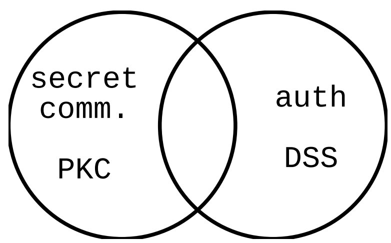
MIT 6.443J / 8.371J / 18.409 / MAS.865
Quantum Information Science
April 27, 2006

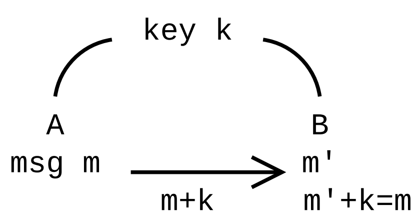
Unconditional Security of QKD

1. Cryptography
2. Quantum Key Distribution: BB84
3. EPR Protocol
4. CSS Code Protocol
5. Secure BB84

1 Crytography



In the Vernam Cipher (one-time pad), Alice and Bob share a secret key k.



Eve has $m + k$, but

$$\begin{aligned} I(m+k, m) &= H(m+k) - H(m+k/m) \\ &= H(m+k) - H(k) \\ &= 0 \end{aligned}$$

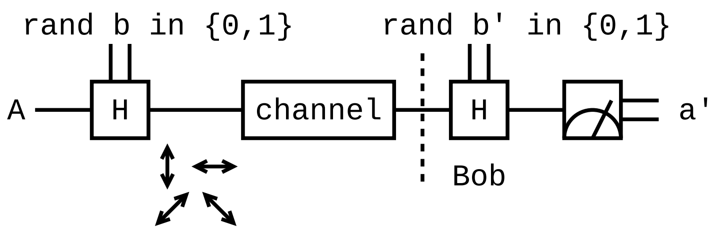
The key k is called a "pad." It is referred to as "one-time" because k can't be reused.

Distribution of k $\Rightarrow$ "security criterion"

$$I(\text{Eve}, \text{key}) = 2^{-l}$$

where resources required $\sim$ poly(l).

2 Quantum Key Distribution: BB84



$$\begin{aligned} a &= |0\rangle, |1\rangle \\ &= \updownarrow, \leftrightarrow \end{aligned}$$

Keep all bits for which $b' = b$. A and B hash obtain key k.

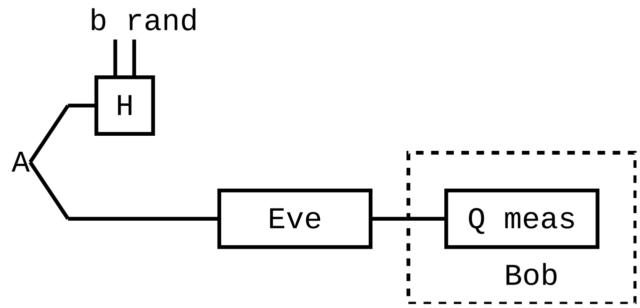
Thm. *Info gain $\Leftrightarrow$ disturbance. In any attempt to distinguish non-orthogonal states $|\psi\rangle$ and $|\phi\rangle$, information gain is only possible at the expense of disturbing the states.*

Proof. WLOG assume

$$\begin{array}{rcl} |\psi\rangle|u\rangle & \rightarrow & |\psi\rangle|v\rangle \\ |\phi\rangle|u\rangle & \rightarrow & |\phi\rangle|u'\rangle \\ \langle\phi|\psi\rangle & = & \langle\phi|\psi\rangle\langle v|v'\rangle \\ 1 & = & \langle v|v'\rangle \\ |v\rangle & = & |v'\rangle \\ & \text{contradiction} & \end{array}$$

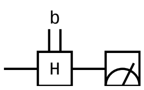
□

Problem: collective attacks



3 EPR Protocol

Perfect EPR Pair ⇒ good key.

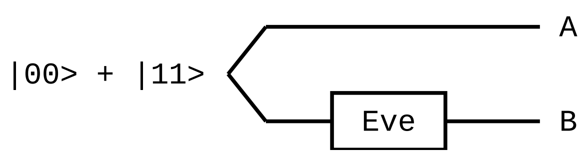
- A announces b
- B does 

- Random checks (test Bell's inequalities)
- Entaglement purification ⇒ m EPR pairs
- Measure, get key

Q: what is Eve's mutual information with k? We want:

$$I \sim e^{-l}$$

⇒ bound Eve's errors
Does classical statistics apply? The most general model for Eve is:



Eve can be treated as an error on the state $|00\rangle + |11\rangle$:

