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Problem Set II Solution

1. Stress intensity = max$\{|\sigma_r - \sigma_\theta|, |\sigma_\theta - \sigma_z|, \sigma_z - \sigma_r|\}$
   For thin wall approximation:

$$\sigma_r = -\frac{P_i + P_o}{2} \tag{1}$$

$$\sigma_z = -\frac{P_i R^2 - P_o(R+t)^2}{(R+t)^2 - R^2} \tag{2}$$

$$\sigma_\theta = \frac{P_i - P_o}{t}(R + \frac{t}{2}) \tag{3}$$

Therefore:

$$S_{thin} = \sigma_\theta - \sigma_r = \frac{P_i - P_o}{t}(R + \frac{t}{2}) + \frac{P_i + P_o}{2} \tag{4}$$

Thick wall solution:
Equilibrium in radial direction gives:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{5}$$

Hook's law:

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta - \nu\sigma_z) \tag{6}$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r - \nu\sigma_z) \tag{7}$$

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_r - \nu\sigma_\theta) \tag{8}$$

Since $\epsilon_\theta = u/r$, $\epsilon_r = \frac{du}{dr}$, we get:

$$\frac{d\epsilon_\theta}{dr} = \frac{1}{r}(\epsilon_r - \epsilon_\theta) \tag{9}$$

For this close end cylinder far from the end, plane stress condition is assumed, i.e., σ_z is const.
Plug Eq 6 and Eq 5into Eq 9, we get

$$\frac{d}{dr}(\sigma_\theta + \sigma_r) = 0 \tag{10}$$

Plug Eq 10 into Eq 5, we get

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2\sigma_r) = 0 \tag{11}$$

With B.C. $\sigma_r(r = R) = -P_i$ and $\sigma_r(r = R + t) = -P_o$, we get:

$$\sigma_r = -P_i(\frac{R}{r})^2 + (1 - (\frac{R}{r})^2)\frac{-P_o(R+t)^2 + P_i R^2}{(R+t)^2 - R^2} \tag{12}$$

$$\sigma_\theta = P_i(\frac{R}{r})^2 + (1 + (\frac{R}{r})^2)\frac{-P_o(R+t)^2 + P_i R^2}{(R+t)^2 - R^2} \tag{13}$$

$$\sigma_\theta - \sigma_r = 2(\frac{R}{r})^2\frac{(P_i - P_o)(R+t)^2}{(R+t)^2 - R^2} \tag{14}$$

Maximum stress intensity is at the location of inner radius:

$$S_{thick} = 2\frac{(P_i - P_o)(R+t)^2}{(R+t)^2 - R^2}$$

The error in thin wall approximation is:

$$|1 - \frac{S_{thin}}{S_{thick}}|$$

Results are tabulated below:

t/R	0.03	0.10	0.15	0.30
$P_i = 2P_o$	1.41%	4.13%	5.67%	8.88%
$P_i = 20P_o$	1.31%	4.09%	5.88%	10.46%

2. We use thin shell approximation to solve this problem. For a region of cylinder far from a junction, stresses are:

$$\sigma_\theta = \frac{PR}{t} \tag{15}$$

$$\sigma_r = -P/2 \tag{16}$$

$$\sigma_x = \frac{PR}{2t} \tag{17}$$

Radial displacement:

$$u_c = \frac{PR^2}{2Et}(2 - \nu + \frac{\nu t}{R}) \tag{18}$$

For the sphere, stresses are:

$$\sigma_\theta = \sigma_\phi = \frac{PR}{2t} \tag{19}$$

$$\sigma_r = -P/2 \tag{20}$$

Radial displacement:

$$u_s = \frac{PR^2}{2Et}(1 - \nu + \frac{\nu t}{R}) \tag{21}$$

At the junction, Note C Eq 30–33 give that at the edge of cylinder:

$$u_0 = u_c + \frac{V_0}{2\beta^3 D} + \frac{M_0}{2\beta^2 D} \tag{22}$$

$$\phi_0 = -\frac{V_0}{2\beta^2 D} - \frac{M_0}{\beta D} \tag{23}$$

at the edge of hemisphere:

$$u_0 = u_s - \frac{2R\lambda}{Et} V_0 + \frac{2\lambda^2}{Et} M_0 \tag{24}$$

$$\phi_0 = -\frac{2\lambda^2}{Et} V_0 + \frac{4\lambda^3}{REt} M_0 \tag{25}$$

where, $\lambda = \beta_s R$, $\beta_s = (\frac{3(1-\nu^2)}{R^2 t^2})^{1/4}$, u_0 is the radial displacement at the junction, ϕ_0 is the slope at the junction. [1]

