

# 22.01 Fall 2015, Problem Set 4 Solutions (Analytical Version)

Due: October 21, 11:59PM on Stellar

October 31, 2015

Complete all the assigned problems, and do make sure to show your intermediate work. Please upload your full problem set in PDF form on the Stellar site. Make sure to upload your work at least 15 minutes early, to account for computer/network issues.

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## 1 Other Worldly Decay

Suppose a parallel universe exists, where for an ensemble of radioactive particles in a box, the probability of one particle decaying in a fixed amount of time  $\Delta t$  is proportional by a constant  $\eta$  to the number of unstable particles remaining in the box. Derive a new expression for the half life of a given other-worldly isotope, as a function of the number of particles in a box ( $N_0$ ) at time  $t_0$ . Graph this expression compared to the normal half-life equation, and comment on the reasons for the differences in the shapes of the curves.

*This time, instead of the probability of an ensemble of particles decaying just being a proportionality constant  $\lambda$ , giving the following equation:*

$$\frac{dN}{dt} = -\lambda N \quad (1)$$

*The probability of decay is instead proportional to the number of atoms left over:*

$$\frac{dN}{dt} = (-\eta N) N \quad (2)$$

*Separating variables, we get:*

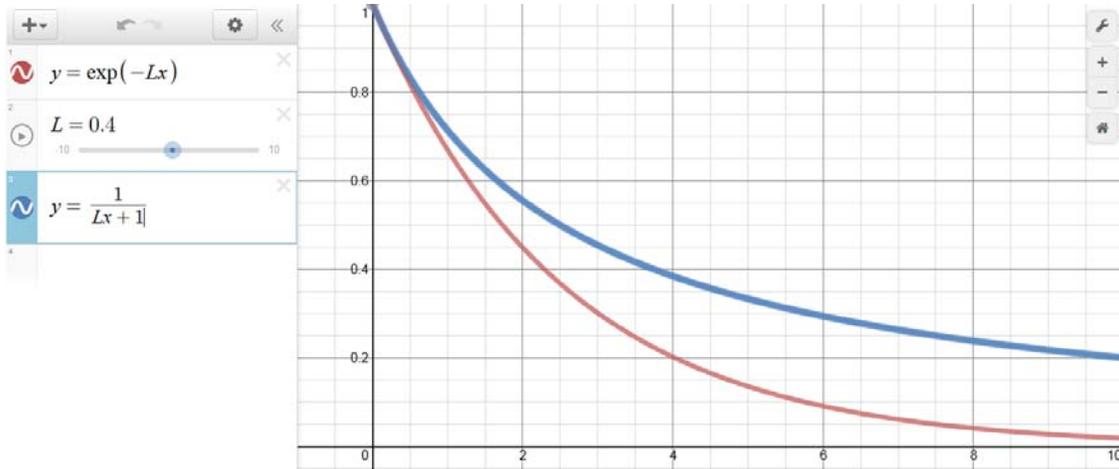
$$\frac{dN}{N^2} = -\eta dt \quad (3)$$

$$\frac{-1}{N} = -\eta t + C; \quad N = \frac{1}{\eta t + C} \quad (4)$$

*Now to find the integration constant: we assume that  $N = N_0$  @  $t = 0$ ;*

$$N = \frac{1}{\eta t + C}; \quad C = \frac{1}{N_0}; \quad N = \frac{1}{\eta t + \frac{1}{N_0}} \quad (5)$$

*Graphed, the functions are as follows:*



## 2 Activity and Half Lives

Back to earth for now... suppose a given smoke detector is rated at an activity of  $0.8\mu\text{Ci}$  of  $^{241}\text{Am}$  upon manufacture. It is powered by a [betavoltaic power module](#), which encases a tritium source ( $^3\text{H}$ ) rated at an activity of  $2\text{Ci}$  upon its manufacture.

