22.101 Applied Nuclear Physics (Fall 2006)
Lecture 5 (9/20/06)

Barrier Penetration

References --
R. D. Evans, *The Atomic Nucleus*, McGraw-Hill, New York, 1955), pp. 60, pp.852.

We have previously observed that one can look for different types of solutions to the wave equation. An application which will turn out to be useful for later discussion of nuclear decay by α -particle emission is the problem of barrier penetration. In this case one looks for positive-energy solutions as in a scattering problem. We consider a one-dimensional system where a particle with mass m and energy E is incident upon a potential barrier with width L and height V_o (V_o is greater than E). Fig. 1 shows that with the particle approaching from the left, the problem separates into three regions, left of the barrier (region I), inside the barrier (region II), and right of the barrier (region III).

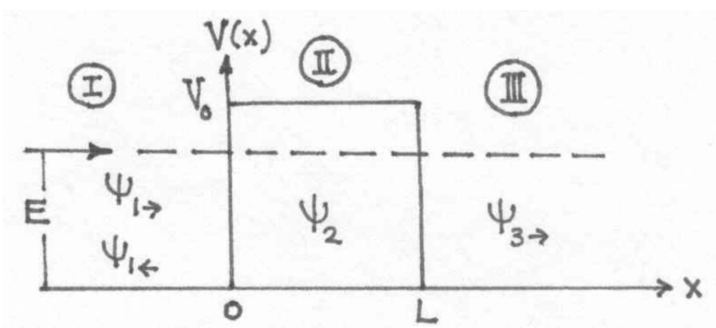

Fig. 1. Particle with energy E penetrating a square barrier of height V_o (V_o > E) and width L.

In regions I and III the potential is zero, so the wave equation (3.1) is of the form

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0\,, \qquad k^2 = 2mE/\hbar^2 \qquad (5.1)$$

where k^2 is positive. The wave functions in these two regions are therefore

$$\psi_1 = a_1 e^{ikx} + b_1 e^{-ikx} \equiv \psi_{1\rightarrow} + \psi_{1\leftarrow} \tag{5.2}$$

$$\psi_3 = a_3 e^{ikx} + b_3 e^{-ikx} \equiv \psi_{3\rightarrow} \tag{5.3}$$

where we have set $b_3 = 0$ by imposing the boundary condition that there is no particle in region III traveling to the left (since there is nothing in this region that can reflect the particle). In contrast, in region I we allow for reflection of the incident particle by the barrier; this means b_1 will be nonzero. The subscripts $\rightarrow$ and $\leftarrow$ denote the wave functions traveling to the right and to the left respectively.

In region II, the wave equation is

$$\frac{d^2\psi(x)}{dx^2} - \kappa^2\psi(x) = 0\,, \qquad \kappa^2 = 2m(|V_o| - E)/\hbar^2 \tag{5.4}$$

So we write the solution in the form

$$\psi_2 = a_2 e^{\kappa x} + b_2 e^{-\kappa x} \tag{5.5}$$

Notice that in region II the kinetic energy, $E - V_o$, is negative, so the wavenumber is imaginary in a propagating wave (another way of saying the wave function is monotonically decaying rather than oscillatory). What this means is there is no wave-like solution in this region. By introducing κ we can think of it as the wavenumber of a hypothetical particle whose kinetic energy is positive, $V_o - E$.

Having obtained the wave function in all three regions we proceed to discuss how to organize this information into a useful form, namely, the transmission and reflection coefficients. We recall that given the wave function ψ, we know immediately the particle density (number of particles per unit volume, or the probability of the finding the particle in an element of volume d^3r about $\underline{r}$), $|\psi(\underline{r})|^2$, and the net current, given by (2.24),

$$\underline{j} = \frac{\hbar}{2mi}\left(\psi^* \underline{\nabla}\psi - \psi\underline{\nabla}\psi^*\right) \tag{5.6}$$

Using the wave functions in regions I and III we obtain

$$j_1(x) = v\left[|a_1|^2 - |b_1|^2\right] \tag{5.7}$$

$$j_3(x) = v|a_3|^2 \tag{5.8}$$

where $v = \hbar k / m$ is the particle speed. We see from (5.7) that j_1 is the net current in region I, the difference between the current going to the right and that going to the left. Also, in region III there is only the current going to the right. Notice that current is like a flux in that it has the dimension of number of particles per unit area per second. This is consistent with (5.7) and (5.8) since $|a|^2$ and $|b|^2$ are particle densities with the dimension of number of particles per unit volume. From here on we can regard a_1, b_1, and a_3 as the amplitudes of the incident, reflected, and transmitted waves, respectively. With this interpretation we define

$$T = \left|\frac{a_3}{a_1}\right|^2, \qquad R = \left|\frac{b_1}{a_1}\right|^2 \tag{5.9}$$

Since particles cannot be absorbed or created in region II and there is no reflection in region III, the net current in region I must be equal to the net current in region III, or $j_1 = j_3$. It then follows that the condition

$$T + R = 1 \tag{5.10}$$

is always satisfied (as one would expect). The transmission coefficient is sometimes also called the Penetration Factor and denoted as P.

