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Quantum Information Science

February 16, 2006

Problem Set #2
 (due in class, 02-Mar-06)

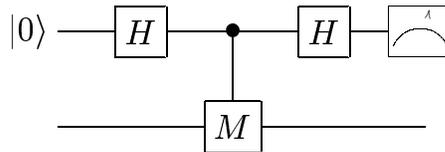
Lecture Topics (2/16, 2/23, 2/28): Stabilizer codes; topological quantum codes; computing on codes

Recommended Reading: Nielsen and Chuang, Section 10.5

Problems:

P1: (Measurements and stabilizers) Stabilizers are one of the most useful way to describe states and transformations in quantum information. In this problem we investigate how measurement is described using the stabilizer formalism. Recall that if an n -qubit state $|\psi\rangle$ is stabilized by $S = \langle g_1, g_2, \dots, g_n \rangle$, then $g|\psi\rangle = |\psi\rangle$ for all $g \in S$. Note that g_1, g_2, \dots, g_n are the *generators* of S .

- (a) Let M be an n qubit measurement operator (expressed as a product of pauli operators, as usual in the stabilizer formalism). Recall that we use the circuit



to measure M on $|\psi\rangle$, and this gives measurement outcomes $k = \{0, 1\}$ (corresponding to the $+1$ and -1 eigenstates, respectively). Show that if M commutes with g_j for all j , then the post-measurement state is stabilized by S . But if M anticommutes with some of the generators, say g_1, g_2, \dots, g_j (and commutes with g_{j+1}, \dots, g_n), then the post-measurement state is stabilized by

$$\langle (-1)^k M, g_1 g_2, g_1 g_3, \dots, g_1 g_j, \dots, g_{j+1}, g_{j+2}, \dots, g_n \rangle. \tag{1}$$

- (b) Let $|\psi\rangle$ be stabilized by $\langle XX, ZZ \rangle$. What is the stabilizer of the state after measuring IY ?
- (c) Let $|\psi\rangle$ be a single qubit state, and suppose we start with a system in the state $|\psi\rangle \otimes |0\rangle$, measure $Y \otimes X$, and then measure $I \otimes Y$. What Pauli operations do we need to perform following each of the measurements to steer the post-measurement state into the $+1$ eigenstate of each measured operator?
- (d) Compute the action of the above series of operations (with Pauli corrections) on the \bar{X} and \bar{Z} operators (the normalizer for the initial state $|0, \psi\rangle$). Describe the overall action in terms of standard gates.

- (e) Suppose we had started with the input state $|\psi\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2}$ instead, and performed the same two measurements. What would have happened then?

P2: (Passive error correction and decoherence free subspaces) Quantum error correction is an example of “active” error correction where one first detects whether an error occurs and then applies a correcting operation. An interesting area of research in quantum information is the possibility of “passive” error correction. We explore in this problem one of these possibilities, Decoherence Free or Noiseless Subspaces (DFS), in the context of stabilizer codes.

Recall the following:

- If S is the stabilizer for the code C , then $\forall V \in S$, and $\forall |\psi\rangle \in C$, $V|\psi\rangle = |\psi\rangle$.
- If $N(G)$ is the normalizer for the group G , then $\forall U \in N$ and $\forall V \in G$, $UVU^\dagger \in G$

As an example, we will use the three qubit bit flip code, C_{bf} , where $|0_L\rangle = |000\rangle$ and $|1_L\rangle = |111\rangle$. The stabilizer S is generated by ZZI and IZZ , that is $S = \langle ZZI, IZZ \rangle$

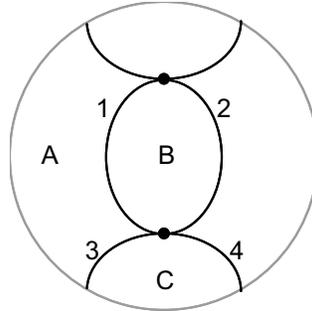
- Using the stabilizer formalism, show that the code can correct the errors $E_1 = XII$, $E_2 = IXI$ and $E_3 = IIX$.
- Now assume that instead of single qubit bit flips, the error is $E_1 = ZIZ$. Show that no error correction or detection is necessary; E_1 causes no errors. This is an example of the DFS condition; we say that $|\psi\rangle$ is in the DFS for errors $\{E_j\}$ iff $\forall j$, $E_j|\psi\rangle = c_j|\psi\rangle$, where c_j is a constant independent of $|\psi\rangle$.
- Show that in general, a code C is a DFS for the set of errors E_i if all the E_i are in the stabilizer of C .
- Show that there is no code for which all single qubit errors form the stabilizer.
- Let U_1 and U_2 be two quantum gates s.t. $U_2U_1 = X_L$ where X_L is a logical bit flip for the three qubit bit flip code (i.e. $X_L|0_L\rangle = |1_L\rangle$ and $X_L|1_L\rangle = |0_L\rangle$). For C_{bf} , we choose two sets of unitaries that satisfy this condition $A = \{U_1 = YYY, U_2 = ZZZ\}$ and $B = \{U_1 = ZYX, U_2 = YZI\}$. Show that $U_1U_2 = X_L$ for both sets A and B .
- We now examine the effect of errors. Assume that instead of the desired circuit, the following operation occurs: $U_2E_1U_1$.
In the QECC model, show that the error $E_1 = XII$ can be detected and corrected for both sets A and B . Also show that in the DFS model, the error $E_1 = ZIZ$ is “passively” corrected for set A and uncorrectable for set B .
- For a general code, given an arbitrary error E , what is the condition on U_1 and U_2 such that U_2EU_1 can be corrected using QECC techniques? Express in terms of stabilizers and normalizers.
- For a general code C which is a DFS for errors $\{E_j\}$, what is the condition on U_1 and U_2 such that the DFS condition will hold for $U_2E_jU_1$, $\forall j$? Express in terms of stabilizers and normalizers.