	Error
$\lvert 00\rangle + \lvert 11\rangle \rightarrow \lvert 00\rangle + \lvert 11\rangle$	I
$\lvert 00\rangle + \lvert 11\rangle \rightarrow \lvert 00\rangle - \lvert 11\rangle$	Z
$\lvert 00\rangle + \lvert 11\rangle \rightarrow \lvert 01\rangle + \lvert 10\rangle$	X
$\lvert 00\rangle + \lvert 11\rangle \rightarrow \lvert 01\rangle - \lvert 10\rangle$	iY

Define:

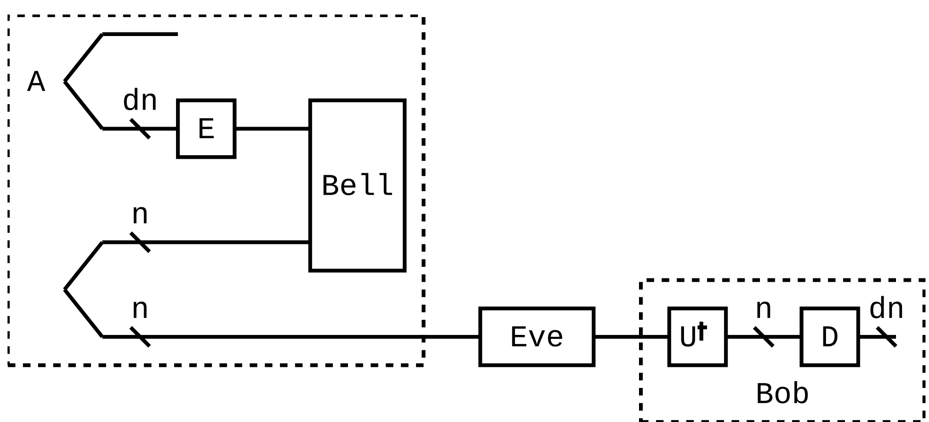
$$\begin{aligned}\Pi_{bf} &= |\beta_{01}\rangle\langle\beta_{01}| + |\beta_{11}\rangle\langle\beta_{11}| \\ \Pi_{pf} &= |\beta_{10}\rangle\langle\beta_{10}| + |\beta_{11}\rangle\langle\beta_{11}|\end{aligned}$$

Claim: we can use classical statistics because $[\Pi_{bf}, \Pi_{pf}] = 0$. Measure the following randomly on random pairs:

$$\begin{aligned}\Pi_{bf}, &\qquad I - \Pi bf \\ \Pi_{pf}, &\qquad I - \Pi pf\end{aligned}$$

Theorem: Random Sampling. *Consider $2n$ bits with $2\mu n$ ones. Measure n bits, obtaining kn ones. Prob$[|k-\mu| > \epsilon] \sim e^{-O(n^2\epsilon)}$ as $n \to \infty$ (Chernoff bound).*

$\Rightarrow$ How to purify?
Let $\delta_n = n - nt$, where t is the estimated number of errors. Let E, D be an encoder pair for a $[[n, \delta_n]]$ QECC. Result: QECC garantees:



$$F(\rho, |\beta_{00}\rangle^{\otimes \delta n})^2 \geq 1 - 2^{-l}$$

Goal: Bound $I(\text{Eve}, \text{key})$

Lemma: High Fidelity $\Rightarrow$ low entropy. *If $F(\rho, |\psi\rangle)^2 > 1 - 2^{-l}$, then $S(\rho) < (n+l)2^{-l}$.*

Proof. If $\langle\psi|\rho|\psi\rangle > 1 - 2^{-l}$, then the maximum eigenvalue of ρ is greater than $1 - 2^{-l}$.

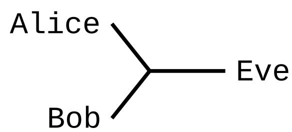
$$S(\rho) < S(\rho_{\max}) = S\left(\begin{bmatrix} 1-2^{-l} & & & \\ & x & & \\ & & x & \\ & & & \ddots \end{bmatrix}\right)$$

where $x = \frac{2^{-l}}{2^n - 1}$.

$$\begin{aligned} S(\rho_{\max}) &= -(1-2^{-l})log(1-2^{-l}) \\ &= -2^{-l}log\frac{2^{-l}}{2^n-1} \\ &\sim (n+l)2^{-l} \end{aligned}$$

□

Now Apply Holevo's theorem.



$$I(\text{Eve}, A\text{and}B) < S(\rho) < O(2^{-l})$$

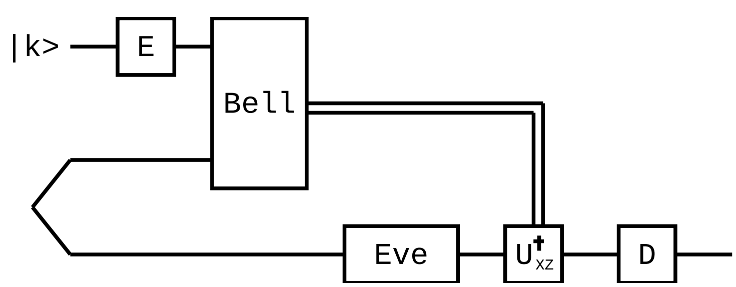
Problems:

1. need efficient codes (CSS works)
2. need quantum memory
3. need quantum computer

The last two are done away with by BB84.