1. Which component of the smoke detector will fail to provide exactly  $1\text{nA}$  of current first, and after how long? Assume that the equation for allowable current through an ionization chamber is as follows:

$$I_{max} = e_c \left( \frac{E_m}{E_i} \right) \left[ \frac{3.7 \cdot 10^{10} \text{ Bq}}{Ci} \right] S(t) \quad (6)$$

where  $I_{max}$  is the saturation current through the ionization chamber,  $e_c$  is the charge of an electron in Coulombs,  $E_m$  is the energy of the ionizing particle (you will have to look this up),  $E_i$  is the ion formation pair energy in air ( $14\text{eV}$ ), and  $S(t)$  is the activity of the  $^{241}\text{Am}$  source.

Here, we want to compare when each component will reach a current of  $1\text{nA}$ . For the  $^{241}\text{Am}$  source, we use the formula in Equation 6, while for the tritium isotope we just equate its activity to the current it can provide directly by beta decay:

$$I_{^{241}\text{Am}} = 10^{-9} \text{ A} = (1.6 \cdot 10^{-19} \text{ C}) \left( \frac{5.638 \cdot 10^6 \text{ eV}}{14 \text{ eV}} \right) \left[ \frac{3.7 \cdot 10^{10} \text{ Bq}}{Ci} \right] [(8 \cdot 10^{-7} \text{ Ci}) e^{-\lambda_{^{241}\text{Am}} t}] \quad (7)$$

$$I_{^3\text{H}} = 10^{-9} \text{ A} = (1.6 \cdot 10^{-19} \text{ C}) \left[ \frac{3.7 \cdot 10^{10} \text{ Bq}}{Ci} \right] (2 \text{ Ci}) e^{-\lambda_{^3\text{H}} t} \quad (8)$$

$$\lambda_{^3\text{H}} = \frac{\ln 2}{(12 \text{ yrs}) \left( 3 \cdot 10^7 \frac{\text{sec}}{\text{yr}} \right)} = 1.93 \cdot 10^{-9} \text{ s}^{-1}; \quad \lambda_{^{241}\text{Am}} = \frac{\ln 2}{(432.2 \text{ yrs}) \left( 3 \cdot 10^7 \frac{\text{sec}}{\text{yr}} \right)} = 5.35 \cdot 10^{-11} \text{ s}^{-1} \quad (9)$$

$$t_{H,1\text{nA}} = 42.7 \text{ yrs}; \quad t_{Am,1\text{nA}} = 402.3 \text{ yrs} \quad (10)$$

Clearly the tritium source will run out sooner.

2. Suppose this first component were to be inexhaustible. How long would it take for the second component to drop to a supplied/allowable current of exactly  $1\text{nA}$ ? For this question, make sure to take the uncertainty due to significant figures into account, and give a maximum, minimum, and expected value.

Here we must account for the fact that the activity of the  $^{241}\text{Am}$  source was specified as  $0.8\mu\text{Ci}$ , which carries an uncertainty of  $\pm 0.05\mu\text{Ci}$ . The expected value is given in part 2.1 above. The minimum and maximum values for the time to Americium failure:

$$t_{Am,min} = 362.1 \text{ yrs}; \quad t_{Am,1\text{nA}} = 440.1 \text{ yrs} \quad (11)$$

### 3 Radioactive Food

Two of the most radioactive foods we eat are bananas and brazil nuts.

1. Look up the specific activities of each of these foods, and write the decay equations for the isotope(s) responsible for each one.

*Bananas' radioactivity comes principally from the decay of potassium-40, which undergoes beta decay about 90% of the time and electron capture (or positron decay) about 10% of the time:*



*The specific activity of bananas is about  $3.5 \frac{\text{nCi}}{\text{kg}}$ . Brazil nuts, in addition to containing even more potassium ( $3.5 \frac{\text{nCi}}{\text{kg}}$ ), also contain non-negligible amounts of radium ( $1 - 7 \frac{\text{nCi}}{\text{kg}}$ ):*



2. Despite being more radioactive, why do Brazil nuts give the body less of a radiation dose compared to bananas?

*It actually appears that by potassium content alone, Brazil nuts would actually give more of a dose compared to bananas per kg of wet matter. The only way one could imagine bananas giving less radiation is because the human body needs water to survive, and because bananas are about 75% water compared to very little water in Brazil nuts. Therefore, bananas are more radioactive per dry weight of product.*

3. How long would you have to keep these two foods before eating to ensure that their radioactivity has dropped to 10% of its original level? When would you consider the "original level" to be established (in other words, when is  $t_0$ )?

