Exercise 4. Partial Differential Equations.

1. Determine whether the following partial differential equations, in which p and q are arbitrary real constants, are elliptic, parabolic, or hyperbolic.

(a) $p^2\frac{\partial^2\psi}{\partial x^2} + q^2\frac{\partial^2\psi}{\partial y^2} = 0$

(b) $(p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y})(p\frac{\partial}{\partial x} - q\frac{\partial}{\partial y})\psi = 1$

(c) $\frac{\partial^2\psi}{\partial x^2} + 4\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} = 0$

(d) $\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} = \psi$

(e) $\frac{\partial^2\psi}{\partial x^2} + p\frac{\partial\psi}{\partial y} = \psi$

(f) $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = x$

(e) $px\frac{\partial\psi}{\partial x} + q\frac{\partial\psi}{\partial y} = 1$

2. Write a computer code function[1] to evaluate the difference stencil in two dimensions for the anisotropic partial differential operator, $\mathcal{L} = 4\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The code function is to operate on a quantity $f(x, y) = f_{ij}$, represented as a matrix of the values at discrete points on a structured, equally-spaced, 2-D mesh with N_x and N_y nodes in the x and y directions, spanning the intervals $0 \le x \le L_x$, $0 \le y \le L_y$. The function should accept parameters $N_x, N_y, L_x, L_y, i, j, f$ and return the corresponding finite-difference expression for $g_{ij} = \mathcal{L}f$ at mesh point i, j.

Write also a test program to construct $f(x, y) = (x^2 + y^2)$ on the mesh nodes, giving f_{ij}, and call your stencil function, with f and the corresponding N_x, N_y, L_x, L_y as arguments, to evaluate g_{ij} and print it.

Submit the following as your solution:

a. Your code in a computer format that is capable of being executed, citing the language it is written in.

b. A brief answer to the following. Will your function work at the boundaries, $x = 0, L_x$, or $y = 0, L_y$? If not, what is needed to make it work there?

c. The values of g_{ij} for four different nodes corresponding to two different interior i and two different interior j, when $N_x = N_y = 10$, $L_x = L_y = 10$.

d. Brief answer to: Are there inefficiencies in using a code like this to evaluate $\mathcal{L}f$ everywhere on the mesh? If so, how might those inefficiencies be avoided?

[1] For OO purists, this could be a "method".