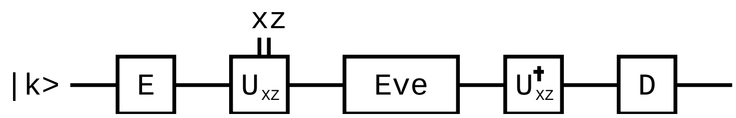
4 CSS Code Protocol

Step 1: EPR → Random Codes The circuit is equivalent to:



$$|\psi\rangle = DU_{xz}^{\dagger}\mathcal{E}_{\text{Eve}}U_{xz}E|k\rangle$$

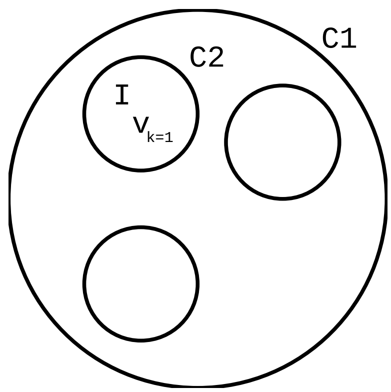
Also equivalent to:



Step 2: Let C_1, C_2 be classical $[n, k_1]$ and $[n, k_2]$ codes correcting up to t errors with $C_2 \subset C_1$. CSS($C1$, $C2$) is a $[[n, k_1, k_2]]$ quantum code with states:

$$|\psi_k\rangle = \frac{1}{|C_2|} \sum_{w \in C_2} |v_k + w\rangle,$$

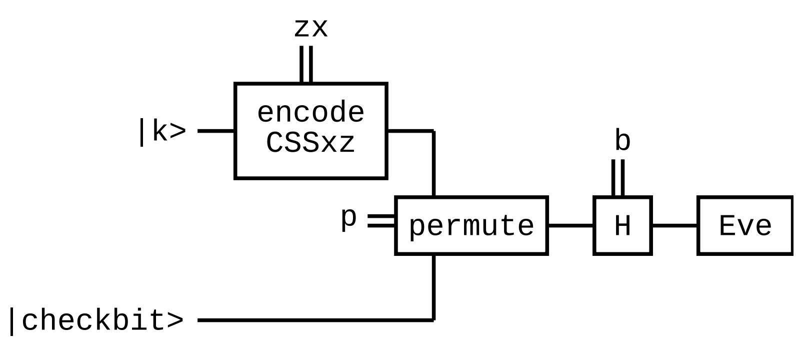
where v_k is a coset representative of C_2 in C_1.



Define: CSS$_{zx}(C_1, C_2)$

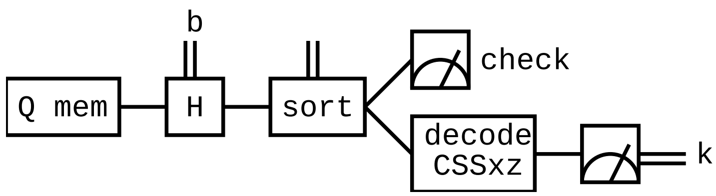
$$|\psi_{kzx}\rangle = \frac{1}{\sqrt{|C_2|}} \sum_{w \in C_2} (-1)^{v_k + w - z}$$

CSS code protocol:



- Alice announces x, z, p, b

- Bob does:



- If error rate $> tn$, abort

5 Secure BB84

1. Remove Quantum Computer Bob doesn't care about z errors.

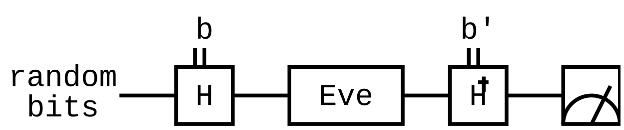
$$\rho = \frac{1}{2^n} \sum_z |\psi_{kxz}\rangle\langle\psi_{kxz}|$$

Alice need not reveal z!

$$\begin{aligned} \rho &= \frac{1}{|C_2|} \sum_{w \in C_2} |v_k + w + x\rangle\langle v_k + w + x| \\ &= |\text{random bit string}\rangle \end{aligned}$$

2. Remove Quantum Memory Double number of qubits and bob measures random b', keep if $b' = b$.

Final Protocol



1. A and B discard if $b_i \neq b'_i$
2. compare check bits, obtain $A : x, B : x + \epsilon$
3. A announces $x - v_k$
4. B computes $x + \epsilon - (x - v_k) = \epsilon + v_k$
5. correction in $C_1 \to v_k$
6. Both compute coset index $v_k \to k$