Due to the continuity of the displacement and slope, the four unknowns u_0, ϕ_0, M_0, and V_0

[1]**Note that the direction of the shear force Q in Note L4 is different from that in Note C. If you assume the direction of Q_{0S} is same as Q_{OC}, the continuity of shear force should give that: $Q_{0S} = -Q_{0C}$.**

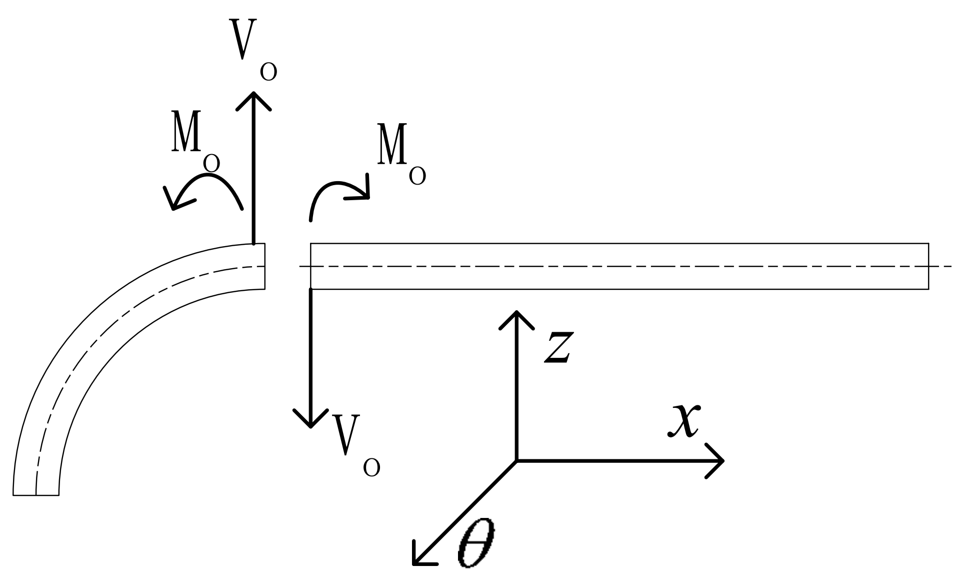

Figure 1: Junction of hemisphere and cylinder

can be solved by above equations. It can be found that

$$M_0 = 0 \tag{26}$$
$$V_0 = -PR^2/(4R\lambda + Et/\beta^3 D) \tag{27}$$
$$u = u_c + \frac{V_0}{2D\beta^3} \tag{28}$$
$$\sigma_x = \frac{PR}{2t} \tag{29}$$
$$\sigma_\theta = \frac{PR}{t} + \frac{EV_0}{2DR\beta^3} \tag{30}$$
$$\sigma_r = -P/2 \tag{31}$$

With mean radius R = inner radius + $t/2$ = 1.155 m, t = 0.11 m, E = 200 GPa, ν = 0.3, we get

(a) At the junction, from Eq 29–Eq 31:
σ_x = 81.38 MPa
σ_θ = 122.06 MPa
σ_r = -7.75 MPa
The maximum stress is the hoop stress: 122.06 MPa.

(b) Radial displacement as a function of radial postion z is: $(R + z)\epsilon_\theta$. Thus, from Eq 18, the radial displacement of cylinder is:

$$\frac{PR}{2Et}(2 - \nu + \frac{\nu t}{R})(R + z)$$

From Eq 21, the radial displacement of hemishpere is:

$$\frac{PR}{2Et}(1 - \nu + \frac{\nu t}{R})(R + z)$$

From Eq 28, the radial displacement of junction is:

$$(\frac{PR}{2Et}(2 - \nu + \frac{\nu t}{R}) + \frac{V_0}{2DR\beta^3})(R + z)$$

Therefore:

	Cylinder	Sphere	Junction
Max. displacement (m)	0.00089	0.00037	0.00063