*The equations for activity of each are as follows:*

$$A_{\text{Banana}} = -\lambda_K N_K; \quad A_{\text{Nut}} = -\lambda_K N_K - \lambda_{\text{Ra}} N_{\text{Ra}} \quad (16)$$

*For bananas this is easy, we just need to know when the number of  ${}^{40}\text{K}$  atoms reaches ten percent of its original value:*

$$\frac{N}{N_0} = 0.1 = e^{-\lambda_K t}; \quad t = 4.2 \text{ billion yrs} \quad (17)$$

*For Brazil nuts, the picture may look more complex, but consider the differences in half lives between radium and potassium. The former has so short a half life compared to the latter, that the activity of Brazil nuts is almost entirely determined by radium:*

$$\frac{N}{N_0} = 0.1 = e^{-\lambda_{\text{Ra}} t}; \quad t = 5,315 \text{ yrs} \quad (18)$$

### 4 Successive Decay Chains and Radioactive Dating

1. For the smoke detector problem above, repeat your analysis in parts 1 and 2, but this time take successive decays of  ${}^{241}\text{Am}$  into account. You may (and should) make an assumption that half-lives below a certain time constant (which you will have to choose) can be considered instantaneous, and write your total ionization chamber saturation current as a sum of all potentially present isotopes.

*This is a bit of a trick question! The half life of Am is 432.2 years, while its next decay product has a half life of 2144000 years. This is so much larger, that the extra radioactivity from neptunium decay can be neglected, and the answer to this problem is the same as in problem 2.*

2. One major source of radioactivity intake for some people is [smoking](#). This is because soils with elevated radon levels create daughter products, which stick to tiny hairs on the tobacco leaves.



$$N_{Po-210}(t) = N_{Pb-per-pack} \frac{\lambda_{Pb-210}}{\lambda_{Po-210} - \lambda_{Pb-210}} (e^{-\lambda_{Pb-210}t} - e^{-\lambda_{Po-210}t}) \quad (25)$$

From these equations, one can define the total number of radioactive disintegrations absorbed by the smoker, assuming (justifiably) that all the metals involved are heavy metals, and bio-accumulate:

$$Disintegrations - per - pack = \int_t^{t_{death}} A_{one-pack}(t) dt \quad (26)$$

Now we just have to define a couple of fixed times,  $t_{start-smoking}$  and  $t_{death}$ , as fixed times in days. Then we can construct an equation accounting for all the disintegrations that will happen during the smoker's lifetime:

$$Lifetime\ Disintegrations = \sum_{i=0}^{\frac{t_{death} - t_{start-smoking}}{86,400 \frac{sec}{day}}} \left[ \int_{86,400i}^{86,400(t_{death} - 86,400i)} A_{one-pack}(t) dt \right] \quad (27)$$

- (c) How long would you have to keep these cigarettes in storage for this radioactivity to reach 1% of its original level, from the day the tobacco was picked? You will need to consider simultaneous production and decay of radioactive isotopes in your answer.

Using the equations above, one can solve for  $t_{1\%}$  using the following equation:

$$0.99 = \frac{\int_0^{t_{1\%}} A_{one-pack}(t) dt}{\int_0^{\infty} A_{one-pack}(t) dt} \quad (28)$$

## 5 Statistics and Certainty

For these problems, consider that you have just measured the activity of a source stamped at  $1\mu Ci$  to be  $0.70\mu Ci$ , by measuring a count rate of 15 counts per second.

1. Taking uncertainty into account, how long would you have to count to ensure 95% confidence in your measurement? What would the standard deviation of your measurement be in this case?