P3: (Topological QEC: Projective Plane Codes) Quantum error correction codes can be constructed using graphs on topological surfaces, where vertices and edges correspond to certain stabilizer operations, and the distance of the code is given by the shortest non-trivial topological chain of errors. In this problem, we explore a specific instance of such a construction, on a projective plane.

- (a) The projective plane in two dimensions, $\mathbb{R}P^2$, can be drawn as a disc in which antipodal points on the boundary are identified. Recall that a cellulation \mathcal{C} of a surface defines sets F , E , and V of faces, edges, and vertices. For each $e \in E$, there corresponds a qubit on which X_e and Z_e are the Pauli X and Z operators. Let $E_f \subset E$ be the set of edges around face $f \in F$, and let $E_v \subset E$ be the set of edges attached to vertex $v \in V$. For the set of all f and v , we define

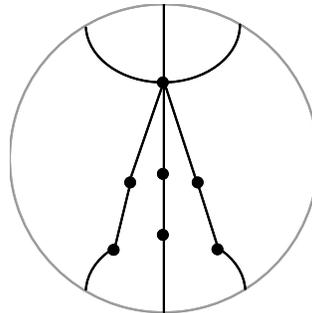
$$A_f = \bigotimes_{e \in E_f} Z_e \quad \text{and} \quad B_v = \bigotimes_{e \in E_v} X_e; \quad (2)$$

these are the stabilizers for the code. Give the three A_f and single unique B_v for the cellulation:



This code encodes four qubits ($|E| = 4$), corresponding to the four edges, labeled 1 through 4. Note there are only three distinct faces, labeled A, B, and C. Verify that the stabilizers commute. What state(s) do they stabilize?

- (b) Give the stabilizers for the cellulation:



What codewords are stabilized by these stabilizers?

- (c) The dual, \mathcal{C}^* , to a cellulation \mathcal{C} places vertices in middle of each face of \mathcal{C} , and edges perpendicular to the edges of \mathcal{C} , connecting the new vertices. Give the dual to the two cellulations above, and the codewords which are stabilized.
- (d) Prove that a cellulation is valid, i.e. it gives a set of stabilizers producing a well-defined code, only when each vertex has distinct edges, and no edge has the same face on both of its sides.
- (e) Can you construct a valid five edge cellulation of the projective plane?

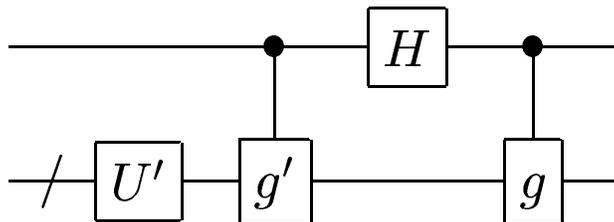
P4: (The Gottesman-Knill Theorem) An important result in quantum computation is that H , CNOT, and the Pauli gates are not universal for quantum computation, and in fact any quantum circuit composed from those gates (together with standard input states and measurements in the computational basis) can be simulated efficiently by a classical computer! This result is known as the *Gottesman-Knill* theorem, and in this problem we prove the essential result behind the theorem.

Let G_n denote the Pauli group on n qubits, that is, matrix multiplication acting on the set of n -fold tensor products of Pauli matrices (including multiplicative factors $\pm 1, \pm i$). By definition, we say the set of U such that $UG_nU^\dagger = G_n$ is the *normalizer* of G_n , and denote it by $N(G_n)$. The following theorem about the normalizer of the Pauli group holds:

Suppose U is any unitary operator on n qubits with the property that if $g \in G_n$ then $UgU^\dagger \in G_n$. Then up to a global phase U may be composed from $O(n^2)$ Hadamard, phase and controlled-NOT gates.

We can construct an inductive proof of this theorem as follows:

- (a) Prove that the Hadamard and phase gates (H and S) can be used to perform any normalizer operation on a single qubit.
- (b) Suppose U is an $n + 1$ qubit gate in $N(G_{n+1})$ such that $UZ_1U^\dagger = X_1 \otimes g$ and $UX_1U^\dagger = Z_1 \otimes g'$ for some $g, g' \in G_n$. Define U' on n qubits by $U'|\psi\rangle \equiv \sqrt{2}\langle 0|U(|0\rangle \otimes |\psi\rangle)$. Use the inductive hypothesis to show that this construction for U :



may be implemented using $O(n^2)$ Hadamard, phase and controlled-NOT gates.

- (c) Show that any gate $U \in N(G_{n+1})$ may be implemented using $O(n^2)$ Hadamard, phase and controlled-NOT gates.