To calculate a_1 and a_3, we apply the boundary conditions at the interfaces, x = 0 and x = L,

$$\psi_1 = \psi_2 , \qquad \frac{d\psi_1}{dx} = \frac{d\psi_2}{dx} \qquad x = 0 \tag{5.11}$$

$$\psi_2 = \psi_3 \ , \qquad \frac{d\psi_2}{dx} = \frac{d\psi_3}{dx} \qquad\qquad x = L \tag{5.12}$$

These 4 conditions allow us to eliminate 3 of the 5 integration constants. For the purpose of calculating the transmission coefficient we need to keep a_1 and a_3. Thus we will eliminate b_1, a_2, and b_2 and in the process arrive at the ratio of a_1 to a_3 (after about a page of algebra),

$$\frac{a_1}{a_3} = e^{(ik-\kappa)L}\left[\frac{1}{2} - \frac{i}{4}\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right)\right] + e^{(ik+\kappa)L}\left[\frac{1}{2} + \frac{i}{4}\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right)\right] \tag{5.13}$$

This result then leads to (after another half-page of algebra)

$$\frac{|a_3|^2}{|a_1|^2} = \left|\frac{a_3}{a_1}\right|^2 = \frac{1}{1 + \dfrac{V_o^2}{4E(V_o - E)}\sinh^2 \kappa L} \equiv P \tag{5.14}$$

with $\sinh x = (e^x - e^{-x})/2$. A sketch of the variation of P with κL is shown in Fig. 2.

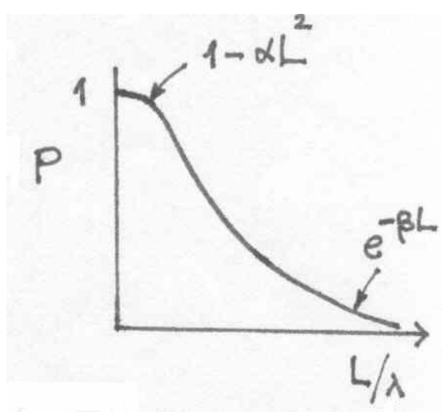

Fig. 2. Variation of transmission coefficient (Penetration Factor) with the ratio of barrier width L to λ, the effective wavelength of the incident particle.

Using the leading expression of sinh(x) for small and large arguments, one can readily obtain simpler expressions for P in the limit of thin and thick barriers,

$$P \sim 1 - \frac{V_o^2}{4E(V_o - E)}(\kappa L)^2 = 1 - \frac{(V_o L)^2}{4E}\frac{2m}{\hbar^2} \qquad \kappa L \ll 1 \tag{5.15}$$

$$P \sim \frac{16E}{V_o}\left(1 - \frac{E}{V_o}\right)e^{-2\kappa L} \qquad \kappa L \gg 1 \tag{5.16}$$

Thus the transmission coefficient decreases monotonically with increasing V_o or L, relatively slowly for thin barriers and more rapidly for thick barriers.

Which limit is more appropriate for our interest? Consider a 5 Mev proton incident upon a barrier of height 10 Mev and width 10 F. This gives $\kappa \sim$ 5 x 10^{12} cm^{-1}, or κL ~ 5. Using (5.16) we find

$$P \sim 16x\frac{1}{2}x\frac{1}{2}xe^{-10} \sim 2x10^{-4}$$

As a further simplification, one sometimes even ignores the prefactor in (5.16) and takes

$$P \sim e^{-\gamma} \tag{5.17}$$

with

$$\gamma = 2\kappa L = \frac{2L}{\hbar}\sqrt{2m(V_o - E)} \tag{5.18}$$

We show in Fig. 3 a schematic of the wave function in each region. In regions I and III, ψ is complex, so we plot its real or imaginary part. In region II ψ is not oscillatory. Although the wave function in region II is nonzero, it does not appear in either the transmission or the reflection coefficient.

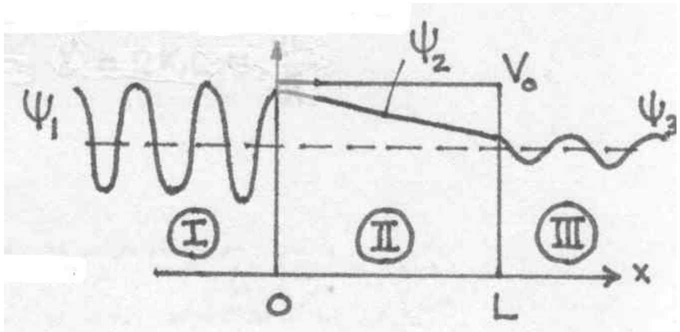

Fig. 3. Particle penetration through a square barrier of height V_o and width L at energy E ($E < V_o$), schematic behavior of wave functions in the three regions.

When the potential varies continuously in space, one can show that the attenuation coefficient γ is given approximately by the expression

$$\gamma \cong \frac{2}{\hbar} \int_{x_1}^{x_2} dx \left[2m\{V(x) - E\} \right]^{1/2} \qquad (5.19)$$

where the limits of integration are indicated in Fig. 4; they are known as the 'classical turning points'. This result is for 1D. For a spherical barrier ($\ell = 0$ or s-wave solution),

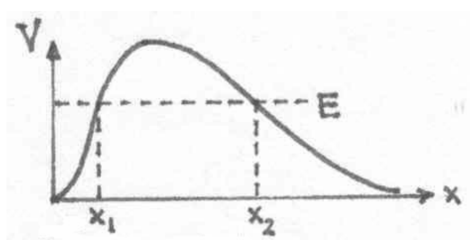

Fig. 4. Region of integration in (5.19) for a variable potential barrier.

one has

$$\gamma \approx \frac{2}{\hbar} \int_{r_1}^{r^2} dr \left[2m\{V(r) - E\} \right]^{1/2} \qquad (5.20)$$

We will use this expression in the discussion of α -decay.