*In this case, the standard deviation of this count rate is as follows:*

$$\sigma = \sqrt{\frac{15\ CPS}{t_{counting}}} \quad (29)$$

*If we measured the specimen to have an activity of  $0.70\mu Ci$ , then that means the uncertainty on this measurement due to significant figures alone is  $0.005\mu Ci$ , or roughly 0.7%. If we want to be 95% confident in this measurement, then we construct our equation as follows:*

$$2\sigma = 2\sqrt{\frac{15\ CPS}{t_{counting}}} = 0.007; \quad t_{counting} = 1.22 \cdot 10^7\ sec \quad (30)$$

*This means that we would have to count for almost five months to ensure this confidence in our measurement!!!*

2. Taking uncertainty into account, when could this source have been calibrated at  $1\mu Ci$ ? A stated calibration activity of  $1\mu Ci$  means that the actual activity could range from  $0.5 - 1.5\mu Ci$ . In the case that the starting activity was  $1.5\mu Ci$ , then the calibration date in days would have been as follows, relative to today:

$$(t_{today} - t_{cal}) = -\ln\left(\frac{0.70\mu Ci}{1.5\mu Ci}\right) (\lambda) \left(\frac{1\ day}{86,400\ sec}\right) = 8.8 \cdot 10^{-6} \lambda\ days\ ago \quad (31)$$

In the case that the actual activity was  $0.5\mu Ci$ , the equation becomes:

$$(t_{today} - t_{cal}) = -\ln\left(\frac{0.70\mu Ci}{0.5\mu Ci}\right) (\lambda) \left(\frac{1 \text{ day}}{86,400 \text{ sec}}\right) = 3.9 \cdot 10^{-6} \lambda \text{ days in the future!!!} \quad (32)$$

## 6 Radioactive Dating with Confidence

For this problem, [consider the methods used to radioactively date the Shroud of Turin](#), thought to be the burial cloth of Jesus of Nazareth.

1. Why did the investigators use carbon as the dating isotope? Consider what other isotopes could have been present, and give at least three reasons that carbon was chosen.

*The investigators used carbon-14 because its half life is 5,730 years, and if the shroud were indeed the burial cloth of Jesus, then a sizeable chunk of radioactivity would be measurable. If the half life were too short or too long, then very little counts would be acquired, greatly increasing the uncertainty of the count rate. The goal here would be to maximize the activity of the isotope being counted. Too short a half life, and it would have all decayed away. Too long, and very few atoms would decay at any given time at all.*

2. How did the investigators use statistics to prove beyond a reasonable doubt that the Shroud of Turin was *not* the burial cloth? What is a  $\chi^2$  test, and how did they arrive at the  $\chi^2$  values in the paper?

*Samples of the shroud were sent to different laboratories, for their own dating procedures and uncertainty to be measured. Each laboratory used their own procedures, showing that each sample (1 - the shroud, and 2-4, the "control" samples known to be from different time periods) were able to be measured within a certain range of ages. The chi-squared test here is applied to measure the "goodness of fit" of each sample's ages, and tells us the confidence of a random measurement of that specimen falling within the uncertainty bounds of the expected, or "theoretical," measured age. The fact that many laboratories were only able to measure the shroud sample with 5% confidence within the specified limits does not mean that they were wrong, as their dating methods showed great confidence for the other, control samples.*

3. Why did the investigators send so many unknown control samples to so many laboratories, and why did they use different cleaning procedures?

*This was done to avoid any chance of introducing bias into the results. Many, many people encounter significant quantities of feelings when dealing in matters of faith and religion, therefore those who would "want" the shroud to be or not to be the real burial cloth could have tampered with the evidence if they knew which piece they had. Keeping everyone guessing completely removes the incentive to tamper with the specimens. Different cleaning procedures were also used, to ensure that if one cleaning procedure artificially reduced the carbon-14 content of a specimen, then it would be discovered, and that method could be eliminated from altering the actual amount of carbon-14 present.*

## References

- [1] C. Litton, "Optimizing Ionization-Type Smoke Detectors," *Fire Technol.*, 15(1) (1979).

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