 p_o \right\} \quad (4.14a)$$

$$= \frac{1}{(r_o/r_i)^2 - 1} \left\{ \left[1 + \frac{(r_o/r_i)^2}{(r/r_i)^2} \right] p_i - \left[1 + \frac{1}{(r/r_i)^2} \right] \left(\frac{r_o}{r_i} \right)^2 p_o \right\} \quad (4.14b)$$

$$z = \frac{2}{(r_o/r_i)^2 - 1} \left[p_i - \left(\frac{r_o}{r_i} \right)^2 p_o \right] \quad \text{Plane Strain!} \quad (4.14c)$$

$$u = \frac{(1 + \nu)(r/r_i) r_i}{(r_o/r_i)^2 - 1} \left\{ \left[(1 - 2\nu) + \frac{(r_o/r_i)^2}{(r/r_i)^2} \right] \frac{p_i}{E} - \left[(1 - 2\nu) + \frac{1}{(r/r_i)^2} \right] \left(\frac{r_o}{r_i} \right)^2 \frac{p_o}{E} \right\} \quad (4.14d)$$

4.2 Stress Approach

In this case we solve directly for the stresses and apply the boundary conditions as before. Thus, re-expressing the stress-strain relations (Eq. 4.3) directly as strain when $\epsilon_z = 0$ we get:

$$\sigma_r = \frac{1+\nu}{E} [(1-\nu)\sigma_r - \nu\sigma_\theta] \quad (4.15a)$$

$$= \frac{1+\nu}{E} [(1-\nu)\sigma_r - \nu\sigma_\theta] \quad (4.15b)$$

$$\sigma_r = \frac{r}{G}, \quad (4.15c)$$

and then substituting them into Eq. 4.4 upon simplification yields

$$(1-\nu)\frac{d}{dr} - \frac{d}{dr} + \frac{\nu}{r} = 0. \quad (4.16)$$

By the first stress-equilibrium relation in Eq. 4.2, this equation is reduced to the simple form

$$\frac{d}{dr}(\sigma_r + \sigma_\theta) = 0. \quad (4.17)$$

Integration of this equation and the second of Eq. 4.2 yields

$$\sigma_r + \sigma_\theta = C_1; \quad \sigma_r = \frac{C_2}{r^2}. \quad (4.18a,b)$$

Substitution of the relation $\sigma_\theta = C_1 - \sigma_r$ in Eq. 4.2, transposition, and integration yield

$$\sigma_r = \frac{C_1}{2} + \frac{C_3}{r^2}, \quad (4.19)$$

where C_1, C_2 and C_3 are constants of integration. Applying the boundary conditions we finally get

$$C_1 = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}, \quad C_2 = 0, \quad C_3 = \frac{(p_i - p_o) r_i^2 r_o^2}{r_o^2 - r_i^2}. \quad (4.20a,b,c)$$

With these values for the constants, Eqs. 4.18, 4.19 and the relation $\sigma_z = (\sigma_r + \sigma_\theta)$, the complete stress solution is explicitly given. This solution can then be substituted in Eqs. 4.15a and 4.15b to determine σ_r and σ_θ and, finally, u is determined by the relation $u = r \epsilon_r$.

NOTE: Since the prescribed boundary conditions of the problem are stresses, it is evident that the stress approach in obtaining the solution involves simpler algebra than the displacement approach.